LECTURE NOTES

Collège de France

Paris, France, November 2015

Lars-Erik Persson

Preface

This "Lecture notes" is a basic material written as a basis for the lectures ***The Hardy inequality-prehistory, history and current status*** and ***The interplay between Convexity, Interpolation and Inequalities*** presented at my visit at Collège de France in November 2015 on invitation by Professor Pierre-Louis Lions.

I cordially thank Professor Pierre-Louis Lions and Collège de France for this kind invitation. I also thank Professor Natasha Samko, Luleå University of Technology, for some related late joint research and for helping me to finalize this material.

I hope this material can serve not only as a basis of these lectures but also as a source of inspiration for further research in this area. In particular, a number of open questions are pointed out.

The material is closely connected to the following books:

[1] A. Kufner and L.E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[2] A. Kufner, L. Maligranda and L.E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[3] L. Larsson, L. Maligranda, J. Pečaric and L.E. Persson, *Multiplicative Inequalities of Carlson Type and Interpolation*, World Scientific Publishing Co., New Jersey-London-Singapore-Beijing-Shanghai-Hong Kong-Chennai, 2006.

[4] C. Niculescu and L.E. Persson, *Convex Functions and their Applications- A Contemporary Approach*. Canad. Math. Series Books in Mathematics, Springer. 2006.

[5] V. Kokilashvili, A. Meskhi and L.E. Persson, *Weighted Norm Inequalities for Integral transforms with Product Weights*, Nova Scientific Publishers, Inc., New York, 2010.

But also some newer results and ideas can be found in this Lecture Notes, in particular from the following manuscript:

[6] L.E. Persson and N. Samko, *Classical and New Inequalities via Convexity and Interpolation*, book manuscript, in preparation.

Collège de France, November, 2015

Lars-Erik Persson


ATLANTIC
OCEAN
NARVIK
KIRUNA
MALMBERGET
GÄLLIVARE
PORJUS
NORRBOTTEN
ARCTIC CIRCLE
HAPARANDA
TORNEÅ
BODEN
LULEÅ
SORSELE
STORAMAN
BASTUTRÄSK
VILHELMINA
GULF OF BOTHNIA

LECTURE II

The interplay between Convexity, Interpolation and Inequalities

Lars-Erik Persson

Luleå University of Technology, Sweden,
and *Narvik University College, Norway*
larserik@ltu.se

Collège de France

Paris, France, November 2015

More information concerning this lecture can be found in the following:

[1] L. Larsson, L. Maligranda, J. Pečaric and L.E. Persson, *Multiplicative Inequalities of Carlson Type and Interpolation*, World Scientific Publishing Co., New Jersey-London-Singapore-Beijing-Shanghai-Hong Kong-Chennai, 2006.

[2] C. Niculescu and L.E. Persson, *Convex Functions and their Applications- A Contemporary Approach*. Canad. Math. Series Books in Mathematics, Springer. 2006.

[3] L.E. Persson and N. Samko, *Classical and New Inequalities via Convexity and Interpolation*, book manuscript, in preparation.

However, this lecture contains also some newer information, which can not be found in these books and the references given there.

1 A. CONVEXITY $\Longrightarrow$ INEQUALITIES

Let I denote a finite or infinite interval on $\mathbb{R}_+$. We say that a function f is convex on I if, for $0 < \lambda < 1$, and all $x, y \in I$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

If this inequality holds in the reversed direction, then we say that the function f is concave.

Examples of convex functions are $f(x) = e^x, x \in \mathbb{R},\ f(x) = x^a, x \geq 0, a \geq 1$ or $a < 0$, and $(1 + x^p)^{1/p}, x \geq 0, p > 1$.

The notion of convexity (concavity) can be defined in a similar way for functions of more variables or even on more general sets. Here we just mention the following two-dimensional one by (the Swedish-Hungarian Professor) Marcel Riesz, which was very crucial when he proved his Riesz convexity theorem, which was very important when interpolation theory was initiated via the famous Riesz-Thorin interpolation theorem, see e.g. the book [B1] by J.Bergh and J. Löfström.

Example A1 Let a and b be complex numbers. Then the function

$$f(\alpha, \beta) = \log \max \frac{\left(|a+b|^{1/\alpha} + |a-b|^{1/\alpha}\right)^{\alpha}}{\left(|a|^{1/\beta} + |b|^{1/\beta}\right)^{\beta}}$$

is convex on the triangle $T : 0 \leq \alpha \leq \beta \leq 1$.

Remark Another inportant student of Riesz was (the Swedish Professor) Lars Hörmander, which has written one of the most important books concerning the notion of convexity and its applications, see [C2].

1.1 Convex functions at a first glance

J. L. W. V. Jensen claimed: ”It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If I am not mistaken in this, the notion ought to find a place in elementary expositions of the theory of real functions”.

The following useful estimates are more or less easy consequences of the convexity (concavity) of the function $f(x) = x^p$:

Example A2 Let $a_1, a_2, ..., a_n$ be positive numbers. Then

$$(a) \qquad \sum_{i=1}^n a_i^p \leq \left(\sum_{i=1}^n a_i\right)^p \leq n^{p-1} \sum_{i=1}^n a_i^p, \ \ p \geq 1,$$

$$(b) \quad n^{p-1} \sum_{i=1}^n a_i^p \leq \left(\sum_{i=1}^n a_i\right)^p \leq \sum_{i=1}^n a_i^p, \ \ 0 < p \leq 1.$$

The next example is a consequence of the convexity of the function $f(x) = e^x$.

Example A3 (Young's inequality) For any $a, b > 0, p, q \in \mathbb{R} \setminus \{0\}, \frac{1}{p} + \frac{1}{q} = 1$, it yields that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p > 1, \tag{1.1}$$

and

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}, \text{ if } p < 1, p \neq \{0\}. \tag{1.2}$$

”Proof of (1.1)”:

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \leq \frac{1}{p} e^{\ln a^p} + \frac{1}{q} e^{\ln b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Example A4 (Two fundamental inequalities” In the book [A3] by E.F. Beckenbach and R. Bellman it is claimed that the following inequalities are ”fundamental relations”: If $x > 0$ and $\alpha \in \mathbb{R}$, then

$$\begin{cases} x^\alpha - \alpha x + \alpha - 1 \geq 0 & \text{for} \quad \alpha > 1 \text{ and } \alpha < 0 \\ x^\alpha - \alpha x + \alpha - 1 \leq 0 & \text{for} \quad 0 < \alpha < 1. \end{cases} \tag{1.3}$$

Remark In particular, they show later in the book that several well-known inequalities follow directly from (1.3) e.g. the AG-inequality, Hölder's inequality, Minkowski's inequality, etc. In the first lecture it was also pointed out that also Bennett's inequalities (even in more precise form) of importance in interpolation theory follows in (1.3).

In [A3] it was given two different proofs of (1.3) but indeed (1.3) follows directly from the fact that the function $f(x) = x^\alpha$ is convex for $\alpha > 1$ and $\alpha < 0$ and concave for $0 < \alpha < 1$. In fact, if $f(x) = x^\alpha$, then the equation for the tangent at $x = 1$ is equal to $l(x) = \alpha(x-1) + 1$ and (1.3) follows directly.

Note also that (1.1) follows directly from (1.3) applied with $x = \frac{a^p}{b^q}$ and $\alpha = \frac{1}{p}$ (the case $0 < \alpha < 1$) and (1.2) follows from (1.3) in the same way by instead applying (1.3) in the cases $\alpha > 1$ and $\alpha < 0$.

We finish this Section by noting that also another useful inequality follows from convexity via Example A1.

Example A5 Let $a, b \in C, 1 < p \leq 2$ and $q = \frac{p}{p-1}$, then

$$(|a+b|^q + |a-b|^q)^{1/q} \leq 2^{1/q} (|a|^p + |b|^p)^{1/p} . \tag{1.4}$$

The inequality is sharp, i.e. $2^{1/q}$ can not be replaced by any smaller number.

One proof of (1.4) is to consider the convex function $f(\alpha, \beta)$ defined in Example A1. By using the parallelogram law

$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2) \tag{1.5}$$

and the triangle inequality

$$\max(|a+b|, |a-b|) \leq |a| + |b|, \tag{1.6}$$

and making some straightforward calculations the proof follows. However, the (scale of) inequalities in (1.4) may also be regarded as natural "intermediate inequalities" for the "endpoint inequalities" (1.5) and (1.6). This can be proved exactly via interpolation (see page 24).

1.2 Convexity and Jensen's inequality

We state Jensen's inequality in the following fairly general form:

Theorem A1 (Jensen's inequality) Let μ be a positive measure on a $\sigma-$algebra $\aleph$ in a set Ω so that $\mu(\Omega)$ =1. If f is a real $\mu-$integrable function, if $-\infty \leq a < f(x) < b \leq \infty$ for all $x \in \Omega$ and if Ψ is convex on (a, b), then

$$\Psi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \Psi(f) d\mu. \tag{1.7}$$

If Ψ is concave, then (1.7) holds in the reversed direction.

Remark If $\Omega = \mathbb{R}_+, n = 2, 3, \ldots,$ $\mu = \sum_{k=1}^n \lambda_k \delta_k$ (δ_k is the unity mass at $t = k$), $\lambda_k > 0$ and $\sum_{k=1}^n \lambda_k = 1$, then Jensen's inequality (1.7) coincides with discrete Jensen's inequality with $f(k) = a_k$. Conversely, if we put the mass $1 - \lambda$ at x and λ at y ($x, y \in I$) and assume that (1.7) holds for positive function Ψ, then, we have that

$$\Psi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Psi(x) + \lambda\Psi(y),$$

i.e. the function Ψ is convex. These considerations show in fact that Jensen's inequality is more or less equivalent to the notion of convexity.

1.3 Hölder type inequalities

Example A6 (Hölder's inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|f\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q},$$

i.e.

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^p d\mu \right)^{1/p} \left(\int_\Omega |g|^q d\mu \right)^{1/q}. \tag{1.8}$$

For the case $0 < p < 1$ (1.8) holds in the reverse direction.

The standard proof of (1.8) is obtained by just applying Young's inequality (1.1) with $a = f(x), b = g(x)$ and integrating. Another proof showing that (1.8) follows directly from Jensen's inequality reads:

We may without loss of generality assume that $\int_\Omega |g| d\mu < \infty$ and apply Jensen's inequality to obtain that

$$\left(\frac{1}{\int_\Omega |g| d\mu} \int_\Omega |fg| d\mu \right)^p \leq \left(\int_\Omega |g| d\mu \right)^{-1} \int_\Omega |f|^p |g| d\mu$$

i.e. that

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |g| d\mu \right)^{1-1/p} \left(\int_\Omega |f|^p |g| d\mu \right)^{1/p}.$$

Put $|f||g|^{1/p} = |f_1|$ and $|g|^{1/q} = |g_1|$ and we find that

$$\int_\Omega |f_1 g_1| d\mu \leq \left(\int_\Omega |f_1|^p d\mu \right)^{1/p} \left(\int_\Omega |g_1|^q d\mu \right)^{1/q}.$$

We just change notation and (1.8) is proved.

Remark We have equality in Hölder's inequality when $g(x) = (f(x))^{p-1}$. In particular, this means that the following important relation

$$\left(\int_\Omega |f|^p d\mu \right)^{1/p} = \sup \int_\Omega |f| \varphi d\mu, \tag{1.9}$$

yields for each $p > 1$, where supremum is taken over all $\varphi \geq 0$ such that $\int_\Omega \varphi^q d\mu = 1$. For the case $0 < p < 1$ (1.9) yields with "sup" replaced by "inf".

The investigations above show that (1.8) can be generalized to a version with finite many functions $f_1, f_2, ..., f_n$ involved:

Let $p_1, p_2, ..., p_n, n = 3, 4, ...,$ be positive numbers such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. Then

$$\int_\Omega |f_1 f_2 \cdots f_n| d\mu \leq \left(\int_\Omega |f_1|^{p_1} d\mu \right)^{1/p_1} \cdots \left(\int_\Omega |f_n|^{p_n} d\mu \right)^{1/p_n}. \tag{1.10}$$

We finish this Section by stating the following further generalization (with infinite many functions involved) to what sometimes is called a "continuous form " of Hölder's inequality.

Example A7 Let $K(x,y)$ be positive and measurable on $(\Omega_1 \times \Omega_2, \mu \times \nu)$, where $\int\limits_{\Omega_2} d\nu = 1$. Then

$$\int\limits_{\Omega_1} \exp\left(\int\limits_{\Omega_2} \log K(x,y)d\nu\right) d\mu \leq \exp \int\limits_{\Omega_2} \log\left(\int\limits_{\Omega_1} K(x,y)d\mu\right) d\nu \tag{1.11}$$

Remark The proof of (1.11) can be performed by using Jensen's inequality in a suitable way. However, we will present a new proof later on by just using the fact that (1.11) is a limit inequality of another useful continuous integral inequality.

Remark By applying (1.11) with $\Omega_2 = Y_1 \cup Y_2 \cup ... \cup Y_n, K(x,y) = |f^{p_i}(x)|$ on Y_i and $\int\limits_{\Omega_2} d\nu = \frac{1}{p_i}, i = 1,2,...,n,$ we get (1.10).

Remark A specialcase of (1.11) was crucial when an interpolation theory for infinite many spaces was created.

1.4 Minkowski type inequalities

Example A8 (Minkowski's inequality) If $p \geq 1$, then

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} \leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}. \tag{1.12}$$

The standard proof of (1.12) is to use Hölder's inequality but here we state another proof based on the (quasi-linearization) formula (1.9). It yields that

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} = \sup_{\|\varphi\|_q=1} \int_\Omega |f+g|\varphi d\mu \leq \sup_{\|\varphi\|_q=1} \int_\Omega (|f|\varphi + |g|\varphi) d\mu \leq$$

$$\leq \sup_{\|\varphi\|_q=1} \int_\Omega |f|\varphi d\mu + \sup_{\|\varphi\|_q=1} \int_\Omega |g|\varphi d\mu = \left(\int_\Omega |f|^p d\mu\right)^{1/p} + \left(\int_\Omega |g|^p d\mu\right)^{1/p}.$$

This proof is easy to generalize and obtain the following more general continuous version of (1.12):

Example A9 (Minkowski's integral inequality) Let the positive kernel $K(x,y)$ be measurable on $(\Omega_1 \times \Omega_2, \mu \times \nu)$. If $p \geq 1$, then

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} K(x,y)d\nu\right)^p d\mu\right)^{1/p} \leq \int_{\Omega_2} \left(\int_{\Omega_1} K^p(x,y)d\mu\right)^{1/p} d\nu. \tag{1.13}$$

For the case $0 < p < 1$, (1.13) holds in the revered direction

Proof. Let $p > 1$. We use again the idea from (1.9) and obtain that

$$I_0 := \left(\int_{\Omega_1} \left(\int_{\Omega_2} K(x,y)d\nu\right)^p d\mu\right)^{1/p} = \sup_{\|\varphi\|_q=1} \int_{\Omega_1} \varphi(x) \int_{\Omega_2} K(x,y)d\nu d\mu,$$

where supremum is taken over all measurable φ such that $\int_a^b \varphi^q(x)dx = 1, q = p/(p-1)$. Hence, by using the Fubini theorem and an obvious estimate, we have that

$$I_0 = \sup_{\|\varphi\|_q=1} \int_{\Omega_2} \int_{\Omega_1} K(x,y)\varphi(x)d\mu d\nu \leq \int_{\Omega_2} \left(\sup_{\|\varphi\|_q=1} \int_{\Omega_1} K(x,y)\varphi(x)d\mu \right) d\nu =$$

$$= \int_{\Omega_2} \left(\int_{\Omega_1} K^p(x,y)d\mu \right)^{1/p} d\nu.$$

For $p = 1$ we have even equality in (1.13) because of the Fubini theorem, so the proof is complete.

The proof of the case $0 < p < 1$ is similar (we just need to use the representation formula (1.9) with "sup" replaced by "inf"). $\square$

By putting pointmasses δ_i in the points y_i and $K(x, y_i) = f_i(x), i = 2, 3, ,$ we obviously get a wellknown generalization of (1.12) with n functions involved.

For applications the following special case of Example A9 is useful e.g. when working with mixed-norm L_p spaces and we need some estimate replacing the Fubini theorem. More exactly, we let $\Omega_1 = \Omega_2 = \mathbb{R}$ with Lebesgue measure and put

$$K(x,y) = \begin{cases} k(x,y)\Psi(y)\Phi^{1/p}(x), & a \leq y \leq x, \\ 0 & , x < y \leq b, \end{cases}$$

where $k(x,y), \Psi(y)$ and $\Phi(x)$ are measurable so that Minkowski's integral inequality can be used.

Example A10 (Minkowski's integral inequality of Fubini type) If $p \geq 1$, then

$$\left(\int_a^b \left(\int_a^x k(x,y)\Psi(y)dy \right)^p \Phi(x)dx \right)^{1/p} \leq \int_a^b \left(\int_y^b \Phi(x)k^p(x,y)dx \right)^{1/p} \Psi(y)dy. \quad (1.14)$$

Remark In fact, our previous continuous form of Hölder's inequality (see Example A7) may be regarded as a limit case of (1.13) but in order to understand this we need to consider (the scale of) powermeans.

1.5 Powermean inequalities

The scale of powermeans $\{\mathcal{P}_\alpha(f;\mu)\}, -\infty < \alpha < \infty$, of a function f on a finite measure space (Ω, μ) is defined as follows:

$$\mathcal{P}_\alpha(f;\mu) := \begin{cases} \left(\frac{1}{\mu(\Omega)} \int\limits_\Omega |f|^\alpha d\mu \right)^{1/\alpha} & , -\infty < \alpha < \infty, \alpha \neq 0, \\ \exp \frac{1}{\mu(\Omega)} \int\limits_\Omega \log |f| d\mu, & \alpha = 0. \end{cases} \quad (1.15)$$

(for the case $\alpha \geq 0$ we assume that $f > 0$ a.e.)

A special case for positive sequences $a = \{a_i\}_{i=1}^n, n \in \mathbb{Z}_+$, is obtained by letting $\mu = \sum\limits_{i=1}^n \delta_i$ and $f_i = a_i, i = 1, 2, ..., n$ $(\Omega = \mathbb{R}_+)$:

$$\mathcal{P}_\alpha(a) := \begin{cases} \left(\frac{1}{n}\sum\limits_{n=1}^{n} a_i^\alpha\right)^{1/\alpha}, -\infty < \alpha < \infty, \alpha \neq 0, \\ \left(\prod\limits_{i=1}^{n} a_i\right)^{1/n}, \ \alpha = 0. \end{cases} \tag{1.16}$$

As a generalization of the usual Harmonic-Geometric-Arithmetic mean inequality we have the following:

Example A11 (The "power mean inequality") The scale of powermeans $\{\mathcal{P}_\alpha(f;\mu)\}$, defined by (1.15), is a non-decreasing function of α (for fixed f and μ).

Proof. First, we let $0 < \alpha < \beta < \infty$. Then, by using Jensen's inequality (1.7) with $\Psi(u) = u^{\beta/\alpha}$, we find that

$$\mathcal{P}_\alpha^\beta(f;\mu) := \left(\frac{1}{\mu(\Omega)}\int\limits_\Omega |f|^\alpha d\mu\right)^{\beta/\alpha} \leq \frac{1}{\mu(\Omega)}\int\limits_\Omega |f|^\beta d\mu = \mathcal{P}_\beta^\beta(f;\mu),$$

and we conclude that $\mathcal{P}_\alpha(f;\mu) \leq \mathcal{P}_\beta(f;\mu)$. Next, let $0 < \alpha < \infty$. Then, by again using Jensen's inequality, now with the convex function $\Psi(u) = \exp u$, we obtain that

$$\mathcal{P}_0^\alpha(f;\mu) = \exp\left(\frac{1}{\mu(\Omega)}\int\limits_\Omega \log |f|^\alpha d\mu\right) \leq \frac{1}{\mu(\Omega)}\int\limits_\Omega |f|^\alpha d\mu = \mathcal{P}_\alpha^\alpha(f;\mu),$$

so that $\mathcal{P}_0(f;\mu) \leq \mathcal{P}_\alpha(f;\mu)$. If $\alpha < 0$, then

$$\mathcal{P}_\alpha(f;\mu) = \left(\mathcal{P}_{-\alpha}\left(\frac{1}{|f|};\mu\right)\right)^{-1}$$

and, moreover, $\mathcal{P}_0(f;\mu) = \left(\mathcal{P}_0\left(\frac{1}{|f|};\mu\right)\right)^{-1}$ and the proof of the remaining cases follows by just using what we already have proved. □

It is also clear that

$$\lim_{\alpha\to 0} \mathcal{P}_\alpha(f;\mu) = \mathcal{P}_0(f;\mu).$$

Remark In particular, by letting $\int\limits_{\Omega 2} d\gamma = 1$, replacing $K(x,y)$ in (1.13) by $K(x,y)^{1/p}$ and letting $p \to \infty$ ($\alpha \to 0_+$), we obtain that (1.11) holds and that this version of Hölder's inequality is a limit case of (1.13).

Remark As we have seen standard Hölder's inequality imply both the standard and the continuous versions of Minkowski's inequality. Moreover, as we see above we have also implication in the reversed direction even on this more general continuous level, remarkable.

The scale of powermeans can be generalized to the following two-parameter scale of general Gini-means $\{\mathcal{G}_{\alpha,\beta}(f;\mu)\}$, $-\infty < \alpha, \beta < \infty$, as follows:

$$\mathcal{G}_{\alpha,\beta}(f;\mu) := \begin{cases} \left(\dfrac{\int_\Omega |f|^\alpha d\mu}{\int_\Omega |f|^\beta d\mu} \right)^{1/(\alpha-\beta)} & , \alpha \neq \beta, \\ \exp \left(\dfrac{\int_\Omega |f|^\alpha \log |f| d\mu}{\int_\Omega |f|^\alpha d\mu} \right) & , \alpha = \beta. \end{cases} \tag{1.17}$$

We note that $\mathcal{G}_{\alpha,0}(f;\mu) = \mathcal{P}_\alpha(f;\mu)$.

Example A12 The scale of general Gini-means $\mathcal{G}_{\alpha,\beta}(f;\mu)$ is non-decreasing in both α and β.

Remark One important step in the proof of Example A12 is to use the following (Peetre-Persson) representation formula:

$$\mathcal{G}_{\alpha,\beta}(f;\mu) = \exp \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{G}_{a,a}(f;\mu) da, \ \alpha < \beta.$$

We also mention the following general version of Beckenbach-Dresher's inequality, which can be proved by just using special cases of Hölder's and Minkowski's inequalities (in both directions):

Example A13 (Beckenbach-Dresher's inequality) Let f and g be positive and measurable functions on the measure space (Ω, μ). Then

$$\left(\frac{\int_\Omega |f+g|^\alpha d\mu}{\int_\Omega |f+g|^\beta d\mu} \right)^{1/(\alpha-\beta)} \leq \left(\frac{\int_\Omega |f|^\alpha d\mu}{\int_\Omega |f|^\beta d\mu} \right)^{1/(\alpha-\beta)} + \left(\frac{\int_\Omega |g|^\alpha d\mu}{\int_\Omega |g|^\beta d\mu} \right)^{1/(\alpha-\beta)}, \tag{1.18}$$

whenever $0 < \beta \leq 1 \leq \alpha < \infty, \alpha \neq \beta$.

1.6 Hilbert type inequalities

Hilbert's discrete inequality reads:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^\infty a_m^2 \right)^{1/2} \left(\sum_{n=1}^\infty b_n^2 \right)^{1/2}, \tag{1.19}$$

where π as the sharp constant. The sharp constant π was found by I. Schur. In Hilbert's version of (1.19) from early 19th the constant 2π appeared instead of π. We remark that the following more general form of (1.19) is nowadays usually referred to in the literature as the Hilbert inequality.

Example A14 Let $\{a_m\}_1^\infty$ and $\{b_n\}_1^\infty$ be sequences of positive numbers. Then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^\infty a_m^p \right)^{1/p} \left(\sum_{n=1}^\infty b_n^q \right)^{1/q}, \tag{1.20}$$

where $p > 1$ and $q = p/(p-1)$. However, Hilbert himself was not close to consider this case (the l_p spaces appeared only around 1910).

There exists also an integral version of (1.20) namely the following:

Example A15 Let $p > 1, q = p/(p-1)$ and let f and g be positive and measurable functions on $(0, \infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(y)dy \right)^{1/q}. \tag{1.21}$$

The constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is sharp in both (1.20) and (1.21). Next we give a simple proof of (1.21), which in particular shows that (1.21) in fact follows from Jensen's inequality (convexity) and even from Hölder's inequality.

Proof of (1.21): By using Hölder's inequality we obtain that

$$\begin{aligned} I := \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy &= \int_0^\infty \int_0^\infty \left(\frac{1}{x+y} \right)^{1/p} \left(\frac{x}{y} \right)^{1/pq} f(x) \left(\frac{1}{x+y} \right)^{1/q} \left(\frac{y}{x} \right)^{1/pq} g(y)dxdy \\ &\leq \left(\int_0^\infty \int_0^\infty \frac{1}{x+y} \left(\frac{x}{y} \right)^{1/q} f^p(x)dxdy \right)^{1/p} \left(\int_0^\infty \int_0^\infty \frac{1}{x+y} \left(\frac{y}{x} \right)^{1/p} g^q(y)dxdy \right)^{1/q} \\ &:= \left(\int_0^\infty I_1(x) f^p(x)dx \right)^{1/p} \left(\int_0^\infty I_2(y) g^q(y)dy \right)^{1/q}, \end{aligned}$$

where

$$I_1(x) = \int_0^\infty \frac{1}{x+y} \left(\frac{x}{y} \right)^{1/q} dy = \left[\frac{x}{y} = z \right] = \int_0^\infty \frac{1}{(1+z)z^{1/p}} dz = \frac{\pi}{\sin \frac{\pi}{p}},$$

and

$$I_2(y) = \int_0^\infty \frac{1}{x+y} \left(\frac{y}{x} \right)^{1/p} dx = \left[\frac{y}{x} = z \right] = \int_0^\infty \frac{1}{(1+z)z^{1/q}} dz = \frac{\pi}{\sin \frac{\pi}{q}}$$

and this completes the proof since $\frac{\pi}{\sin \frac{\pi}{p}} = \frac{\pi}{\sin \frac{\pi}{q}}$.

The sharpness of (1.21) follows since the only inequality we have used in this proof is Hölder's inequality.

$\square$

Remark It is completely clear historically that the original motivation for G.H. Hardy when he after 10 years of research finally proved his inequality in 1925 (see (1.29)) was to find an elementary proof of Hilbert's inequality (1.19) (Hilbert's original proof was very different and more difficult than that above). There is now a huge number of results generalizing (1.20) and (1.21) e.g. by replacing the kernel $k(x, y) = 1/x + y$ with some other kernel with similar homogeneity properties. In several cases such results are referred to as Hardy-Hilbert-type inequalities.

The close connection between (1.21) and Hardy-type inequalities discussed later on (see Section 1.8) is the following equivalence:

Example A16 Let $p > 1,\ q = p/(p-1)$ and let f and g be positive and measurable functions on $(0,\infty)$. Then the Hilbert inequality (1.21) holds if and only if the following Hardy-type inequality holds:

$$\left(\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy\right)^{1/p} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)dx\right)^{1/p}. \tag{1.22}$$

Proof. Assume that (1.22) holds. Then, by Hölder's inequality,

$$\int_0^\infty g(y)\int_0^\infty \frac{f(x)}{x+y}dxdy \leq \left(\int_0^\infty g^q(y)dy\right)^{1/q}\left(\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy\right)^{1/p}$$

$$\leq \left(\int_0^\infty g^q(y)dy\right)^{1/q} \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)dx\right)^{1/p},$$

so (1.21) holds.

Assume now that (1.21) holds and use it with $g(y) = \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^{p-1}$:

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy = \int_0^\infty \int_0^\infty \frac{f(x)\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^{p-1}}{x+y}dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy\right)^{1/q}$$

and since $1 - 1/q = 1/p$ we see that (1.22) holds. The proof is complete. □

1.7 Carlson type inequalities and interpolation

In 1934 (the Swedish Professor) Fritz Carlson presented and proved the following inequalities:

Example A17 (a) Let $\{a_n\}_1^\infty$ be a sequence of real numbers. Then

$$\sum_{n=1}^\infty |a_n| \leq \sqrt{\pi}\left(\sum_{n=1}^\infty a_n^2\right)^{1/4}\left(\sum_{n=1}^\infty n^2 a_n^2\right)^{1/4}. \tag{1.23}$$

(b) If $f(x)$ is a measurable function on $(0,\infty)$, then

$$\int_0^\infty |f(x)|dx \leq \sqrt{\pi}\left(\int_0^\infty f^2(x)dx\right)^{1/4}\left(\int_0^\infty x^2 f^2(x)dx\right)^{1/4}. \tag{1.24}$$

The constant $\sqrt{\pi}$ in both (1.23) and (1.24) is sharp (in (1.24) we even have equality when $f(x) = \frac{1}{1+bx^2}$).

Remark Carlson himself obviously thought that he had discovered some inequalities which are independent of other inequalities e.g. Hölder's inequality. In fact, he remarked that by Hölder's inequality (and exponents 4, 4 and 2) we find that if $x > 1$, then, for positive a_n,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sqrt{a_n}\sqrt{n^x a_n}\frac{1}{n^x} \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/4} \left(\sum_{n=1}^{\infty} n^{2x} a_n^2\right)^{1/4} \left(\sum_{n=1}^{\infty} \frac{1}{n^x}\right)^{1/2},$$

and since $C = C(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \to \infty$ when $x \to 1_+$ Carlson concluded that (1.23) could not follow from Hölder's inequality.

Hence, it must have been a very big surprise for F. Carlson when G.H.Hardy two years later in 1936 presented two proofs which both show that (1.23) in fact even follows from the Schwarz inequality (used in the standard case with two terms).

Proof I of (1.23) By Schwarz inequality we find that ($\alpha, \beta > 0$ will be chosen in a suitable way later on)

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = \left(\sum_{n=1}^{\infty} a_n \sqrt{\alpha + \beta n^2}\frac{1}{\sqrt{\alpha + \beta n^2}}\right)^2 \leq \sum_{n=1}^{\infty} a_n^2(\alpha + \beta n^2) \sum_{n=1}^{\infty} \frac{1}{\alpha + \beta n^2}.$$

Put $S = \sum_{n=1}^{\infty} a_n^2$ and $T = \sum_{n=1}^{\infty} n^2 a_n^2$.
We note that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha + \beta n^2} \leq \frac{1}{\alpha} \int_0^{\infty} \frac{1}{1 + \left(\sqrt{\frac{\beta}{\alpha}}x\right)^2} dx = \frac{1}{\sqrt{\alpha\beta}} \frac{\pi}{2}.$$

Therefore

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 \leq \frac{1}{\sqrt{\alpha\beta}}(\alpha S + \beta T)\frac{\pi}{2} = [\text{ choose } \alpha = \mathrm{T}, \beta = \mathrm{S}]$$

$$= \pi\sqrt{ST} = \pi \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/2},$$

and the proof is complete.

Proof II of (1.23) Let again $S = \sum_{n=1}^{\infty} a_n^2$ and $T = \sum_{n=1}^{\infty} n^2 a_n^2$. Consider

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx.$$

If T converges so does S and, by the Parseval relation, we have that $S = \frac{2}{\pi}\int_0^{\pi} f^2(x)dx$ and $T = \frac{2}{\pi}\int_0^{\pi} (f')^2(x)dx$.

Moreover, $f(0) > 0$ and

$$\int_0^{\pi} f(x)dx = 0 \tag{1.25}$$

so that there exists $\varepsilon, 0 < \varepsilon < \pi$, such that $f(\varepsilon) = 0$.

Thus,by Schwarz inequality,

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = f^2(0) = f^2(0) - f^2(\varepsilon) = 2\int_{\varepsilon}^{0} f(x)f'(x)dx$$

$$\leq 2\sqrt{\int_0^{\pi} f^2(x)dx}\sqrt{\int_0^{\pi} (f')^2(x)dx} = 2\sqrt{\frac{\pi S}{2}}\sqrt{\frac{\pi T}{2}} = \sqrt{ST} = \pi\sqrt{\sum_{n=1}^{\infty} a_n^2}\sqrt{\sum_{n=1}^{\infty} n^2 a_n^2},$$

and the proof is complete.

Remark In the application of Schwarz inequality equality holds if and only if $(f'(x))^2 = b^2 f^2(x)$ i.e. when $f(x) = ae^{bx}$ but in view of (1.25) this is possible only if $a = 0$ which implies that $a_n = 0, n = 1, 2, ...$, i.e., that $f \equiv 0$.

We also present another proof showing that Carlson's inequality also follows from Hilbert's inequality (1.19)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2},$$

but with constant $\sqrt{2\pi}$ instead of the sharp one $\sqrt{\pi}$. In fact, (1.19) implies that

$$\left(\sum_{m=1}^{\infty} a_m\right)^2 = \sum_{m=1}^{\infty} a_m \sum_{n=1}^{\infty} a_n = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m+n}{m+n} a_m a_n = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m a_m a_n}{m+n} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m n a_n}{m+n}$$

$$\leq \pi \left(\sum_{m=1}^{\infty} (m a_m)^2\right)^{1/2} \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} + \pi \left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} (n a_n)^2\right)^{1/2}$$

$$= 2\pi \left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2} \left(\sum_{m=1}^{\infty} m^2 a_m^2\right)^{1/2},$$

and the proof is complete.

Remark In Appendix II we present some general forms of Carlson's inequalities and its close relation to interpolation theory (e.g. the Peetre $\pm$ method and beyond). Here we just describe one such basic relation:

Let A_0 and A_1 denote two Banach spaces. We say that the space X in an intermediate space if $A_0 \bigcap A_1 \subset X \subset A_0 + A_1$. The Peetre J-functional, which is very central for the development of real interpolation theory, is defined as follows:

$$J(t, f) = J(t, f; A_0, A_1) := \max\left(\|f\|_{A_0}, t\|f\|_{A_1}\right), t > 0.$$

If X is an intermediate space of the compatible Banach couple (A_0, A_1), then we say that X is of class $C_J(\theta; A_0, A_1)$ if

$$\|f\|_X \leq Ct^{-\theta} J(t, f) \tag{1.26}$$

for some $C > 0,\ t > 0$ and where $0 < \theta < 1$.

We have the following connection to a generalized form of Carlson's inequality:

Example A18 Let (A_0, A_1) be a compatible Banach couple and let X be an intermediate space. Then X is of the class $C_J(\theta; A_0, A_1)$ if and only if the following generalized form of Carlson's inequality holds:

$$\|f\|_X \leq C\|f\|_{A_0}^{1-\theta}\|f\|_{A_1}^{\theta}. \tag{1.27}$$

Proof. Assume that (1.26) holds. Then by using this inequality with $t = \|f\|_{A_0}/\|f\|_{A_1}$ we find that

$$\|f\|_X \leq C \max\left(t^{-\theta}\|f\|_{A_0}, t^{1-\theta}\|f\|_{A_1}\right) = C\|f\|_{A_0}^{1-\theta}\|f\|_{A_1}^{\theta},$$

i.e. (1.27) holds. Assume now that (1.27) holds. Then from the arguments above we see that

$$\|f\|_X \leq C \max\left(t^{-\theta}\|f\|_{A_0}, t^{1-\theta}\|f\|_{A_1}\right)$$

indeed holds for $t = \|f\|_{A_0}/\|f\|_{A_1}$, where the two terms are equal. Moreover, since $t^{-\theta}\|f\|_{A_0}$ is decreasing and $t^{1-\theta}\|f\|_{A_1}$ is increasing, this holds also for other values of t as well. The proof is complete. $\square$

Remark The classical Carlson inequality (1.24) means that the space L_1 is of the class $J(\frac{1}{2}; L_2, L_2(x^2))$ and a later generalization of V.I. Levin is equivalent to that L_1 is also of the class $J\left(\frac{\lambda q}{p\mu+\lambda q}; L_p(x^{p-1-\lambda}), L_q(x^{q-1+\mu})\right)$, see Example II:6 in Appendix II. A much more general multidimensional form is presented in Appendix II and a PhD thesis of L. Larson.

Remark There are many open questions for which spaces (1.27) holds or, equivalently, as we have seen here that X is of the class $C_J(\theta; A_0, A_1)$ in concrete situations. And such information is useful in real interpolation theory.

1.8 Four classical inequalities (by Hardy, Carleman and Pólya-Knopp)

The information in this Section and Appendix III in mainly taken from a paper 2010 by L.E. Persson and N. Samko.

A) Hardy's inequality (continuous form)

If f is non-negative and p-integrable over $(0, \infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx), \quad p > 1. \tag{1.28}$$

B) Hardy's inequality (discrete form)

If $\{a_n\}_1^\infty$ is a sequence of non-negative numbers, then

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{n=1}^n a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p, \quad p > 1. \tag{1.29}$$

Remark The dramatic more than 10 years period of research until Hardy stated in 1920, and proved in 1925, his inequality (1.28) was described in a paper 2007 by A.Kufner, L.Maligranda and L.E.Persson. It is historically clear that Hardy's original motivation when he discovered his inequalities was to find a simple proof of Hilbert's double series inequality (1.19).

Remark It is clear that (1.28) $\Rightarrow$ (1.29). More exactly, by applying (1.28) with step functions we obtain (1.29). This was pointed out to Hardy in a private letter from F. Landau already in 1921 and here Landau even included a proof of (1.29) so it should not be wrong to even call (1.29) the Hardy-Landau inequality.

C). Carleman's inequality:

If $\{a_n\}_1^\infty$ is a sequence of positive numbers, then

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdots a_n} \quad \leq \quad e \sum_{n=1}^{\infty} a_n. \tag{1.30}$$

Remark This inequality was proved by (the Swedish Professor) Torsten Carleman in 1922 in connection to this important work on quasi-analytical functions. Carleman's idea of proof was to find maximum of $\sum_{i=1}^{n}(a_1 \cdots a_i)^{1/i}$ under the constraint $\sum_{i=1}^{n} a_i = 1, n \in Z_+$. However, (1.30) is in fact a limit inequality (as $p \to \infty$) of the inequalities (1.29) according to the following:

Replace a_i with $a_i^{1/p}$ in the Hardy discrete inequality (1.29) and we obtain that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{n=1}^{n} a_i^{1/p}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n, \; p > 1.$$

Moreover, when $p \to \infty$ we have that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{1/p}\right)^p \to \left(\prod_{i=1}^{n} a_i\right)^{1/n} \quad and \quad \left(\frac{p}{p-1}\right)^p \to e.$$

In view of the fact that Carleman and Hardy had a direct cooperation at that time is may be a surprise that Carleman did not mention this fact in his paper.

Remark In 1954 (the Swedish Professor) Lennart Carleson presented another proof of (a generalized form of) (1.30). Also Carleson used convexity in a crucial way in his proof.

D) Pólya-Knopp's inequality

If f is a positive and integrable function on $(0, \infty)$, then

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x lnf(y)dy\right)dx \leq e\int_0^\infty f(x)dx. \tag{1.31}$$

Remark Sometimes (1.31) is referred to as the Knopp inequality with reference to his 1928 paper. But it is clear that it was known before and in his 1925 paper Hardy informed that G. Pólya had pointed out the fact that (1.31) is in fact a limit inequality (as $p \to \infty$) of the inequality (1.28) and the proof is literally the same as that above that (1.29) implies (1.30), see Remark above. Accordingly, nowadays (1.31) is many times referred to as Pólya-Knopp's inequality and we have adopted this terminology.

ALL these inequalities (1.28)-(1.31) follows easily directly from Jensen's inequality (which was NOT discovered by Hardy himself and others) according to the following:

Basic Observation We note that for $p > 1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.32}$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1 \cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{1.33}$$

where $\quad f(x) = g(x^{1-1/p})x^{-1/p}$.

Proof of (1.33) and ,thus, of (1.32) and the other inequalities (1.28)-(1.31) above:
By Jensen's inequality and Fubini's theorem we have that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x} = \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

Remark In 1928 G.H.Hardy himself proved the first generalization of his inequality (1.28) namely the following:

The inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a dx \tag{1.34}$$

holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $a < p-1, p > 1$.

But, in fact, this is no genuine generalization of (1.28) since by making the substitution $f(x) = g\left(x^{\frac{p-a-1}{p}}\right)x^{-\frac{(1+a)}{p}}$, we find that also (1.34) is equivalent to (1.33) so indeed (1.28) and (1.34) are equivalent for each $a < p-1$. In Appendix III we give some further consequences of this idea.

Remark Note that (1.33) holds also when $p = 1$ (with equality). Hence, by raising to power $1/p$ we get inequalities holding for L_p spaces with $p = 1$ and $p = \infty$. By using interpolation we get Hardy type inequalities for all interpolation spaces between L_1 and L_∞ and not only for L_p spaces.

Remark In Appendix IV we present some further inequalities connected to convexity/concavity.

2 B. INTERPOLATION THEORY $\Longrightarrow$ INEQUALITIES

B1 Riesz-Thorin's interpolation theorem and the complex interpolation method

The examples in this Section together with our introductory Example A1 were guiding when M.Riesz and others were looking for a more general way to handle situations as in our Examples below (e.g. Hausdorff-Young's inequality). The famous paper by M.Riesz from 1926 was later on extended by his student G.O. Thorin in 1938 to what is today called the Riesz-Thorin interpolation theorem.

Theorem B1 Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty, p_0 \neq p_1$, and assume that

$$T : L_{p_0}(\mu) \xrightarrow{M_0} L_{q_0}(\mu) \ \text{ i.e. } \ \|Tf\|_{L_{q_0}(\mu)} \leq M_0 \|f\|_{L_{p_0}(\mu)}$$

and

$$T : L_{p_1}(\mu) \xrightarrow{M_1} L_{q_1}(\mu) \ \text{ i.e. } \ \|Tf\|_{L_{q_1}(\mu)} \leq M_1 \|f\|_{L_{p_1}(\mu)}.$$

Then

$$T : L_{p_\theta}(\mu) \xrightarrow{M_\theta} L_{q_\theta}(\mu) \ \text{ i.e. } \ \|Tf\|_{L_{q_\theta}(\mu)} \leq M_\theta \|f\|_{L_{p_\theta}(\mu)},$$

where $M_\theta \leq M_0^{1-\theta} M_1^\theta$, and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, 0 \leq \theta \leq 1$.

Remark An important special case for several applications is when $p_0 = q_0 = 2$ and $p_1 = 1, q_1 = \infty$. Then

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \frac{\theta}{1} = \frac{1+\theta}{2} \quad (1 \leq p_\theta \leq 2),$$
$$\frac{1}{q_\theta} = \frac{1-\theta}{2} + \frac{\theta}{\infty} = \frac{1-\theta}{2} \quad (2 \leq q_\theta \leq \infty).$$

Note that $\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1$ (p_θ and q_θ are conjugate indices).

Example B1 (Hausdorff-Young's inequality). Consider the Fourier transform

$$Ff(x) = \int_{\mathbb{R}^n} e^{-ixy} f(y)dy.$$

Then we have boundedness
$F : L_2 \to L_2$ with the norm $M_0 = (2\pi)^{n/2}$ (the Parseval identity) and
$F : L_1 \to L_\infty$ with the norm $M_1 \leq 1$.

By using Theorem B1 and the remark after, we find that

$$F : L_p \xrightarrow{C} L_q, \ 1 \leq p \leq 2, \ \frac{1}{p} + \frac{1}{q} = 1$$

with the norm $M_0 \leq M_0^{1-\theta} M_1^\theta = (2\pi)^{\frac{n}{2}(1-\theta)} = (2\pi)^{n/q}$. Equivalently, we can formulate it as Hausdorff-Young's inequality:

$$\|Ff\|_{L_q} \leq (2\pi)^{n/q} \|f\|_{L_p}, \ 1 \leq p \leq 2, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{2.1}$$

Remark The original proof of (2.1) even for $n = 1$ was very complicated and made in many steps. Hence, the proof above was early very convincing to show the power of using interpolation theory for proving inequalities.

Remark Unfortunately, interpolation theory does not always give sharp inequalities in the intermediate estimates. For example, the constant $C = (2\pi)^{n/q}$ in (2.1) is not sharp. In fact, Babenko and Beckner proved that the sharp constant in (2.1) is equal to $(A_p)^n(2\pi)^{n/q}$, where $A_p = \left(p^{1/p}/q^{1/q}\right)^{1/2}$.

Example B2 (Hausdorff-Young's inequality - discrete form). Let

$$(Tf)_n = c_n = \int\limits_0^{2\pi} f(x)e^{-ixy}dx, n = 0, \pm 1, \pm 2,$$

and let $\sum_{-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier series of the function g. Then

$$\left(\sum_{-\infty}^{\infty} |c_n|^q\right)^{1/q} \leq (2\pi)^{1/q} \left(\int\limits_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \tag{2.2}$$

where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Parseval's relation,

$$\left(\sum_{-\infty}^{\infty} |c_n|^2\right)^{1/2} \leq \sqrt{2\pi} \left(\int\limits_0^{2\pi} |f(x)|^2 dx\right)^{1/2}$$

i.e. $T : L_2 \xrightarrow{\sqrt{2\pi}} l_2$. Moreover,

$$|c_n| \leq \int\limits_0^{2\pi} |f(x)|dx, \ \text{ i.e., } \ \sup_n |c_n| \leq 1 \cdot \int\limits_0^{2\pi} |f(x)|dx$$

i.e. $T : L_1 \xrightarrow{1} l_\infty$. Hence, by using Theorem B1 and the Remark after, we find that

$$T : L_p \xrightarrow{C} l_q, \ 1 < p \leq 2, \ \frac{1}{p} + \frac{1}{q} = 1,$$

with $C \leq (2\pi)^{1/q}$, so (2.2) holds. $\square$

Example B3 (Young's inequality) Here we consider the convolution operator

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = k * f(x), \ \text{ with } \ k \in L_q(\mathbb{R}^n), 1 \leq q \leq \infty.$$

Then we have boundedness $T : L_{q'} \to L_\infty, q' = q/(q-1)$, with the norm $\leq \|k\|_{L_q}$ (by the Hölder inequality), and $T : L_1 \to L_q$ with the norm $\leq \|k\|_{L_q}$ (by the generalized Minkowski inequality, see Example A9).

Hence, by using Theorem B1, we obtain the boundedness

$$T : L_p \xrightarrow{C} L_r,$$

where $\frac{1}{p} = \frac{1-\theta}{q'} + \frac{\theta}{1}$ and $\frac{1}{r} = \frac{1-\theta}{\infty} + \frac{\theta}{q}$ (which gives that $1 \leq p \leq q'$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$) with the norm $C \leq \|k\|_{L_q}$. Thus, we have proved the Young inequality: if $1 \leq q \leq \infty, 1 \leq p \leq q', q' = q/(q-1)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, then

$$\|k * f(x)\|_{L_r} \leq \|k\|_{L_q} \|f\|_{L_p}. \tag{2.3}$$

Remark Beckner found the following sharp form of the Young inequality (2.3):

$$\|k * f(x)\|_{L_r} \leq (A_p A_q A_{r'})^n \|k\|_{L_q} \|f\|_{L_p},$$

where $A_s = [s^{\frac{1}{s}} / s'^{\frac{1}{s'}}]^{1/2}$ and where $(A_p A_q A_{r'})^n$ is the best constant.

In our last Example we derive Hölder's inequality by using bilinear interpolation (Calderon theorem).

Example B4 (Hölder's inequality) We note that the multiplication operator $T(f,g) = fg$ is a bilinear bounded operator from $L_\infty \times L_1$ into L_1 and from $L_1 \times L_\infty$ into L_1, and moreover

$$\|T(f,g)\|_{L_1} \leq \|f\|_{L_\infty} \|g\|_{L_1}$$

$$\|T(f,g)\|_{L_1} \leq \|f\|_{L_1} \|g\|_{L_\infty}$$

By using the interpolation theorem for bilinear operators in complex spaces (Calderon theorem) we find that

$$T : [L_1, L_\infty]_\theta \times [L_\infty, L_1]_\theta \to [L_1, L_1]_\theta$$

with the norm ≤ 1. Hence, since $[L_1, L_\infty]_\theta \equiv L_p(p = 1/(1-\theta)), [L_\infty, L_1]_\theta \equiv [L_1, L_\infty]_{1-\theta} \equiv L_q(q = 1/\theta)$ and $[L_1, L_1]_\theta \equiv L_1$, we obtain the Hölder inequality

$$\|fg\|_{L_1} = \|T(f,g)\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q}, \frac{1}{p} + \frac{1}{q} = 1, p > 1.$$

Remark As we have seen in part A, several inequalities can be derived from Hölder's inequality. Hence, Example B4 indeed shows that interpolation also implies all these inequalities.

Remark The complex interpolation method is a further development of the Riesz-Thorin interpolation theorem with the basic idea taken from Thorin's proof and the log-convexity of a special function (J.L.Lions, A.P.Calderon, S.G.Krein, etc.). For definitions applications, historical remarks see the books in Appendix 1 e.g. [B1] by (the Swedish Professors) Jöran Bergh and Jörgen Löfström, students of (the Swedish Professor) Jaak Peetre.

B2 The Marcinkiewicz interpolation theorem and the real interpolation method

Let $f^*(t)$ denote the non-decreasing rearrangement of a function f on a measure space (Ω, μ).

The Lorenz spaces $L_{p,r} = L_{p,r}(\mu), 1 \le p \le \infty, 1 \le r \le \infty$, are defined by

$$\|f\|_{L_{p,r}} = \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right)^{1/r}, \; 1 \le r < \infty,$$

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t).$$

It is easy to see that $L_{p,p} = L_p$ and $L_{p,r_1} \subset L_{p,r_2}, r_1 \le r_2$. By using Marcinkiewicz interpolation theorem we can get similar "intermediate" (strong) inequalities as in Theorem B1 between-spaces even if we only have so called weak estimates at the endpoint spaces i.e. that the spaces $L_{q_0}(\mu)$ and $L_{q_1}(\mu)$, can be replaced by $L_{q_0,\infty}(\mu)$ and $L_{q_1,\infty}(\mu)$, respectively (but with less good constants and some restrictions on the parameters). However, by using the real interpolation method (shortly described below) we can get the following more general result:

Theorem B2 Suppose that p_0, p_1, q_0 and q_1 are positive or infinite numbers and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, 0 < \theta < 1$. If $p_0 \ne p_1$, then

$$(L_{p_0,q_0}, L_{p_1,q_1})_{\theta,q} = L_{p,q}.$$

This formula is also true in the case $p_0 = p_1 = p$, provided

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.4}$$

Remark There exists also a concrete (but fairly complicated) description also in the off-diagonal case, i.e. when q does not satisfy the interpolation relation (2.4).

By using Theorem B2 instead of theorem B1 we can derive in a sense more precise versions of the inequalities in Examples B1-B3, e.g. the following:

Example B5 (Paley's inequality) It yelds that

$$\|Ff\|_{L_{q,p}} \le C\|f\|_{L_p}, \; 1 < p \le 2, \; \frac{1}{p} + \frac{1}{q} = 1. \tag{2.5}$$

Remark According to the results above we see that if $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$f \in L_p \Rightarrow Ff \in L_q \tag{2.6}$$

$$f \in L_p \Rightarrow Ff \in L_{q,p}, \tag{2.7}$$

respectively. Since $L_{q,p}$ is continuously and properly imbedded in L_q we see that (2.7) is a sharper criteria than (2.6). Moreover, the following sharper criteria than (2.6) can be proved:

$$f \in L_p \Rightarrow \int_0^\infty |Ff|^q \left(h \left(\max \left(|Ff|, \frac{1}{|Ff|} \right) \right) \right)^{(2-p)/(p-1)} dx < \infty, \tag{2.8}$$

holds for some function $h \ge 1, 1/th(t) \in L_1(1,\infty)$ such that $h(x)x^a$ is a decreasing or an increasing function of x for some real number $a (1 < p \le 2)$.

It is possible to prove that (2.7) and (2.8) are, in a sense, equivalent.

Obviously inspired by considerations as above the real interpolation method was developed by J.Peetre, J.L.Lions and others. This method can be described in some different (but of course equivalent) ways. Here we will describe it via the so called Peetre K-functional $K(t,f) = K(t,f;A_0,A_1)$ as follows:

$$K(t,f) := \inf_{f=f_0+f_1} \left(\|f_0\|_{A_0} + t\|f_1\|_{A_1} \right), t > 0.$$

Here A_0 and A_1 are compatible Banach (or quasi-Banach) spaces so that the spaces $A_0 \cap A_1$ and $A_0 + A_1$ can be defined ($K(t,f)$ is obviously an equivalent norm in $A_0 + A_1$ for all $t > 0$) while the previously mentioned J-functional is an equivalent norm on the space $A_0 \cap A_1$. One way to describe the real interpolation space $(A_0, A_1)_{\theta,q}, 0 < \theta < 1, q \geq 1$, is via the condition

$$\|f\|_{A_0,A_1{}_{\theta,q}} := \left(\int_0^\infty \left(t^{-\theta} K(t,f) \right)^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{2.9}$$

Remark The relation to the case with interpolation between Lebesgue and/or Lorentz spaces (see e.g. Theorem B2) is obvious by using Hardy type inequalities and e.g. the relations

$$K\left(t,f;L_1,L_\infty\right) = \int_0^t f^*(s)ds,$$

$$K\left(t,f;L_p,L_\infty\right) \approx \left(\int_0^{t^p} \left(f^*(s)\right)^p ds \right)^{1/p},$$

and

$$K\left(t,f;L_{p_0,q_0},L_{p_1,q_1}\right) \approx \left(\int_0^{t^\alpha} \left(s^{1/p_0} f^*(s) \right)^{q_0} \frac{ds}{s} \right)^{1/q_0} + \left(\int_{t^\alpha}^\infty \left(s^{1/p_1} f^*(s) \right)^{q_1} \frac{ds}{s} \right)^{1/q_1},$$

$0 < p_0 < p_1 < \infty, \frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$. The last estimate is sometimes called Holmstedt's formula after J. Peetre's student T. Holmstedt.

B3 Some more connections between interpolation, inequalities and convexity/concavity

It is a dual equivalent way to describe the real interpolation method, where we instead consider "sums" of elements from the space $A_0 \cap A_1$ equipped with the (Peetre) J-functional

$$J(t,f) = \sup \left(\|f\|_{A_0}, t\|f\|_{A_1} \right), t > 0.$$

It is proved that these methods are equivalent so we can talk about the real interpolation method.

Remark It is well known that both the J- and the K- functionals are quasi-concave functions φ in the sense that $\varphi(t)$ is non-decreasing and $\frac{\varphi(t)}{t}$ is non-increasing i.e. that

$$\varphi(t) \leq \max\left(1, \frac{t}{s}\right)\varphi(s).$$

It is also known that almost all theorems connected to the functional (2.9) holds also when $t^{-\theta}$ is replaced by a more general "function parameter" λ (in simplest case when $\lambda(t)t^{-\varepsilon}$ is non-decreasing and $\lambda(t)t^{-1+\varepsilon}$ is non-increasing for some $0 < \varepsilon < 1/2$.

Moreover, it was mentioned in the book [B1] by J. Bergh and J. Löfström that the embedding $(A_0, A_1)_{p,q_1} \subset (A_0, A_1)_{p,q_2}$ can be illustrated in the following sharp way:

Example B6 (Bergh's inequality). Let φ be a quasi-concave function on $\mathbb{R}_+$. If $0 < p \leq q \leq \infty$ and $0 < \alpha < 1$, then

$$\left(\int_0^\infty (t^{-\alpha}\varphi(t))^q \frac{dt}{t}\right)^{1/q} \leq p^{1/p}q^{-1/q}[\alpha(1-\alpha)]^{1/p-1/q}\left(\int_0^\infty (t^{-\alpha}\varphi(t))^p \frac{dt}{t}\right)^{1/p}. \tag{2.10}$$

The inequality is sharp with equality for $\varphi(t) = \min(1, t)$.

More generally, we say that $f \in Q_\beta$ if $f(t)t^{-\beta}$ is non-increasing on $\mathbb{R}_+$, $f \in Q^\alpha$ if $f(t)t^{-\alpha}$ is non-decreasing on $\mathbb{R}_+$ and $f \in Q^\alpha_\beta$ if $f \in Q^\alpha \bigcap Q_\beta$ on $\mathbb{R}_+$.

The following generalization of Bergh's in equality is due to J.Bergh, V.Burenkov and L.E.Persson:

Example B7

(a) Let $\alpha_0 < \alpha$ and $f \in Q^{\alpha_0}$. If $0 < p \leq q \leq \infty$, then

$$\left(\int_0^\infty (t^{-\alpha}f(t))^q \frac{dt}{t}\right)^{1/q} \leq p^{1/p}q^{-1/q}(\alpha-\alpha_0)^{1/p-1/q}\left(\int_0^\infty (t^{-\alpha}f(t))^p \frac{dt}{t}\right)^{1/p}.$$

(b) Let $\alpha < \alpha_1$ and $f \in Q_{\alpha_1}$. If $0 < p \leq q \leq \infty$, then

$$\left(\int_0^\infty (t^{-\alpha}f(t))^q \frac{dt}{t}\right)^{1/q} \leq p^{1/p}q^{-1/q}(\alpha_1-\alpha)^{1/p-1/q}\left(\int_0^\infty (t^{-\alpha}f(t))^p \frac{dt}{t}\right)^{1/p}.$$

(c) Let $\alpha_0 < \alpha < \alpha_1$ and $f \in Q^{\alpha_0}_{\alpha_1}$. If $0 < p \leq q \leq \infty$, then

$$\left(\int_0^\infty (t^{-\alpha}f(t))^q \frac{dt}{t}\right)^{1/q} \leq p^{1/p}q^{-1/q}\left[\frac{(\alpha-\alpha_0)(\alpha_1-\alpha)}{(\alpha_1-\alpha_0)}\right]^{1/p-1/q}\left(\int_0^\infty (t^{-\alpha}f(t))^p \frac{dt}{t}\right)^{1/p}.$$

The constants in each of the inequalities in (a) to (c) are sharp.

Remark The proof is completely different from the original (variational) one J. Bergh had in mind. In fact, first we prove the inequalities (a) and (b) and by using a technique from interpolation theory we find that (c) can be obtained by finding an optimal breaking point and using what we have proved in (a) and (b).

B4 Some more inequalities related to interpolation and convexity

We first recall the following elementary inequality mentioned in Example A5:

$$(|a+b|^q + |a-b|^q)^{1/q} \leq 2^{1/q}(|a|^p + |b|^p)^{1/p}, \tag{2.11}$$

where $a, b \in \mathbb{C}, 1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. We will see in the next section that (2.11) directly implies Clarkson's inequalities. Thus, in view what we have noticed in our previous Section it is natural that (2.11) is called Clarkson-Haussdorff-Young's inequality. It can easily be proved by using interpolation: Indeed, let L_p^2 be the 2-dimensional complex L_p space. Clearly L_p^2 can be identified with C^2 endowed with the p-norm

$$\|(a, b\|_p = (|a|^p + |b|^p)^{1/p}.$$

We consider the elementary operator T from C^2 to C^2, defined by

$$T(a, b) = (a + b, a - b).$$

By the triangle inequality, $T : l_1^2 \to l_\infty^2$ has norm 1 and, by the parallelogram law, $T : l_2^2 \to l_2^2$ has norm $2^{1/2}$. Therefore, by using Theorem B1 and the remark after, we find that, for $1 \leq p \leq 2$, $T : l_p^2 \to l_q^2$ with norm $\leq 2^{1/q}$, i.e. that (2.11) holds.

By using (2.11) and other elementary inequalities in Section A1 we obtain the following more general result:

Example B8 (Clarkson-Haussdorff-Young's inequality with general parameters). Let $a, b \in \mathbb{C}, r \in \mathbb{R}, r \neq 0$ and $s \in \mathbb{R}_+$. Then

$$(|a+b|^r + |a-b|^r)^{1/r} \leq 2^\gamma(|a|^s + |b|^s)^{1/s}, \tag{2.12}$$

where $\gamma = \frac{1}{r} - \frac{1}{s} + \frac{1}{q}, q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2, \frac{1}{r'} + \frac{1}{r} = 1$.

Proof. Put $A_r = (|a+b|^r + |a-b|^r)^{1/r}$ and $B_s = (|a|^s + |b|^s)^{1/s}$. By using the inequalities in Section 1.1 and (2.11) we in particular have that $B_p \leq B_q$ and $A_p \leq A_q, q \leq p$, and $A_q 2^{-1/q} \leq A_p 2^{-1/p}$ and $B_q 2^{-1/q} \leq B_p 2^{-1/p}, q \leq p$, and it follows that:

$$\begin{gathered}
r \leq 2, s \leq 2 : A_r \leq 2^{1/r} B_2 \leq 2^{1/r} B_s, \\
r \leq 2, s \geq 2 : A_r \leq 2^{1/r} B_2 \leq 2^{1/r - 1/s + 1/2} B_s, \\
r > 2, s \leq r' : A_r \leq 2^{1/r} B_{r'} \leq 2^{1/r} B_s, \\
r > 2, s \geq r' : A_r \leq 2^{1/r} B_{r'} \leq 2^{1/r - 1/s + 1/r'} B_s.
\end{gathered}$$

The proof follows by combining these inequalities. □

By using (2.11) and (2.12) we obtain fairly general versions of Clarkson's classical inequalities. The classical forms reads:

Example B9 (Clarkson's classical inequalities) Let $p > 1, q = p/(p-1)$. Then

$$\left(\|f(x)+g(x)\|^p_{L_p(\mu)} + \|f(x)-g(x)\|^p_{L_p(\mu)}\right)^{1/p} \le 2^{1/q}\left(\|f(x)\|^p_{L_p(\mu)} + \|g(x)\|^p_{L_p(\mu)}\right)^{1/p}, p \ge 2, \tag{2.13}$$

$$\left(\|f(x)+g(x)\|^p_{L_p(\mu)} + \|f(x)-g(x)\|^p_{L_p(\mu)}\right)^{1/p} \le 2^{1/p}\left(\|f(x)\|^q_{L_p(\mu)} + \|g(x)\|^q_{L_p(\mu)}\right)^{1/q}, p \ge 2, \tag{2.14}$$

$$\left(\|f(x)+g(x)\|^q_{L_p(\mu)} + \|f(x)-g(x)\|^q_{L_p(\mu)}\right)^{1/q} \le 2^{1/q}\left(\|f(x)\|^p_{L_p(\mu)} + \|g(x)\|^p_{L_p(\mu)}\right)^{1/p}, 1 < p \le 2. \tag{2.15}$$

Proof. All these inequalities follow from (2.11). Let $q \ge 2$. Then

$$\|f(x)+g(x)\|^q_{L_q(\mu)} + \|f(x)-g(x)\|^q_{L_q(\mu)} = \int_\Omega \left(|f(x)+g(x)|^q + |f(x)-g(x)|^q\right) d\mu \le$$

$$[(2.11)] \le 2\int_\Omega \left(|f(x)|^p + |g(x)|^p\right)^{q/p} d\mu \le \left[(a^p+b^p)^{1/p} \le 2^{1/p-1/q}(a^q+b^q)^{1/q}\right]$$

$$\le 2^{q/p}\int_\Omega \left(|f(x)|^q + |g(x)|^q\right) d\mu = 2^{q/p}\left(\|f(x)\|^q_{L_q(\mu)} + \|g(x)\|^q_{L_q(\mu)}\right).$$

We conclude that

$$\left(\|f(x)+g(x)\|^q_{L_q(\mu)} + \|f(x)-g(x)\|^q_{L_q(\mu)}\right)^{1/q} \le 2^{1/p}\left(\|f(x)\|^q_{L_q(\mu)} + \|g(x)\|^q_{L_q(\mu)}\right)^{1/q},$$

i.e. (2.13) is proved by just interchanging the roles of p and q. Obviously (2.14) is a special case of (2.13) since $\left[(a^p+b^p)^{1/p} \le 2^{1/p-1/q}(a^q+b^q)^{1/q}\right], p \le q$. The proof of (2.15) is similar so we omit the details. □

By applying Example B8 with the parameters r and s interchanged and by making the substitutions $a_1 = a+b$ and $b_1 = a-b$ we obtain the following estimates in the opposite direction:

Example B10 Let $a, b \in \mathbb{C}, r \in \mathbb{R}_+$ and $s \in \mathbb{R}, s \ne 0$. Then

$$(|a+b|^r + |a-b|^r)^{1/r} \ge 2^\gamma(|a|^s + |b|^s)^{1/s},$$

where $\gamma = \frac{1}{r} - \frac{1}{s} + \frac{1}{q}, q = \min(2, r)$ if $s \le 2$ and $q = \min(s', r)$ if $s > 2, \frac{1}{s'} + \frac{1}{s} = 1$.

Remark In particular, by applying Examples B8 and B10 with $r = s = p$ we obtain the following well known estimates: If $1 < p \le \infty$, then

$$\min(2^{1/p}, 2^{1/p'})(|a|^p + |b|^p)^{1/p} \le (|a+b|^p + |a-b|^p)^{1/p} \le \max(2^{1/p}, 2^{1/p'})(|a|^p + |b|^p)^{1/p}.$$

Note that for the case $p = 2$ both inequalities reduces to equalities and we just have the parallelogram law.

Remark By using examples B8 and B10 and argue as in the proof of Example B9 we can prove a number of both known and new inequalities by using the same interpolation technique.

For example we have the following generalization of Example B8:

Example B11 Consider the operator $T : \bar{a} \to \left(\sum_1^n a_i, ..., \sum_1^n \varepsilon_i a_i, ..., \sum_1^n -a_i\right)$, where $\varepsilon_i = \pm 1, 1 \leq i \leq n$ (each coordinate of the vector $T(\bar{a}) \in \mathbb{R}^m, m = 2^n$, is equal to a sum of the type $\sum_1^n \varepsilon_i a_i$). It yields that

$$\left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left|\sum_1^n \varepsilon_i a_i\right|^r\right)^{1/r} \leq n^{1/q-1/s} \left(\sum_1^n |a_i|^s\right)^{1/s}, \tag{2.16}$$

for every $r \in \mathbb{R}, r \neq 0, s > 0, q = \min(2, s)$ for $r \leq 2$ and $q = \min(r', s)$ for $r > 2$.

Remark For the case $r > 0$ inequality (2.16) can be rewritten as

$$\left(\int_0^1 \left|\sum_1^n \varphi_i(t) a_i\right|^r dt\right)^{1/r} \leq n^{1/q-1/s} \left(\sum_1^n |a_i|^s\right)^{1/s},$$

where $\varphi_i(t) = \operatorname{sign}(\sin(2^i \pi t))$ are the usual Rademacher functions. F or the case $n = 2$ the last estimate coincide with a result of Koskela from 1969, and for the case $s = r', r > 2$ and $s = r \leq 2$ another proof was done by Williams and Wells from 1978.

Proof of Example B11 It is easy to see that for the operator $T : l_1^{(n)} \to l_\infty^{(m)}$, we have that $M_1 = 1$ and $T : l_2^{(n)} \to l_2^{(m)}$ has the norm $M_2 = 2^{n/2}$ and, thus, the operator $T : l_p^n \to l_{p'}^m, 1 \leq p \leq 2$, has norm $\leq \left(2^{n/2}\right)^{2/p'}$, i.e.

$$\left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left|\sum_1^n \varepsilon_i a_i\right|^{p'}\right)^{1/p'} \leq \left(\sum_1^n |a_i|^p\right)^{1/p}, 1 \leq p \leq 2. \tag{2.17}$$

We put

$$A_r = \left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left|\sum_1^n \varepsilon_i a_i\right|^r\right)^{1/r} \text{ and } B_s = \left(\frac{1}{n} \sum_1^n |a_i|^s\right)^{1/s}.$$

Then, according to the basic estimate (2.17) and the monotonicity properties of A_r and B_s, we have the following estimates:

$$r \leq 2, s \leq 2 : A_r \leq \left(\sum_1^n |a_i|^2\right)^{1/2} \leq \left(\sum_1^n |a_i|^s\right)^{1/s} = n^{1/s} B_s,$$

$$r \leq 2, s \geq 2 : A_r \leq \left(\sum_1^n |a_i|^2\right)^{1/2} \leq n^{1/2-1/s} \left(\sum_1^n |a_i|^s\right)^{1/s} = n^{1/2} B_s,$$

$$r>2, s\le r' : A_r \le \left(\sum_1^n |a_i|^{r'}\right)^{1/r'} \le \left(\sum_1^n |a_i|^{s}\right)^{1/s} = n^{1/s}B_s,$$

$$r>2, s\ge r' : A_r \le \left(\sum_1^n |a_i|^{r'}\right)^{1/r'} \le n^{1/r'-1/s}\left(\sum_1^n |a_i|^{s}\right)^{1/s} = n^{1/r'}B_s.$$

The proof of (2.16) follows by combining these inequalities.

$\square$

Example B12 Consider the Littlewood-Walsh matrices $A_{2^n} = (\varepsilon_{ij}), 1 \le i,j \le 2^n$, defined recursively in the following way:

$$A_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, ..., A_{2^n} = \begin{bmatrix} A_{2^{n-1}} & A_{2^{n-1}} \\ A_{2^{n-1}} & -A_{2^{n-1}} \end{bmatrix}, \ n = 2,3, ...$$

Consider the operator

$$T : \bar{a} \to \left(\sum_1^{2^n} \varepsilon_{1j}a_j, \sum_1^{2^n} \varepsilon_{2j}a_j, ..., \sum_1^{2^n} \varepsilon_{2^n j}a_j\right)$$

from $\mathbb{R}^{2^n}$ to $\mathbb{R}^{2^n}$, and discuss as in the proof of Example B11 and we find that

$$\left(\sum_1^{2^n}\left|\sum_1^{2^n} \varepsilon_{ij}a_j\right|^r\right)^{1/r} \le 2^{n(1/r-1/s+1/q)}\left(\sum_1^{2^n}|a_j|^s\right)^{1/s},$$

for every $r\in\mathbb{R}, r\neq 0, s>0, q=\min(2,s)$ for $r\le 2$ and $q=\min(r',s)$ for $r>2$.

Example B13 We note that for the operator

$$T : \bar{a} \to \left(\sum_1^n a_i, a_1-a_2, a_1-a_3, a_1-a_4, ..., a_1-a_n, ..., a_{n-1}-a_n\right)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m, m=\frac{n(n-1)}{2}+1$, we have $T : l_1^{(n)} \to l_\infty^{(m)}$ with the norm $M_1 = 1$ and $T : l_2^{(n)} \to l_2^{(m)}$ with the norm $M_2 = n^{1/2}$ and, thus, by using interpolation and the similar monotonicity arguments as in the previous examples, we find that if $r\in\mathbb{R}, r\neq 0$ and $s>0$, then

$$\left(\left|\sum_1^n a_i\right|^r + \sum_{1\le i<j\le n}|a_i-a_j|^r\right)^{1/r} \le C\left(\sum_1^n |a_i|^s\right)^{1/s},$$

where $C = m^{1/r-1/2}n^{1/2+1/q-1/s}, q=\min(2,s)$ for $r\le 2$ and $C = n^{1/r+1/q-1/s}, q=\min(r',s)$ for $r>2$. Here $m = n(n-1)/2$.

Remark In particular, for the case $r=s=p\ge 2$, this estimate reads

$$\left|\sum_1^n a_i\right|^p + \sum_{1\le i<j\le n}|a_i-a_j|^p \le n^{p-1}\sum_1^n |a_i|^p.$$

For the case $n = 3$ (the Swedish Professor) Harold Shapiro proved (by interpolation) this and some similar estimates already in 1973 in a talk at the meeting of the Swedish Mathematical Society (This talk by Shapiro was my really first experience of interpolation theory).

3 APPENDIX I Some books in inequalities, interpolation and convexity

A. Inequalities

[A1] R.P. Agarwal, *DIFFERENCE EQUATIONS AND INEQUALITIES*, Marcel Dekker, Inc., New York/Basel/Hong Kong, 1992.

[A2] D. Bainov and P. Simeonov, *INTEGRAL INEQUALITIES AND APPLICATIONS*, Kluwer Academic Publishers, Dordresht, 1992.

[A3] E.F. Beckenbach and R. Bellman, *INEQUALITIES*, Springer Verlag, New York, 1961.

[A4] P. B. Borwein and T. Erdélyi, *POLYNOMIALS AND POLYNOMIAL INEQUALITIES*, Graduate Texts in Mathematics, Springer Verlag, New York, 1995.

[A5] J. M. Borwein and P.B. Borwein, *Pi AND THE AGM*, Canadian Mathematical Society series of monographs and advanced texts, Wiley, 1987.

[A6] P.S. Bullen, *A DICTIONARY OF INEQUALITIES*, Pitman Monographs and Surveys in Pure and Applied Mathematics 97, 1998.

[A7] P.S. Bullen, *HANDBOOK OF MEANS AND THEIR INEQUALITIES, Mathematics and its Applications*, Kluwer Academic Publishers, Dordresht, 2003.

[A8] P.S. Bullen, D.S. Mitrinovic and P.M. Vasic, *MEANS AND THEIR INEQUALITIES*, D. Reidel Publishing Company, Dordresht/Boston/Lancaster/Tokyo, 1988.

[A9] D. Bainov and P. Simeonov, *NTEGRAL INEQUALITIES AND APPLICATIONS*, Kluwer Academic Publishers, Dordresht/Boston/London, 1992.

[A10] M.J. Claud and B.C. Drashman, *INEQUALITIES. WITH APPLICATIONS TO ENGINEERING*, Springer Verlag, New York, 1998.

[A11] G. Duveat and J-L. Lions, *INEQUALITIES IN MECHANICS AND PHYSICS*, Springer Verlag, Berlin/New York, 1976.

[A12] D.J.H. Garling, *INEQUALITIES: A JOURNEY INTO LINEAR ANALYSIS*, Cambridge University Press, Cambridge, 2007.

[A13] G.H. Hardy, J.E. Littlewood and G. Polya, *INEQUALITIES*, Cambridge University Press,1934.

[A14] V. Kokilashvili and M. Krbec, *WEIGHTED INEQUALITIES IN LORENTZ AND OR-*

LICZ SPACES, World Scientific, Singapore/New York/London/Hong Kong, 1991.

[A15] A. Kufner and L.E. Persson, *WEIGHTED INEQUALITIES OF HARDY TYPE*, World Scientific, New Jersey/London/ Singapore/Hong Kong, 2003.

[A16] A. Kufner, L. Maligranda and L.E. Persson, *THE HARDY INEQUALITY. ABOUT ITS HISTORY AND SOME RELATED RESULTS*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[A17] L. Larson, L. Maligranda, J. Pečaric and L.E. Persson, *MULTIPLICATIVE INEQUALITIES OF CARLSON TYPE AND INTERPOLATION*, World Scientific, New Jersey/London/ Singapore/Beijing/Shanghai/Hong Kong/Taipei/Chennai, 2006.

[A18] E.H. Lieb and M. Loss, *ANALYSIS. GRADUATE STUDIES IN MATHEMATICS* 14, American Mathematical Society, Second Edition, RI, 2001.

[A19] A.W. Marshall and I. Olkin, *THEORY OF MAJORIZATION AND ITS APPLICATIONS*, Academic Press, San Diego, 1979.

[A20] A. Meskhi, V. Kokilashvili and L.E. Persson, *WEIGHTED NORM INEQUALITIES FOR INTEGRAL TRANSFORMS WITH PRODUCT KERNELS*, Nova Scientific Publishers, Inc., New York, 2010.

[A21] D.S. Mitrinovic, *ANALYTIC INEQUALITIES*, Springer Verlag, Berlin/Heidelberg/New York, 1979.

[A22] D.S. Mitrinovic, J.E. Pečaric and A.M. Fink, *CLASSICAL AND NEW INEQUALITIES IN ANALYSIS*, Kluwer Academic Publishers, Dordresht/Boston/London, 1973.

[A23] D.S. Mitrinovic, J.E. Pečaric and A.M. Fink, *INEQUALITIES INVOLVING FUNCTIONS AND THEIR INTEGRALS AND DERIVATIVES*, Kluwer Academic Publishers, Dordresht/Boston/London, 1991.

[A24] D.S. Mitrinovic, J.E. Pečaric and V. Volonec, *RECENT ADVANCES IN GEOMETRIC INEQUALITIES*, Kluwer Academic Publishers, Dordresht/Boston/London, 1989.

[A25] B. Opic and A. Kufner, *HARDY-TYPE INEQUALITIES*, Pitman Research Notes in Mathematics Series, Vol. 219, Longman Scientific and Technical, Harlow, 1990.

[A26] B.G. Pachpatte, *MATHEMATICAL INEQUALITIES*, North Holland Mathematical Library, Elsevier, 2005.

[A27] M.J. Steele, *THE CAUCHY-SCHWARZ MASTER CLASS. AN INTRODUCTION TO THE ART OF MATHEMATICAL INEQUALITIES*, MAA Problem Book Series. Mathematical Association of America, DC, Cambridge University Press, Cambridge, 2004.

[A28] W. Walter, *DIFFERENTIAL AND INTEGRAL INEQUALITIES*, Springer Verlag, New York/Berlin, 1070.

[A29] A. Zygmund, *TRIGONOMETRIC SERIES*, Vols. I, II, Third Edition, Cambridge University Press, Cambridge, 2002.

B. Interpolation

[B1] J. Bergh and J. Löfström, *INTERPOLATION SPACES. AN INTRODUCTION*, Springer Verlag, Berlin, 1976.

[B2] C. Bennett and R. Sharpley, *INTERPOLATION OF OPERATORS*, Pure and Applied Mathematics, Academic Press, Inc., Boston, MA, 1988.

[B3] Yu. A. Brudnyi and N. Ya Krugljak, *INTERPOLATION FUNCTIORS ANDINTERPOLATION SPACES* 1, North Holland, Amsterdam, 1991.

[B4] S.G. Krein, Yu. I. Petunin and E.M. Semenov, *INTERPOLATION OF LINEAR OPERATORS*, Nauka, Moscow, 1978 (in Russian); English translation in Amer. Math. Soc., Providence, 1982.

[B5] L. Maligranda, *INDICES AND INTERPOLATION*, Dissertationes Math. (Rozprawy Mat.) 234, 1985.

[B6] L. Maligranda, *ORLICZ SPACES AND INTERPOLATION*, Sem. Math. 5, Univ of Campinas, Campinas SP, Brazil, 1989.

[B7] L. Tartar, *AN INTRODUCTION TO SOBOLEV SPACES AND INTERPOLATION*, Lecture Notes of the Unione Matematica Italiano 3, Springer Berlin, UMI, Bologna, 2007.

[B8] H. Triebel, *INTERPOLATION THEORY, FUNCTION SPACES,DIFFERENTIAL OPERATORS*, second edition, Johann Ambrosius Barth, Heidelberg,1995.

C. Convexity

[C1] W. Fenchel, *CONVEXITY AND ITS APPLICATIONS*, Birkhäuser Verlag, Boston, 1983.

[C2] L. Hörmander, *NOTION OF CONVEXITY*, Birkhäuser Verlag, Boston/Basel/Berlin, 1994.

[C3] M.A. Krasnoselskii and Ja. B. Rutickii, *CONVEX FUNCTIONS AND ORLICZ SPACES* (translated from the first Russian edition by Leo F. Baron), P. Noordhoff Ltd., Groningen, 1961.

[C4] C.P. Niculescu and L.E. Persson, *CONVEX FUNCTIONS-BASIC THEORY AND APPLICATIONS*, Universitaria Press, 2003.

[C5] C.P. Niculescu and L.E. Persson, *CONVEX FUNCTIONS AND THEIR APPLICATIONS-A CONTEMPORY APPROACH*, Canad. Math. Series Books in Mathematics, Springer, 2006.

[C6] J.E. Pečaric, F. Proschan and Tong, *CONVEX FUNCTIONS; PARTIAL ORDERINGS AND STATISTICAL APPLICATIONS*, Mathematics in Science and Engineering, Vol. 187, Academic Press, San Diego, 1992.

[C7] R.R. Phelps, *CONVEX FUNCTIONS; MONOTONE OPERATORS AND DIFFERENTIABILITY*, Lecture Notes in Mathematics 1364, Springer Verlag, Berlin/Heidelberg, 1989, 1993.

[C8] R.T. Rockafellar, *CONVEX ANALYSIS*, Princeton University Press, 1970.

4 APPENDIX II More on Carlson's inequality and interpolation

The book [A15] is devoted to Carlson's inequality, its extensions and applications. In particular, four (out of nine) Chaperts are used to describe the close connections between Carlson type inequalities and interpolation theory. In this Appendix we briefly describe some parts of this development (also some complementary comments are given).

4.1 Examples of Carlson type inequalities

Example II:1 (V.I. Levin) If $a_k \geq 0, k = 1, 2, ...,$ and m is a positive integer, then

$$\left(\sum_{k=1}^{\infty} a_k\right)^{(m+1)(2m+1)} < C \prod_{j=0}^{2m} \sum_{k=1}^{\infty} k^j a_k^{m+1},$$

$$C = \prod_{j=0}^{2m+1} j^{2j-2m-1} = (2m+1)^{2m+1} \prod_{j=0}^{2m} \binom{2m}{j}$$

unless all $a_k : s$ are zero, where C is the sharp constant.

Remark (i) It is interesting to note that the sharp constant is an integer in every case covered by Example II:1.
(ii) In the cases $m = 1$ and $m = 2$ we get the following sharp inequalities:

$$\left(\sum_{k=1}^{\infty} a_k\right)^6 < 54 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2,$$

$$\left(\sum_{k=1}^{\infty} a_k\right)^{15} < 3 \cdot 10^5 \sum_{k=1}^{\infty} a_k^3 \sum_{k=1}^{\infty} k a_k^3 \sum_{k=1}^{\infty} k^2 a_k^3 \sum_{k=1}^{\infty} k^3 a_k^3 \sum_{k=1}^{\infty} k^4 a_k^3.$$

Thus there exists an inequality with the speed of light as the sharp constant.

An elementary observation: If α and β are real numbers, let

$$I(\alpha, \beta) = \int_0^{\infty} x^{\alpha} f^{\beta}(x) dx.$$

By using Hölder's inequality we easily arrive at

$$I(\alpha, \beta) \leq I(\alpha_1, \beta_1)^{1-\theta} I(\alpha_2, \beta_2)^{\theta},$$

where $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ and $\beta = (1-\theta)\beta_1 + \theta\beta_2$.
Hence, we are lead to the following result by (the Swedish professor) Bo Kjellberg:

Example II:2 (The Kjellberg convexity principle) Suppose that $m \geq 2$, that

$$I(\alpha_j, \beta_j) < \infty, j = 1, 2, ..., m$$

$j = 1, 2, ...m$ and that (α, β) is a point in the convex hull of the points $(\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)$. Then

$$I(\alpha, \beta) < \infty.$$

Remark Let $\int_0^\infty f^2(x)dx < \infty$ and $\int_0^\infty x^2 f^2(x)dx < \infty$. Then, by using this principle and trivial estimates, we find that the integral

$$I(\alpha, \beta) < \infty \text{ for } \{(\alpha, \beta) : -1 < \alpha \leq 0, \tfrac{2\alpha}{3} + \tfrac{2}{3} \leq \beta \leq 2\alpha + 2 \text{ or } 0 \leq \alpha \leq 1, \tfrac{2\alpha}{3} + \tfrac{2}{3} \leq \beta \leq 2\}.$$

The "Carlson point" is $\alpha = 0, \beta = 1$ (and the "corner points" are $(0, 2), 2.2)$ and $(-1, 0)$).

Moreover, (the Swedish Professor) Arne Beurling proved the following variant:

Example II:3 (Beurling's variant) It yields that

$$\int_{-\infty}^{\infty} |f(x)|dx \leq \sqrt{2\pi} \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{1/4}.$$

Remark In fact, this inequality is equivalent to Carlson's original continuous inequality (a simple proof can be found in [A15]).

E. Landau proved the following sharpening of Carlson's inequality:

Example II:4 (Landau's sharpening) It yields that

$$\left(\sum_{k=1}^{\infty} a_k \right)^4 < \pi^2 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} \left(k - \frac{1}{2} \right) a_k^2.$$

We may ask if some similar holds for the continuous case ?

Example II:5 (S. Barza, J. Pečaric and L.E. Persson): Suppose that $a > 0$. Then, for every non-negative, measurable function f on $\mathbb{R}_+$, it holds that

$$\left(\int_0^{\infty} |f(x)|dx \right)^4 \leq 4\pi^2 \int_0^{\infty} f^2(x)dx \int_0^{\infty} (x-a)^2 f^2(x)dx,$$

and the constant $4\pi^2$ is sharp.

Remark Note that, by Carlson's continuous inequality, the constant in the above inequality is not sharp if we were allowed to put $\alpha = 0$. In fact, at the point $\alpha = 0$ the sharp constant jumps from $4\pi^2$ to $2\pi^2$!

4.2 Examples of further relations between interpolation and Carlson type inequalities

The next result by V.I. Levin was very fundamental when J. Peetre proved his "parameter theorem" which was very important for the development of the real J-interpolation method.

Example II:6 (Levin's extention of Carlson's continuous inequality) Suppose that $p > 1, q > 1, s > 0$ and $t > 0$, and that λ and μ are any real numbers. If

$$s = \frac{\mu}{p\mu + q\lambda} \text{ and } t = \frac{\lambda}{p\mu + q\lambda}, \tag{4.1}$$

then

$$\int_0^\infty f(x)dx \leq C \left(\int_0^\infty x^{p-1-\lambda} f^p(x) dx \right)^s \left(\int_0^\infty x^{q-1+\mu} f^q(x) dx \right)^t \tag{4.2}$$

for all non-negative functions f, where the sharp constant C is given by

$$C = \left(\frac{1}{ps} \right)^s \left(\frac{1}{qt} \right)^t \left(\frac{1}{\mu + \lambda} B \left(\frac{s}{1-s-t}, \frac{t}{1-s-t} \right) \right)^{1-s-t}.$$

Conversely, in order for the existence of a constant C such that (4.2) holds it is necessary that s and t are defined by (4.1).

Also the next multidimensional version of Levin's result has influenced the close connection between interpolation and Carlson type inequalities in both directions (see the PhD thesis of L. Larsson from 2003). We need the following notations: Let S be a measurable subset of the unit sphere in $\mathbb{R}^n$ and define the infinite cone Ω by

$$\left\{ x \in \mathbb{R}^n; 0 < |x| < \infty, \frac{x}{|x|} \in S \right\}, \tag{4.3}$$

Suppose that the positive, measurable functions w, w_0 and w_1, defined on Ω, are homogeneous of degrees γ, γ_0 and γ_1, respectively (we say that $v : \Omega \longrightarrow \mathbb{R}_+$ is homogeneous of degree α if, for all $t > 0$ and $x \in \Omega$, it holds that $v(tx) = t^\alpha v(x)$). Suppose that $0 < p < p_0, p_1 < \infty$, and fix $\theta \in (0, 1)$. Define

$$d = \gamma + \frac{n}{p}, \; d_0 = \gamma_0 + \frac{n}{p_0}, \; d_1 = \gamma_1 + \frac{n}{p_1}$$

and q by the relation

$$\frac{1}{q} = \frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1}. \tag{4.4}$$

Example II:7 (A multidimensional version by S. Barza, V.I. Burenkov, J. Pečarić and L.E. Persson): Let $0 < p < p_0, p_1 < \infty$, Then the Carlson type inequality

$$\|fw\|_{L_p(\Omega,dx)} \leq C \|fw_0\|_{L_{p_0}(\Omega,dx)}^{1-\theta} \|fw_1\|_{L_{p_1}(\Omega,dx)}^{\theta} \tag{4.5}$$

holds for some constant C if and only if

$$d = (1-\theta)d_0 + \theta d_1,$$

$$d_0 \neq d_1,$$

and, with q defined by (4.4) and S by (4.3),

$$\frac{w}{w_0^{1-\theta}w_1^{\theta}} \in L_q(S,\sigma).$$

Here, σ is used to denote surface area measure on S. In (4.5) we may use

$$C = (1-\theta)^{-\frac{1-\theta}{p_0}}\theta^{-\frac{\theta}{p_1}}\left(\frac{B\left(\frac{(1-\theta)q}{p_0},\frac{\theta q}{p_1}\right)}{p_0p_1|d_0-d_1|}\right)^{1/q}\left(\frac{1}{p}-\frac{1}{q}\right)^{-\frac{1}{q}}\left\|\frac{w}{w_0^{1-\theta}w_1^{\theta}}\right\|_{L_q(S,\sigma)}, \tag{4.6}$$

and this is the sharp constant. Equality in (4.5) holds with the constant given by (4.6) if and only if f satisfies

$$|f(x)| = H\tilde{f}(rx)$$

for almost every x, for some $H \geq 0, r > 0$, where

$$\tilde{f} = \left((1-k)\frac{w^p}{w_0^{p_0}}\right)^{1/(p_0-p)}$$

and k is defined by

$$\left((1-k)^{1/p_0}\frac{w}{w_0}\right)^{r_0} = \left(k^{1/p_1}\frac{w}{w_1}\right)^{r_1}$$

where

$$\frac{1}{r_i} = \frac{1}{p} - \frac{1}{p_i},\ i = 0,1.$$

Remark In his PhD thesis from 2003 Leo Larsson extended the sufficient part of Example II:7 (for instance in (4.4) we can have "$\geq$" instead of "$=$," more general weights, etc.). Interpolation arguments were used in a fundamental way and he also pointed out that his new results implied applications both concerning embeddings and interpolation.

Next we will shortly describe how Carlson type inequalities has in a crucial way influenced the development of the Peetre $\pm$ method (which make it possible to also interpolate between Orlicz spaces under special restrictions).

The first observation is the following easily proved result (see the book [A15]):

Lemma The following two statements are equivalent:

(A) There is a constant C_1 such that, for all sequences $\{a_k\}_{k=1}^{\infty}$ of non-negative numbers, the inequality

$$\left(\sum_{k=1}^{\infty} a_k\right)^4 \leq C_1 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2$$

holds

(B) There is a constant C_2 such that, for all sequences $\{b_\ell\}_{\ell=0}^{\infty}$ of non-negative numbers, the inequality

$$\left(\sum_{\ell=0}^{\infty} b_\ell\right)^4 \leq C_2 \sum_{\ell=0}^{\infty} \frac{b_\ell^2}{2^\ell} \sum_{\ell=0}^{\infty} 2^\ell b_\ell^2 \tag{4.7}$$

holds.

Next we define the two variable function $\psi = \psi(s,t)$:

$$\psi(s,t) = \begin{cases} s\varphi\left(\frac{s}{t}\right), s,t > 0, \\ 0 \qquad s = 0 \text{ or } t = 0, \end{cases}$$

where φ is a concave function on $\mathbb{R}_+$.

Important: Each concave function φ has the properties that $\varphi(t)$ is non-decreasing and $\frac{\varphi(t)}{t}$ is non-increasing (such functions are called quasi-concave and is a special case of more general quasi-monotone functions, which were important when created "interpolation with a parameter-function" in real interpolation theory (with $\varphi(t)$ more general quasi-monotone function than $\varphi(t) = t^\theta, 0 < \theta < 1$).

With $\varphi(t) = t^{1/2}$ the Carlson type inequality in the form (4.7) can be written

$$\|\{b_\ell\}\|_{\ell_1} \leq C_2^{1/4} \psi\left(\left\|\left\{\frac{b_\ell}{\varphi(2^\ell)}\right\}\right\|_{\ell_2}, \left\|\left\{2^\ell \frac{b_\ell}{\varphi(2^\ell)}\right\}\right\|_{\ell_2}\right).$$

We can consider this inequality for any concave function φ and the corresponding ψ and the ℓ_2-norms can be replaced by ℓ_r-norms for any r. The inequality under consideration is thus

$$\|\{b_\ell\}\|_{\ell_1} \leq C \psi\left(\left\|\left\{\frac{b_\ell}{\varphi(2^\ell)}\right\}\right\|_{\ell_p}, \left\|\left\{2^\ell \frac{b_\ell}{\varphi(2^\ell)}\right\}\right\|_{\ell_q}\right). \tag{4.8}$$

We will return to this inequality shortly but first we need some definitions. Let $\mathcal{P}$ denote the class of concave functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\lim_{t\to 0_+} \varphi(t) = 0 \text{ and } \lim_{t\to\infty} \varphi(t) = \infty.$$

We also define the following subclasses of $\mathcal{P}$. Define

$$s_\varphi(t) := \sup_{s>0} \frac{\varphi(st)}{\varphi(s)},\ t > 0.$$

- $\mathcal{P}_+$ is the set of concave functions φ for which

$$\lim_{t\to 0^+} s_\varphi(t) = 0.$$

- $\mathcal{P}_-$ is the set of concave functions φ for which

$$\lim_{t\to\infty} \frac{s_\varphi(t)}{t} = 0.$$

- $\mathcal{P}_\pm = \mathcal{P}_+ \cap \mathcal{P}_-$.

- $\mathcal{P}_0$ is the set of concave functions φ satisfying

$$\lim_{t\to 0^+} \varphi(t) = 0 \ \text{ and } \lim_{t\to\infty} \frac{\varphi(t)}{t} = 0.$$

The following result is important for the Peetre $\pm$ method of interpolation:

Example II:8 (A Carlson type inequality by J. Gustavsson and J. Peetre): Suppose that $1 < p, q \leq \infty$. Then, if $\varphi \in \mathcal{P}_{\pm}$, there exists a constant such that the inequality

$$\sum_{k=1}^{m} a_k \leq C\psi\left(\left\|\left\{\frac{a_k}{\varphi(2^k)}\right\}\right\|_{\ell_p}, \left\|\left\{\frac{2^k a_k}{\varphi(2^k)}\right\}\right\|_{\ell_q}\right) \tag{4.9}$$

holds for some finite constant independent of m.

Remark The inequality (4.9) holds also for the cases $p = q = 1, \varphi \in \mathcal{P}; p = 1, q > 1, \varphi \in \mathcal{P}_-$; and $p > 1, q = 1, \varphi \in \mathcal{P}_+$, but not e.g. for $\varphi(t) = \min(1, t)$.

Remark This result was crucial when Gustavsson-Peetre further developed the original Peetre $\pm$method (for the definition see the book [A15], page 147) so that we have a natural interpolation between special Orlicz spaces.

More recently N. Krugljak, L. Maligranda and L.E. Persson developed and complemented the Peetre $\pm$ method in various ways. Their first result was the following:

Example II:9 (a) The implication in Example II:8 is in fact an equivalence in the class $\mathcal{P}_0$.

(b) The crucial inequality (4.9) can equivalently be rewritten on the following more symmetric form:

$$\sum_{k} \psi(a_k, b_k) \leq C\psi\left(\left\|\left\{\sum_{(a_k,b_k)\in S_m} a_k\right\}\right\|_{\ell_p}, \left\|\left\{\sum_{(a_k,b_k)\in S_m} b_k\right\}\right\|_{\ell_q}\right) \tag{4.10}$$

where for $m \in \mathbb{Z}_+$, S_m denotes the sector in $\mathbb{R}^2_+$, which is bounded by the lines $y = 2^m x$ and $y = 2^{m+1}x$.

This means that in order to extend the Peetre $\pm$ method for a more general class we must look for a more general Carlson type inequality (4.9) or, equivalently, (4.10). One main idea was to use the so called Brudnyi-Krugljak construction $\{t_m\}$ and consider a block-version of Carlson's inequality (where 2^k in the interval $\chi = t_m - t_{m-1}$ are put together in the same "block").

Example II:10 (A block-version of Carlson's inequality). Suppose that $1 < p, q \leq \infty$ and $\varphi \in \mathcal{P}_0$. Let $\{\chi_m\}$ be the intervals arising from the Brudnyi-Krugljak construction associated to φ. Then there is a constant C such that, for any sequence $\{a_k\}$ of non-negative numbers,

$$\sum a_k \leq C\psi\left(\left\|\left\{\sum_{2^k\in\chi_m} \frac{a_k}{\varphi(2^k)}\right\}_m\right\|_{\ell_p}, \left\|\left\{\sum_{2^k\in\chi_m} \frac{2^k a_k}{\varphi(2^k)}\right\}_m\right\|_{\ell_q}\right).$$

The best constant $C \leq (1 + \sqrt{2})^2$.

Remark This inequality can also equivalently be rewritten on a form corresponding to (4.10). The information in Examples II:9 and II:10 implies that interpolation between Orlicz spaces can be performed in more general situations namely when "$\varphi \in \mathcal{P}_0$" instead of "$\varphi \in \mathcal{P}_\pm$". More information can be found in the book [A15], where the probably most clear definitions and consequences of the Peetre $\pm$ method, the Brudnyi-Krugljak construction and relations to other interpolation methods (e.g. the orbit method) can be found. Concerning the Brudnyi-Krugljak construction I also want to mention previous works by K.I. Oskolkov and (the Swedish professor) Svante Janson.

5 Appendix III Some further Hardy type inequalities via convexity

We start with the following complement of our Basic Observation in Section 1.8.

Basic Observation III It yields that

$$\int_0^\infty exp\left(\frac{1}{x}\int_0^x lnf(y)dy\right)dx \leq e\int_0^\infty f(x)dx \tag{5.1}$$

$$\Leftrightarrow$$

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x lng(y)dy\right)\frac{dx}{x} \leq 1\cdot\int_0^\infty g(x)\frac{dx}{x}, \tag{5.2}$$

where $f(x) = \frac{g(x)}{x}$.

Remark According to Basic Observation III, Jensen's inequality and Fubini's theorem we see that also the limit Pólya-Knopp's inequality (5.1) (= (1.31)) follows directly from Jensen's inequality via (5.2).

In the same way as above, we can also prove the following more general statement:

Proposition III Let f be a measurable function on $\mathbb{R}_+$ and let Φ be a convex function on $D_f = \{f(x)\}$. Then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \leq \int_0^\infty \Phi\big(f(x)\big)\frac{dx}{x}. \tag{5.3}$$

If Φ instead is positive and concave, then the reversed inequality holds.

Example III:1 Consider the convex function $\Phi(u) = u^p, p \geq 1$ or $p < 0$. Then (5.3) reads

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p\frac{dx}{x} \leq \int_0^\infty f^p(x)\frac{dx}{x}, \tag{5.4}$$

i.e. according to Basic Observation, we obtain Hardy's original inequality (1.28) for $p > 1$, but also that it indeed holds also for $p < 0$ (if we assume that $f(x) > 0$ a.e.). For $0 < p < 1$ (5.4) holds in the reversed direction.

Example III:2 Consider the convex function $\Phi(u) = e^u$ and replace $f(y)$ with $ln\ f(y)$. Then (5.3) reads

$$\int_0^\infty exp\left(\frac{1}{x}\int_0^x lnf(y)dy\right)\frac{dx}{x} \leq \int_0^\infty f(x)\frac{dx}{x},$$

i.e. according to Basic Observation III, we obtain also Pólya-Knopp's inequality (1.31) directly without going via some limit argument.

More generally, the following statement follows more or less directly from (5.4):

Example III:3 Let f be a measurable and non-negative function on $(0, \infty)$. Then

a) the inequality (1.34) holds whenever $a < p - 1$ and $p > 1$ or $p < 0$.

b) the inequality

$$\int_0^{\infty}\left(\frac{1}{x}\int_x^{\infty}f(y)dy\right)^p x^a dx \le \left(\frac{p}{a+1-p}\right)^p \int_0^{\infty} f^p(x)x^a dx \tag{5.5}$$

holds whenever $a > p-1, \quad p > 1$ or $p < 0$.

c) For the case $0 < p < 1$ the inequalities (1.34) and (5.5) hold in the reversed direction whenever $a < p-1$ and $a > p-1$, respectively.

In fact, statement a) follows by just using the substitution $f(t) = g(t^{\frac{p-1-a}{p}})t^{-\frac{1+a}{p}}$, and making some straightforward calculations as in Basic Observation to see that (1.34) is equivalent to (5.4) and, thus, to (1.32) . Moreover, b) follows from a) by making some other obvious standard substitutions.

The proof of c) is the same as that of a) and b) and the fact that now the inequality (5.4) holds in the reversed direction (see Example III:1).

All inequalities above are sharp.

The case including finite intervals

It is also known that the Hardy inequality (1.28) holds for finite intervals, e.g. that

$$\int_0^{\ell}\left(\frac{1}{x}\int_0^{x}f(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^{\ell} f^p(x)dx, \quad p > 1. \tag{5.6}$$

holds for any $\ell, 0 < \ell \le \infty$.

In this Appendix we shall point out an improved variant of (5.6) with sharp constant also for $\ell < \infty$. In fact, guided by the result in our previous Section we will see that also some weighted variants of (5.6) are equivalent to (5.7) below. More details can be found in a recent paper from 2012 by L.E. Persson and N. Samko.

We begin by giving the following auxiliary result of independent interest.

Lemma III:1 Let g be a non-negative and measurable function on $(0,\ell), 0 < \ell \le \infty$.

a) If $p < 0$ or $p \ge 1$, then

$$\int_0^{\ell}\left(\frac{1}{x}\int_0^{x}g(y)dy\right)^p \frac{dx}{x} \le 1\cdot\int_0^{\ell} g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}. \tag{5.7}$$

(in the case $p < 0$ we assume that $g(x) > 0, 0 < x \le \ell$)

b) If $0 < p \le 1$, then (5.7) holds in the reversed direction.

c) The constant $C = 1$ is sharp in both a) and b).

Proof. By using Jensen's inequality with the convex function $\Psi(u) = u^p, p \ge 1, p < 0$, and reversing the order of integration, we find that

$$\int_0^{\ell}\left(\frac{1}{x}\int_0^{x}g(y)dy\right)^p \frac{dx}{x} \le \int_0^{\ell}\frac{1}{x}\int_0^{x}g^p(y)dy\frac{dx}{x} = \int_0^{\ell}g^p(y)\left(\int_y^{\ell}\frac{1}{x^2}dx\right)dy =$$

$$= \int_0^{\ell}g^p(y)\left(\frac{1}{y}-\frac{1}{\ell}\right)dy = \int_0^{\ell}g^p(y)\left(1-\frac{y}{\ell}\right)\frac{dy}{y}.$$

The only inequality in this proof holds in the reversed direction when $0 < p \leq 1$ so the proof of b) follows in the same way.

Concerning the sharpness of the inequality (5.7) we first let $\ell < \infty$ and assume that

$$\int_0^\ell \left(\frac{1}{x} \int_0^x g(y)dy \right)^p \frac{dx}{x} \leq C \cdot \int_0^\ell g^p(x) \left(1 - \frac{x}{\ell} \right) \frac{dx}{x} \tag{5.8}$$

for all non-negative and measurable functions g on $(0, \ell)$ with some constant $C, 0 < C < 1$. Let $p \geq 1$ and $\varepsilon > 0$ and consider $g_\varepsilon(x) = x^\varepsilon$ (for the case $p < 0$ we assume that $\varepsilon < 0$). By inserting this function into (5.8) we obtain that

$$C \geq (\varepsilon p + 1)^{1-p},$$

so that, by letting $\varepsilon \to 0_+$ we have that $C \geq 1$. This contradiction shows that the best constant in (5.7) is $C = 1$. In the same way we can prove that the constant $C = 1$ is sharp also in the case b). For the case $\ell = \infty$ the sharpness follows by just making a limit procedure with the result above in mind. The proof is complete. $\square$

Remark For the case $\ell = \infty$ (5.7) coincides with the inequality (5.4) and, thus, the constant $C = 1$ is sharp, which in its turn, implies the well-known fact that the constant $C = \left(\frac{p}{p-1}\right)^p$ in (1.28) is sharp for $p > 1$ and as we see above also for $p < 0$.

A generalization of (1.34) for the interval $(0, \ell), 0 < \ell \leq \infty, p \geq 1$, reads:

$$\int_0^\ell \left(\frac{1}{x} \int_0^x f(y)dy \right)^p x^a dx \leq \left(\frac{p}{p-1-a} \right)^p \int_0^\ell f^p(x) x^a \left[1 - \left(\frac{x}{\ell} \right)^{\frac{p-a-1}{p}} \right] dx, \tag{5.9}$$

where $a < p - 1, p \geq 1$.

In our next theorem we will give a proof of (5.9) based on the fact that (5.9) in fact is equivalent to (5.7) and it directly follows that the constant $\left(\frac{p}{p-1-a}\right)^p$ in (5.9) is sharp. More generally, we will present and prove a recent equivalence theorem, namely that all the inequalities in our next Theorem are equivalent to the sharp basic inequality (5.7) or its reversed version:

Theorem III:1 Let $0 < \ell \leq \infty$, let $p \in \mathbb{R}_+ \setminus \{0\}$ and let f be a non-negative function. Then

a) the inequality (5.9) holds for all measurable functions f, each $\ell, 0 < \ell \leq \infty$ and all a in the following cases:

$$(a_1) \qquad p \geq 1, a < p - 1,$$
$$(a_2) \qquad p < 0, a > p - 1.$$

b) for the case $0 < p < 1, a < p - 1$, inequality (5.9) holds in the reversed direction under the conditions considered in a).

c) the inequality

$$\int_\ell^\infty \left(\frac{1}{x} \int_x^\infty f(y)dy \right)^p x^{a_0} dx \leq \left(\frac{p}{a_0 + 1 - p} \right)^p \int_\ell^\infty f^p(x) x^{a_0} \left[1 - \left(\frac{\ell}{x} \right)^{\frac{a_0+1-p}{p}} \right] dx \tag{5.10}$$

holds for all measurable functions f, each $\ell, 0 \leq \ell < \infty$ and all a_0 in the following cases:

$$(c_1) \qquad p \geq 1, a_0 > p-1,$$
$$(c_2) \qquad p < 0, a_0 < p-1.$$

d) for the case $0 < p \leq 1,\ a > p-1$, inequality (5.10) holds in the reversed direction under the conditions considered in c).

e) All inequalities above are sharp.

f) Let $p \geq 1$ or $p < 0$. Then, the statements in a) and c) are equivalent for all permitted a and a_0 because they are in all cases equivalent to (5.7) via substitutions.

g) Let $0 < p < 1$. Then, the reversed inequalities the statements in b) and d) are equivalent for all permitted a and a_0.

Proof. First we prove that (5.9) in the case (a_1) in fact is equivalent to (5.7) via the relation

$$f(x) = g(x^{\frac{p-a-1}{p}})x^{-\frac{a+1}{p}}.$$

In fact, with $f(x) = g(x^{\frac{p-a-1}{p}})x^{-\frac{a+1}{p}}$ and $\ell_0 = \ell^{\frac{p}{p-a-1}}$, in (5.9) we get that

$$\begin{aligned} RHS \quad &= \left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0} g^p(x^{\frac{p-a-1}{p}}) \left[1 - \left(\frac{x}{\ell_0}\right)^{\frac{p-1-a}{p}}\right] \frac{dx}{x} = \\ &= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell^{\frac{p-a-1}{p}}} g^p(y) \left[1 - \frac{y}{\ell_0^{\frac{p-1-a}{p}}}\right] \frac{dy}{y} = \\ &= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell} g^p(y) \left[1 - \frac{y}{\ell}\right] \frac{dy}{y}, \end{aligned}$$

where $y = x^{\frac{p-a-1}{p}}, dy = x^{-\frac{a+1}{p}} \left(\frac{p-1-a}{p}\right) dx$, and

$$\begin{aligned} LHS \quad &= \int_0^{\ell_0} \left(\frac{1}{x} \int_0^x g(y^{\frac{p-a-1}{p}}) y^{-\frac{a+1}{p}} dy\right)^p x^a dx = \\ &\left(\frac{p}{p-1-a}\right)^p \int_0^{\ell_0} \left(\frac{1}{x^{\frac{p-a-1}{p}}} \int_0^{x^{\frac{p-a-1}{p}}} g(s)ds\right)^p \frac{dx}{x} = \\ &= \left(\frac{p}{p-1-a}\right)^{p+1} \int_0^{\ell} \left(\frac{1}{y} \int_0^y g(s)ds\right)^p \frac{dy}{y}. \end{aligned}$$

Since we have only equalities in the calculations above we conclude that (5.7) and (5.9) are equivalent and, thus, by Lemma III:1, a) is proved for the case (a_1).

For the case (a_2) all calculations above are still valid and, according to Lemma III:1, (5.7) holds also in this case and a) is proved also for the case (a_2) .

For the case $0 < p \leq 1, a < p-1$, all calculations above are still true and both (5.7) and (5.9) hold in the reversed direction according to Lemma III:1. Hence also b) is proved.

For the proof of c) we consider (5.9) with $f(x)$ replaced by $f(1/x)$, with a replaced by a_0 and with ℓ replaced by $\ell_0 = 1/\ell$:

$$\int_0^{\ell_0} \left(\frac{1}{x} \int_0^x f(1/y)dy \right)^p x^{a_0} dx \leq$$

$$\leq \left(\frac{p}{p-1-a_0} \right)^p \int_0^{\ell_0} f^p(1/x) x^{a_0} \left[1 - \left(\frac{x}{\ell_0} \right)^{\frac{p-a_0-1}{p}} \right] dx.$$

Moreover, by making the variable substitution $y = 1/s$, we find that

$$LHS \quad = \int_0^{\ell_0} \left(\frac{1}{x} \int_{1/x}^{\infty} \frac{f(s)}{s^2} ds \right)^p x^{a_0} dx = \int_{\ell}^{\infty} \left(y \int_y^{\infty} \frac{f(s)}{s^2} ds \right)^p y^{-a_0-2} dy =$$

$$= \int_{\ell}^{\infty} \left(\frac{1}{y} \int_y^{\infty} \frac{f(s)}{s^2} ds \right)^p y^{-a_0-2+2p} dy =$$

[put $\frac{f(s)}{s^2} = g(s)$]

$$= \int_{\ell}^{\infty} \left(\frac{1}{y} \int_y^{\infty} g(y) \right)^p y^{2p-a_0-2} dy,$$

and

$$RHS \quad = \left(\frac{p}{p-1-a_0} \right)^p \int_{\ell}^{\infty} f^p(y) y^{-a_0} \left[1 - \left(\frac{\ell}{y} \right)^{\frac{p-a_0-1}{p}} \right] y^{-2} dy =$$

$$= \left(\frac{p}{p-1-a_0} \right)^p \int_{\ell}^{\infty} g^p(y) y^{2p-a_0-2} \left[1 - \left(\frac{\ell}{y} \right)^{\frac{p-a_0-1}{p}} \right] dy.$$

Now replace $2p - a_0 - 2$ by a and g by f and we have $a_0 = 2p - a - 2$, so that $p - 1 - a_0 = a + 1 - p$. Hence, it yields that

$$\int_0^{\ell} \left(\frac{1}{x} \int_x^{\infty} f(s)ds \right)^p x^a dx \leq \left(\frac{p}{a+1-p} \right)^p \int_{\ell}^{\infty} f^p(x) x^a \left[1 - \left(\frac{\ell}{x} \right)^{\frac{a+1-p}{p}} \right] dx$$

and, moreover,

$$a_0 < p - 1 \Leftrightarrow 2p - a - 2 < p - 1 \Leftrightarrow a > p - 1.$$

We conclude that c) with the conditions (c_1) and (c_2) are in fact equivalent to a) with the conditions (a_1) and (a_2), respectively, and also c) is proved.

The calculations above hold also in the case d) and the only inequality holds in the reversed direction in this case so also d) is proved.

Next we note that the proof above only consists of suitable substitutions and equalities to reduce all inequalities to the sharp inequality (5.7) and we obtain a proof also of the statements e), f) and g) according to Lemma III:1. The proof is complete. $\square$

The case with piecewise constant $p = p(x)$.

By using similar arguments as before in this Appendix, we can derive the following result:

Theorem III:2 Let $a > 0$ and

$$p(x) = \begin{cases} p_0, & 0 \leq x \leq a, \\ p_1, & x > a, \end{cases}$$

where $p_0, p_1 \in \mathbb{R} \setminus \{0\}$. Moreover, let $\alpha < 1, 0 < a \leq \ell \leq \infty$. Then

$$\int_0^\ell \left(\frac{1}{x} \int_0^x f(t)dt \right)^{p(x)} x^\alpha \frac{dx}{x} \leq \frac{1}{1-\alpha} \int_0^\ell (f(x))^{p(x)} x^\alpha \left(1 - \left(\frac{x}{\ell} \right)^{1-\alpha} \right) \frac{dx}{x} + \tag{5.11}$$

$$+ \max\{0, \frac{a^{\alpha-1} - \ell^{\alpha-1}}{1-\alpha}\} \int_0^\ell \left[(f(x))^{p_1} - (f(x))^{p_0} \right] dx,$$

whenever $p(x) \geq 1$ or $p(x) < 0$.

For the case $0 < p(x) < 1$ (5.11) holds in the reversed direction. The inequality (5.11) is sharp in the sense that the constant $C = \frac{1}{1-\alpha}$ in front of the first integral on the right hand side of (5.11) does not hold in general with any $C < \frac{1}{1-\alpha}$.

Remark By using Theorem III:2 with $p_0 = p_1 = p$ we obtain the following weighted generalization of our basic inequality (5.7):

$$\int_0^\ell \left(\frac{1}{x} \int_0^x g(t)dt \right)^p x^\alpha \frac{dx}{x} \leq \frac{1}{1-\alpha} \int_0^\ell (f(x))^p x^\alpha \left(1 - \left(\frac{x}{\ell} \right)^{1-\alpha} \right) \frac{dx}{x} \tag{5.12}$$

for any $\alpha < 1$. For the case $0 < p \leq 1$ (5.12) holds in the reverse direction. The inequality is sharp in both cases (For the case when $p < 0$ we also assume that $f(x) > 0$ a.e.). This, in particular, means that all power weighted Hardy type inequalities presented in Appendix III can be derived from Theorem III:2.

Remark It is obvious from the proof above that Theorem III:2 can be generalized to the situation when $p(x) = p_i,\ a_i \leq x \leq a_{i+1},\ a_0 = 0,\ a_{N+1} \leq \infty,\ i = 0, 1, ..., N,\ N \in \mathbb{Z}_+$. The only difference is that the second term on the right hand side in (5.11) will be more complicated.

6 Appendix IV Some further inequalities connected to convexity/concavity

First we state the classical Hermite-Hadamard inequality, which relates three arithmetic means with a convex function involved.

Example IV:1 (Hermite-Hadamard's inequality). If f is convex on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{6.1}$$

Proof. Integrate the convexity inequality

$$f(x) \leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

and the right hand side inequality (6.1) follows. Moreover, by using the definition of convexity, we find that

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx\right)$$

$$\frac{1}{2}\int_0^1 \left[f\left(\frac{a+b-t(b-a)}{2}\right) + f\left(\frac{a+b+t(b-a)}{2}\right)\right] dt$$

$$\geq \frac{1}{2}\int_0^1 2f\left(\frac{a+b}{2}\right) dt = f\left(\frac{a+b}{2}\right).$$

The proof is complete. □

Remark The proof above shows that both inequalities in (6.1) hold in the reverse direction if f is concave.

Remark Let $f(x) = e^x$. Then, by (6.1),

$$e^{\frac{a+b}{2}} \leq \frac{e^b - e^a}{b-a} \leq \frac{e^a + e^b}{2}$$

so that

$$\sqrt{xy} \leq \frac{x-y}{\log x - \log y} \leq \frac{x+y}{2}, x \neq y, x, y > 0,$$

which is the usual geometric-logarithmic-arithmetic mean inequality.

In 2004 C.Niculescu and L.E.Persson generalized (6.1) to the case with a general measure μ as follows:

Example IV:2 If f is a convex on $[a,b]$, then

$$f(x_\mu) \le \frac{1}{\mu[a,b]} \int\limits_a^b f(x)d\mu(x) \le \frac{b-x_\mu}{b-a} f(a) + \frac{x_\mu - a}{b-a} f(b),$$

where

$$x_\mu := \frac{1}{\mu[a,b]} \int\limits_a^b x d\mu(x).$$

(x_μ is called the barycenter of $[a,b]$ with respect to μ).

Example IV:3 (Favard's inequality). Let $p \ge 1$ and let f be a non-negative and concave function on $[0,1]$. Then,

$$\int\limits_0^1 f(x)dx \ge \frac{(p+1)^{1/p}}{2} \left(\int\limits_0^1 f^p(x)dx \right)^{1/p}. \tag{6.2}$$

Example IV:4 (Grüss-Barnes inequalities). Let f and g be non-negative and concave functions on $[0,1]$. If $p,q \ge 1$, then

$$\int\limits_0^1 f(x)g(x)dx \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{6} \left(\int\limits_0^1 f^p(x)dx \right)^{1/p} \left(\int\limits_0^1 g^q(x)dx \right)^{1/q}. \tag{6.3}$$

and if $0 < p,\ q \le 1$, then

$$\int\limits_0^1 f(x)g(x)dx \le \frac{(p+1)^{1/p}(q+1)^{1/q}}{3} \left(\int\limits_0^1 f^p(x)dx \right)^{1/p} \left(\int\limits_0^1 g^q(x)dx \right)^{1/q}. \tag{6.4}$$

Remark We note that (6.3) follows at once from the special case $p = q = 1$ and the Favard inequality (6.2).

These results were further generalized by L.Maligranda, J.Pečaric and L.E.Persson in 1994 as follows:

Example IV:5 Let f and g be non-negative and concave functions on $[0,1]$ and let $p,q \ge 1$. Then

$$\int\limits_0^1 (1-x)f(x)g(x)dx \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{12} \left(\int\limits_0^1 f^p(x)dx \right)^{1/p} \left(\int\limits_0^1 g^q(x)dx \right)^{1/q} + \frac{f(0)g(0)}{6}$$

and

$$\int\limits_0^1 xf(x)g(x)dx \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{12} \left(\int\limits_0^1 f^p(x)dx \right)^{1/p} \left(\int\limits_0^1 g^q(x)dx \right)^{1/q} + \frac{f(1)g(1)}{6}.$$

Equality in these inequalities occurs if either
1^o. $f(x) = 1 - x, g(x) = x$ or $f(x) = x, g(x) = 1 - x$ or
2^o. $f(x) = g(x) = x$ or
3^o $f(x) = g(x) = 1 - x$.

Remark A number of generalizations when f and g are replaced by $f_1, f_2, ..., f_n, n = 3, 4, ...$ are stated and proved in the same paper.