Analytic Integrated Assessment and Uncertainty

Christian Traeger

ARE, UC Berkeley

College de France, 10/29/15

I GAUVAL: An Analytic IAM (Integrated Assessment Model)

II Optimal Carbon Tax: Quantification in Closed Form

III Uncertainty: A Teaser

IV Smart Cap, COP, and "Optimal Compromise"
(Cooling the Climate Debate)

Contributions

GAUVAL: An integrated assessment model (IAM) with closed-form solution for opt carbon tax and welfare loss

- (At least) As realistic as the numeric "DICE" model
- Analytic insights into quantitative assessment
- Avoids curse of dimensionality in numeric stochastic IAMs

GAUVAL explains

- detailed discounting sensitivities (certain and uncertain)
- relation between shocks and epistemological uncertainty
- why the marginal damage curve is mostly flat

$\hookrightarrow$ Std. Cap not so good $\Rightarrow$ use "smart cap" instead

- The Smart Cap: A better emission control mechanism
- Implications for the COP negotiations
- (Based on work with Larry Karp)

Contributions

GAUVAL: An integrated assessment model (IAM) with closed-form solution for opt carbon tax and welfare loss

- (At least) As realistic as the numeric "DICE" model
- Analytic insights into quantitative assessment
- Avoids curse of dimensionality in numeric stochastic IAMs

GAUVAL explains

- detailed discounting sensitivities (certain and uncertain)
- relation between shocks and epistemological uncertainty
- why the marginal damage curve is mostly flat

$\hookrightarrow$ Std. Cap not so good $\Rightarrow$ use "smart cap" instead

- The Smart Cap: A better emission control mechanism
- Implications for the COP negotiations
- (Based on work with Larry Karp)

Contributions

GAUVAL: An integrated assessment model (IAM) with closed-form solution for opt carbon tax and welfare loss

- (At least) As realistic as the numeric "DICE" model
- Analytic insights into quantitative assessment
- Avoids curse of dimensionality in numeric stochastic IAMs

GAUVAL explains

- detailed discounting sensitivities (certain and uncertain)
- relation between shocks and epistemological uncertainty
- why the marginal damage curve is mostly flat

$\hookrightarrow$ Std. Cap not so good $\Rightarrow$ use "smart cap" instead

- The Smart Cap: A better emission control mechanism
- Implications for the COP negotiations
- (Based on work with Larry Karp)

Contributions

GAUVAL: An integrated assessment model (IAM) with closed-form solution for opt carbon tax and welfare loss

- (At least) As realistic as the numeric "DICE" model
- Analytic insights into quantitative assessment
- Avoids curse of dimensionality in numeric stochastic IAMs

GAUVAL explains

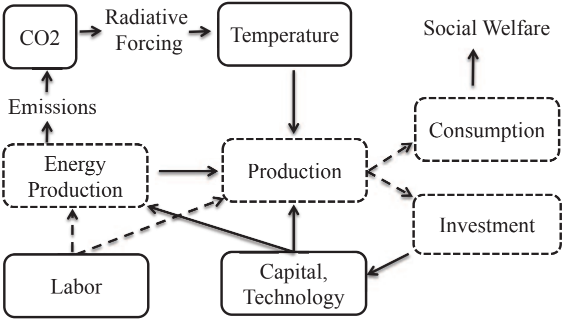
- detailed discounting sensitivities (certain and uncertain)
- relation between shocks and epistemological uncertainty
- why the marginal damage curve is mostly flat

$\hookrightarrow$ Std. Cap not so good $\Rightarrow$ use "smart cap" instead

- The Smart Cap: A better emission control mechanism
- Implications for the COP negotiations
- (Based on work with Larry Karp)

What is an Integrated Assessment Model (IAM) ?

- Joint representation of climate system & economy
- Integrates cause and effect of climate change
- Matches stylized market and climatic observations



Modeling Progress w.r.t. Literature

Closest are Golosov et. al (2014, E), Gerlagh & Liski (2012).

GAUVAL adds a full climate change model consisting of:

- carbon cycle (also in Golosov, Gerlagh & Liski)
- radiative forcing
- ocean-atmosphere temperature dynamics

↪ First analytic model with realistic temperature dynamics

GAUVAL adds general disentangled risk attitude

- unit elasticity only for intertemporal substitutability (good approximation)
- risk aversion calibrated to long-run risk literature (in macro and finance, clear evidence that larger than IES)

↪ Better calibrate of discount rate *and* risk premia

(numeric IAM applications: Crost & Traeger (2014), Jensen & Traeger (2014))

Modeling Progress w.r.t. Literature

Closest are Golosov et. al (2014, E), Gerlagh & Liski (2012).

GAUVAL adds a full climate change model consisting of:

- carbon cycle (also in Golosov, Gerlagh & Liski)
- radiative forcing
- ocean-atmosphere temperature dynamics

↪ First analytic model with realistic temperature dynamics

GAUVAL adds general disentangled risk attitude

- unit elasticity only for intertemporal substitutability (good approximation)
- risk aversion calibrated to long-run risk literature (in macro and finance, clear evidence that larger than IES)

↪ Better calibrate of discount rate *and* risk premia

(numeric IAM applications: Crost & Traeger (2014), Jensen & Traeger (2014))

Results

First Theory Result: Characterization of a class of IAMs with closed-form solution (see paper).

Calibration:

- Damage function close to DICE (initially slightly less convex, then more convex)
  - → damage parameter ξ_0 (semi-elasticity of output to exp temperature increase)
- Carbon cycle taken from DICE:
  - → Carbon transition matrix Φ
- Temperature dynamics calibrated to Magicc 6.0:
  - → "Heat" transition matrix σ and, in particular: speed of atmospheric temperature response to forcing σ^{forc}
- Time preference, output, and consumption rate are based on 2015 IMF forecast

Results

First Theory Result: Characterization of a class of IAMs with closed-form solution (see paper).

Calibration:

- Damage function close to DICE (initially slightly less convex, then more convex)
  - → damage parameter ξ_0 (semi-elasticity of output to exp temperature increase)
- Carbon cycle taken from DICE:
  - → Carbon transition matrix Φ
- Temperature dynamics calibrated to Magicc 6.0:
  - → "Heat" transition matrix σ and, in particular: speed of atmospheric temperature response to forcing σ^{forc}
- Time preference, output, and consumption rate are based on 2015 IMF forecast

Results

First Theory Result: Characterization of a class of IAMs with closed-form solution (see paper).

Calibration:

- Damage function close to DICE (initially slightly less convex, then more convex)
  - → damage parameter ξ_0 (semi-elasticity of output to exp temperature increase)
- Carbon cycle taken from DICE:
  - → Carbon transition matrix $\mathbf{\Phi}$
- Temperature dynamics calibrated to Magicc 6.0:
  - → "Heat" transition matrix $\boldsymbol{\sigma}$ and, in particular: speed of atmospheric temperature response to forcing σ^{forc}
- Time preference, output, and consumption rate are based on 2015 IMF forecast

Results

First Theory Result: Characterization of a class of IAMs with closed-form solution (see paper).

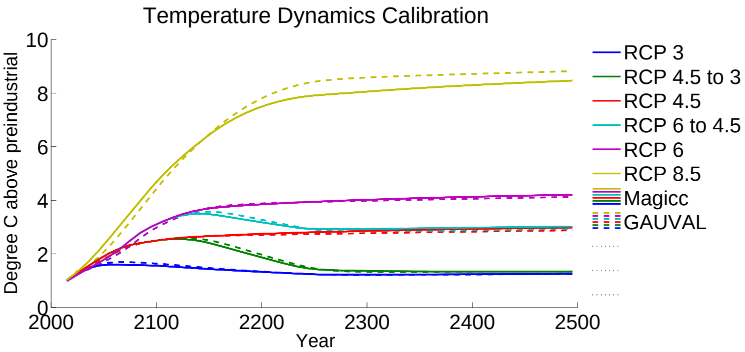
Calibration:

- Damage function close to DICE (initially slightly less convex, then more convex)
  - → damage parameter ξ_0 (semi-elasticity of output to exp temperature increase)
- Carbon cycle taken from DICE:
  - → Carbon transition matrix $\boldsymbol{\Phi}$
- Temperature dynamics calibrated to Magicc 6.0:
  - → "Heat" transition matrix $\boldsymbol{\sigma}$ and, in particular: speed of atmospheric temperature response to forcing σ^{forc}
- Time preference, output, and consumption rate are based on 2015 IMF forecast

Temperature Dynamics

Calibration of Atmosphere-Ocean Temperature Dynamics

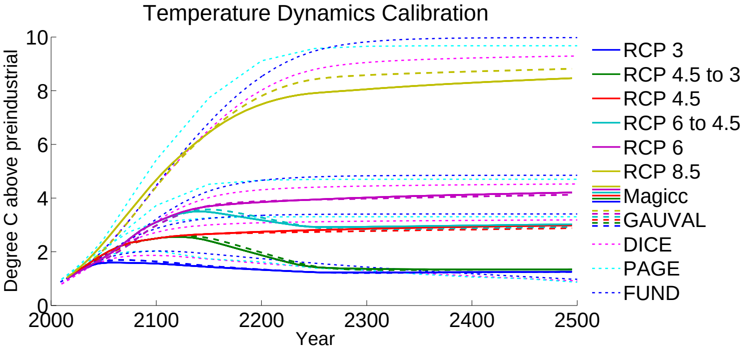
- Match Magicc 6.0 for IPCC's RCP scenarios,
  Magicc6.0 emulates AOGCMS ("big models") used in Assessment Reports by the Intergovernmental Panel on Climate Change IPCC



Temperature Dynamics

Calibration of Atmosphere-Ocean Temperature Dynamics

- Match Magicc 6.0 for IPCC's RCP scenarios,
  Magicc6.0 emulates AOGCMS ("big models") used in Assessment Reports by the Intergovernmental Panel on Climate Change IPCC



The Social Cost of Carbon: Formula

The optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{\text{damages}} \underbrace{\left[(\mathbf{1} - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc}}_{\text{climate dynamics}} \underbrace{\left[(\mathbf{1} - \beta\mathbf{\Phi})^{-1}\right]_{1,1}}_{\text{carbon dynamics}}$$

- discount factor β
- production Y_t
- preindustrial carbon M_{pre}
- damage parameter ξ_0 (semi-elasticity of net production)
- temperature dynamics $\boldsymbol{\sigma}$ and, in particular:
- speed of atmospheric temperature response to forcing σ^{forc}
- carbon dynamics Φ (transition matrix)

  $\left[(\mathbf{1} - \beta\mathbf{\Phi})^{-1}\right]_{1,1}$ interpretation by Neumann series expansion:
  ∞ sum over β discounted emission persistence & return to atmosphere

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1} = 57 \frac{\$}{tC} ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57$/ton carbon or 16 $/t$CO_2$
- Increases with output ("policy ramp")
- Proportion to damage (semi-) elasticity ξ_0
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \underbrace{\frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%}}_{26 \frac{\$}{tC}} \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} = 57 \ \frac{\$}{tC} \ ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO$_2$
- Increases with output ("policy ramp")
- Proportion to damage (semi-) elasticity ξ_0
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \underbrace{\frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%}}_{26 \frac{\$}{tC}} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1}}_{1.4} \underbrace{\sigma^{forc}}_{0.42} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} = \text{“15”} \frac{\$}{tC} \, ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO_2
- Increases with output (“policy ramp”)
- Proportion to damage (semi-) elasticity ξ_0
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \underbrace{\frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%}}_{26 \frac{\$}{tC}} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1}}_{1.4} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1}}_{3.7} = \mathbf{57} \; \frac{\$}{tC} \; ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO_2
- Increases with output ("policy ramp")
- Proportion to damage (semi-) elasticity ξ_0
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

Discounting: A Sensitivity Result

Second Theory Result:

A carbon cycle whose transition matrix $\mathbf{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \frac{\mathbf{1}}{\boldsymbol{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Discounting: A Sensitivity Result

Second Theory Result:

A carbon cycle whose transition matrix $\mathbf{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \boldsymbol{\frac{1}{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Recall Ramsey equation "$r = \rho + \eta g$".

Countering wide-spread belief (e.g. Nordhaus 2007, JEL):

- SCC is (very) sensitive to composition of cons. disc. rate r:
- *not* sensitive to growth term, highly sensitive to p.r.t.p. ρ

Discounting: A Sensitivity Result

Second Theory Result:

A carbon cycle whose transition matrix $\mathbf{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \frac{\mathbf{1}}{\boldsymbol{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Reduce pure time preference from $\rho = 1.75\%$ to $\rho = 0.1\%$

- Normative: Stern Review
- Descriptive: Long-run risk model
- Both: Disentangle individual and generational time pref

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}}_{\cancel{1.4}\ 2} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1}-\beta\mathbf{\Phi})^{-1}\right]_{1,1}}_{\cancel{3.7}\ 26} = \cancel{\mathbf{57}}\ \mathbf{660}\ \frac{\$}{tC}$$

A Glimpse of Uncertainty

Summary/"Teaser":

I analyze uncertainty governing carbon flows and temperature response uncertainty

- Better information over temperature response to emissions ("climate sensitivity") is much more valuable than learning about carbon flows ("missing sink")

I analyze and compare shocks, epistemological uncertainty, and anticipated learning

- Crucial role: uncertainty distribution's cumulants (≈ moments) weighted by intertemporal risk aversion (≈ difference between Arrow Pratt risk aversion and desire for intertemporal smoothing (Traeger (2015))
- "Learning shocks" are similar to fully persistent shocks, ↪ Learning model most sensitive to time preference

A Glimpse of Uncertainty

Summary/“Teaser”:

I analyze uncertainty governing carbon flows and temperature response uncertainty

- Better information over temperature response to emissions (“climate sensitivity”) is much more valuable than learning about carbon flows (“missing sink”)

I analyze and compare shocks, epistemological uncertainty, and anticipated learning

- Crucial role: uncertainty distribution’s cumulants (≈ moments) weighted by intertemporal risk aversion (≈ difference between Arrow Pratt risk aversion and desire for intertemporal smoothing (Traeger (2015))
- “Learning shocks” are similar to fully persistent shocks, ↪ Learning model most sensitive to time preference

A Glimpse of Uncertainty

Summary/"Teaser":

I analyze uncertainty governing carbon flows and temperature response uncertainty

- Better information over temperature response to emissions ("climate sensitivity") is much more valuable than learning about carbon flows ("missing sink")

I analyze and compare shocks, epistemological uncertainty, and anticipated learning

- Crucial role: uncertainty distribution's cumulants (≈ moments) weighted by intertemporal risk aversion
(≈ difference between Arrow Pratt risk aversion and desire for intertemporal smoothing (Traeger (2015))
- "Learning shocks" are similar to fully persistent shocks,
↪ Learning model most sensitive to time preference

To Paris: Flat Marginal Damages!

The $SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1}$

is independent of

- the level of CO_2 (& T, in present and future)

↪ Flat marginal damage curve! (SCC_t not function of E_t or M_t)

Add technological and macroeconomic uncertainty:

↪ Optimal policy keeps price fix, *NOT* quantity

Why flat marginal damages? Three effects balance each other

- Falling marginal impact of CO_2 on temperature T
- Increasing marginal impact of T on production
- Decreasing marginal impact of consumption on welfare

To Paris: Flat Marginal Damages!

The $SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \mathbf{\Phi})^{-1} \right]_{1,1}$

is independent of

- the level of CO_2 (& T, in present and future)

$\hookrightarrow$ Flat marginal damage curve! (SCC_t not function of E_t or M_t)

Add technological and macroeconomic uncertainty:

$\hookrightarrow$ Optimal policy keeps price fix, *NOT* quantity

Why flat marginal damages? Three effects balance each other

- Falling marginal impact of CO_2 on temperature T
- Increasing marginal impact of T on production
- Decreasing marginal impact of consumption on welfare

To Paris: Flat Marginal Damages!

The $SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1}$

is independent of

- the level of CO_2 (& T, in present and future)

$\hookrightarrow$ Flat marginal damage curve! (SCC_t not function of E_t or M_t)

Add technological and macroeconomic uncertainty:

$\hookrightarrow$ Optimal policy keeps price fix, *NOT* quantity

Why flat marginal damages? Three effects balance each other

- Falling marginal impact of CO_2 on temperature T
- Increasing marginal impact of T on production
- Decreasing marginal impact of consumption on welfare

To Paris: Instrument Choice & Consequences

COPs including Paris:

- Countries negotiate quantity target
- Re-negotiation periods long = involve major technological and macroeconomic uncertainties

↪ Negotiating a *quantity target* is very *inefficient*

How about negotiating a tax? Theory:

- Static world:
  - Gentle slope of $MD(E_t) \equiv SCC(E_t) << MB(E_t)$
    MD =marginal damages & MB= marginal benefits from emissions

  ↪ Tax quite efficient
- However, climate change is a dynamic problem:
  - Technological progress shifts $MB(E_t)$ & $MD(E_t)$ curves
  - Slope of $MD(E_t)$ vs $MB(E_t)$ *not* the relevant measure of tax vs quantity performance (Karp & Traeger 2015)

↪ Tax/negotiating carbon price not great either ☹

To Paris: Instrument Choice & Consequences

COPs including Paris:

- Countries negotiate quantity target
- Re-negotiation periods long = involve major technological and macroeconomic uncertainties

↪ Negotiating a *quantity target* is very *inefficient*

How about negotiating a tax? Theory:

- Static world:
  - Gentle slope of $MD(E_t) \equiv SCC(E_t) << MB(E_t)$
    MD =marginal damages & MB= marginal benefits from emissions

  ↪ Tax quite efficient
- However, climate change is a dynamic problem:
  - Technological progress shifts $MB(E_t)$ & $MD(E_t)$ curves
  - Slope of $MD(E_t)$ vs $MB(E_t)$ *not* the relevant measure of tax vs quantity performance (Karp & Traeger 2015)

↪ Tax/negotiating carbon price not great either ☹

To Paris: Instrument Choice & Consequences

COPs including Paris:

- Countries negotiate quantity target
- Re-negotiation periods long = involve major technological and macroeconomic uncertainties

$\hookrightarrow$ Negotiating a *quantity target* is very *inefficient*

How about negotiating a tax? Theory:

- Static world:
  - Gentle slope of $MD(E_t) \equiv SCC(E_t) << MB(E_t)$
    MD =marginal damages & MB= marginal benefits from emissions

  $\hookrightarrow$ Tax quite efficient
- However, climate change is a dynamic problem:
  - Technological progress shifts $MB(E_t)$ & $MD(E_t)$ curves
  - Slope of $MD(E_t)$ vs $MB(E_t)$ *not* the relevant measure of tax vs quantity performance (Karp & Traeger 2015)

$\hookrightarrow$ Tax/negotiating carbon price not great either ☹

To Paris: The "Optimal Compromise" is a Smart Cap

Cooling Down the Climate Debate: The Smart Cap!

National implementation: See Karp & Traeger (2015)

- Idea: Trade certificates whose quantity denomination is a function of the certificate price

↪ Efficient for any slope of MD curve

Practical implications for negotiations:

- A compromise between tax and cap advocates (and more efficient than either)
- Uses existing cap and trade markets/institutions
- Enables a compromise in negotiations
  - If abatement turns out cheaper: agree to do more
  - If abatement turns out expensive: agree to do less

↪ Eases practical compromise

To Paris: The "Optimal Compromise" is a Smart Cap

Cooling Down the Climate Debate: The Smart Cap!

National implementation: See Karp & Traeger (2015)

- Idea: Trade certificates whose quantity denomination is a function of the certificate price

↪ Efficient for any slope of MD curve

Practical implications for negotiations:

- A compromise between tax and cap advocates (and more efficient than either)
- Uses existing cap and trade markets/institutions
- Enables a compromise in negotiations
  - If abatement turns out cheaper: agree to do more
  - If abatement turns out expensive: agree to do less

↪ Eases practical compromise

Conclusions

GAUVAL

- DICE-style realism in closed form
- Decoding of optimal carbon tax contributions
- explains & quantifies uncertainty contributions
- Deterministic SCC impact: carbon cycle >> temperature
- Uncertainty impact on welfare:
  Climate sensitivity uncert >> carbon flow uncertainty
- "Choice" of discount rate remains major issue

Implications for COPs & Paris:

- Negotiate a SMART Cap
- Ease the compromise & be more efficient:
  Make quantity target a function of abatement price

Conclusions

GAUVAL

- DICE-style realism in closed form
- Decoding of optimal carbon tax contributions
- explains & quantifies uncertainty contributions
- Deterministic SCC impact: carbon cycle >> temperature
- Uncertainty impact on welfare:
  Climate sensitivity uncert >> carbon flow uncertainty
- "Choice" of discount rate remains major issue

Implications for COPs & Paris:

- Negotiate a SMART Cap
- Ease the compromise & be more efficient:
  Make quantity target a function of abatement price

Conclusions

GAUVAL

- DICE-style realism in closed form
- Decoding of optimal carbon tax contributions
- explains & quantifies uncertainty contributions
- Deterministic SCC impact: carbon cycle >> temperature
- Uncertainty impact on welfare:
  Climate sensitivity uncert >> carbon flow uncertainty
- "Choice" of discount rate remains major issue

Implications for COPs & Paris:

- Negotiate a SMART Cap
- Ease the compromise & be more efficient:

  Make quantity target a function of abatement price

Extensions

Extensions that the model can handle

- model ambiguity
- incorporate adaptation
- limited substitutability of environmental goods
- become regional
- model sea level rise, ocean acidification, geoengeneering
- endogenize non-CO_2 GHGs
- ...

What the model cannot do

- Certain non-linearities and interactions simply not allowed

Accompanying paper will analyzes uncertainty impact on tax

Economy

Structure of the Economy:

- log-utility (deterministic)
- Cobb-Douglas production, using the additional
- Energy composite: ***general function*** of **energy sources**, each produced with labor (control) and exog. technology
- Emissions endog. from dirty energy sectors, exog. LUCF
- Resources, assumption: if scarce then essential
- Decadal time step, because capital structure:
  10 years w/o depreciation, 20 years: full depreciation.

I will derive the class of analytically solvable

- Damage functions: fraction of global output loss as a function of atmospheric temperature increase

Economy

Structure of the Economy:

- log-utility (deterministic)
- Cobb-Douglas production, using the additional
- Energy composite: ***general function*** of **energy sources**, each produced with labor (control) and exog. technology
- Emissions endog. from dirty energy sectors, exog. LUCF
- Resources, assumption: if scarce then essential
- Decadal time step, because capital structure: 10 years w/o depreciation, 20 years: full depreciation.

I will derive the class of analytically solvable

- Damage functions: fraction of global output loss as a function of atmospheric temperature increase

Climate System

Climate system:

- Carbon cycle: I will use DICE 2013

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi} \boldsymbol{M}_t + \boldsymbol{e}_1 \left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exogenous} \right) \tag{1}$$

  first unit vector $\boldsymbol{e}_1$ send emissions to atmospheric layer

- Radiative forcing (direct greenhouse effect)

$$F_t = \eta \, \frac{\log \frac{M_{1,t} + G_t}{M_{pre}}}{\ln 2} \,. \tag{2}$$

  - Standard in numeric & new to analytic IAMs
  - G_t: exogenous non-CO_2 forcing

I will derive the class of analytically solvable models for the

- Atmosphere-ocean temperature dynamics

New to analytically tractable models.

Climate System

Climate system:

- Carbon cycle: I will use DICE 2013

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{e}_1\left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exogenous}\right) \tag{1}$$

  first unit vector $\boldsymbol{e}_1$ send emissions to atmospheric layer

- Radiative forcing (direct greenhouse effect)

$$F_t = \eta \, \frac{\log \frac{M_{1,t}+G_t}{M_{pre}}}{\ln 2} \; . \tag{2}$$

  - Standard in numeric & new to analytic IAMs
  - G_t: exogenous non-CO_2 forcing

I will derive the class of analytically solvable models for the

- Atmosphere-ocean temperature dynamics

New to analytically tractable models.

Damages & Temperature Dynamics: Functional Forms

- Golosov et al. & others solve because $\Leftrightarrow$
- Linear-in-state model, which are solved by affine value fct

Proposition 1:

An affine value function of the form

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t$$

solves GAUVAL if

1. $k_t = \log K_t$, $\boldsymbol{\tau}_t$ is vector of $\tau_i = \exp(\xi_i T_i)$, $i \in \{1, ..., L\}$
2. Damages: $D(T_{1,t}) = 1 - \exp[-\xi_0 \exp[\xi_1 T_{1,t}] + \xi_0]$, $\xi_0 \in \mathbb{R}$,

   Damage parameter ξ_0 is the semi-elasticity of net production to transformed atmospheric temperature $\tau_{1,t} = \exp(\xi_1 T_{1,t})$.

Damages & Temperature Dynamics: Functional Forms

- Golosov et al. & others solve because $\Leftrightarrow$
- Linear-in-state model, which are solved by affine value fct

Proposition 1:

An affine value function of the form

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t$$

solves GAUVAL if

1. $k_t = \log K_t$, $\boldsymbol{\tau}_t$ is vector of $\tau_i = \exp(\xi_i T_i)$, $i \in \{1, ..., L\}$
2. Damages: $D(T_{1,t}) = 1 - \exp[-\xi_0 \exp[\xi_1 T_{1,t}] + \xi_0]$, $\xi_0 \in \mathbb{R}$,

   Damage parameter ξ_0 is the semi-elasticity of net production to transformed atmospheric temperature $\tau_{1,t} = \exp(\xi_1 T_{1,t})$.

Damages & Temperature Dynamics: Functional Forms

- Golosov et al. & others solve because $\Leftrightarrow$
- Linear-in-state model, which are solved by affine value fct

Proposition 1:

An affine value function of the form

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t$$

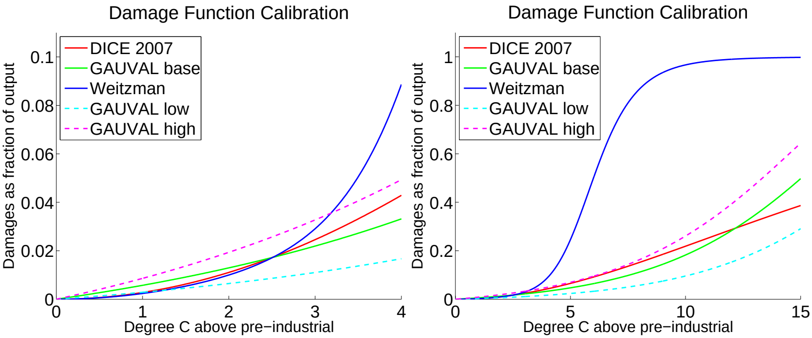
solves GAUVAL if

1. $k_t = \log K_t$, $\boldsymbol{\tau}_t$ is vector of $\tau_i = \exp(\xi_i T_i)$, $i \in \{1, ..., L\}$
2. Damages: $D(T_{1,t}) = 1 - \exp[-\xi_0 \exp[\xi_1 T_{1,t}] + \xi_0]$, $\xi_0 \in \mathbb{R}$,
3. Temperature: $T_{i,t+1} = \frac{1}{\xi_i} \log\Big((1-\sigma_{i,i+1}-\sigma_{i,i-1})\exp[\xi_i T_{i,t}]$ $+\sigma_{i,i+1}\exp[\xi_i w_i^{-1} T_{i-1,t}] + \sigma_{i,i-1}\exp[\xi_i w_{i+1} T_{i+1,t}]\Big)$, with weighting matrix $\boldsymbol{\sigma}$ capturing heat exchange
4. Parameters: $\xi_1 = \frac{\log 2}{s} \approx \frac{1}{4}$ and $\xi_{i+1} = w_i \xi_i = \frac{T_{eq}^{i-1}}{T_{eq}^{i}} \xi_i$.

Testing the Necessary Assumptions

Damage assumption & calibration: One free parameter ξ_0

- Match Nordhaus' DICE damage calibration points: $T = 0°\text{C}$ and $T = 2.5°\text{C} \Rightarrow$ green line ($\xi_0 \approx 0.022$)



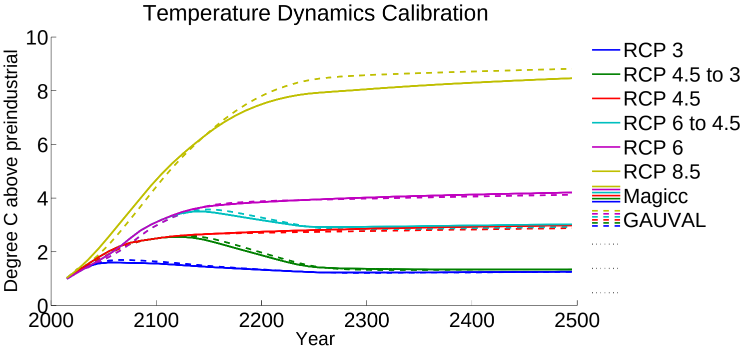
The dashed lines are $\xi_0 \pm 50\%$

Testing Necessary Assumptions

Atmosphere-Ocean Temperature dynamics calibration:

- Match Magicc6.0 for IPCC's RCP scenarios,

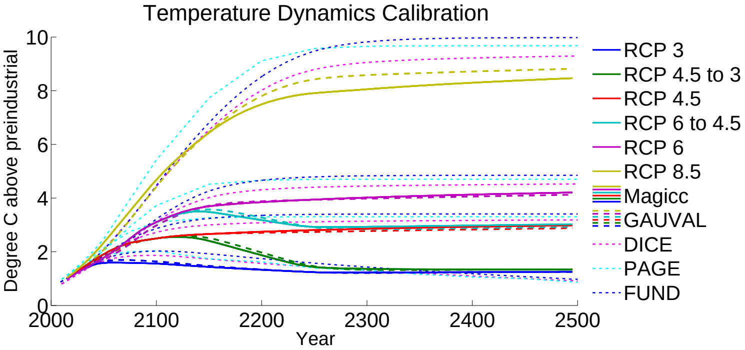
  Magicc6.0 emulates AOGCMS ("big models") used in Assessment Reports by the Intergovernmental Panel on Climate Change IPCC



Testing Necessary Assumptions

Atmosphere-Ocean Temperature dynamics calibration:

- Match Magicc6.0 for IPCC's RCP scenarios,
  Magicc6.0 emulates AOGCMS ("big models") used in Assessment Reports by the Intergovernmental Panel on Climate Change IPCC



The Social Cost of Carbon: Formula

The optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{\text{damages}} \underbrace{\left[(\mathbf{1} - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc}}_{\text{climate dynamics}} \underbrace{\left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1}}_{\text{carbon dynamics}}$$

- discount factor β
- production Y_t
- preindustrial carbon M_{pre}
- damage parameter ξ_0 (semi-elasticity of net production)

- temperature dynamics $\boldsymbol{\sigma}$ and, in particular:
- speed of atmospheric temperature response to forcing σ^{forc}

- carbon dynamics Φ (transition matrix)

  $\left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1}$ interpretation by Neumann series expansion:
  ∞ sum over β discounted emission persistence & return to atmosphere

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1} = 57 \frac{\$}{tC} ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO_2
- Increases with output ("policy ramp")
- damages $\xi_0 \rightarrow \pm 50\%$ implies tax $\pm 50\%$
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%} \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} = 57 \, \frac{\$}{tC} \, ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO_2
- Increases with output ("policy ramp")
- damages ξ_0 → ±50% implies tax ±50%
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \underbrace{\frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%}}_{26 \frac{\$}{tC}} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1}}_{1.4} \underbrace{\sigma^{forc}}_{0.42} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} = \text{“}15\text{”} \; \frac{\$}{tC} \; ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57$/ton carbon or 16 $/t$CO_2$
- Increases with output (“policy ramp”)
- damages $\xi_0 \rightarrow \pm 50\%$ implies tax $\pm 50\%$
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

↪ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

The Social Cost of Carbon: Quantitative

Quantifying the optimal carbon tax:

$$SCC_t = \underbrace{\frac{\beta Y_t}{M_{pre}} \underbrace{\xi_0}_{2.2\%}}_{26\,\frac{\$}{tC}} \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}}_{1.4} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}}_{3.7} = \mathbf{57}\ \frac{\$}{tC}\ ,$$

Quantitative: The optimal carbon tax in 2015 in USD

- is 57\$/ton carbon or 16 \$/tCO_2
- Increases with output ("policy ramp")
- damages $\xi_0 \to \pm 50\%$ implies tax $\pm 50\%$
- temperature response delay: cuts tax by 60%
- temperature persistence: increases tax by 40%

$\hookrightarrow$ Together: temperature dynamics cut tax by 30%

- Carbon persistence: Increases tax by factor 3.7

skip welfare

Discounting: A Sensitivity Result

Proposition 2:

A carbon cycle whose transition matrix $\boldsymbol{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \boldsymbol{\frac{1}{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Recall Ramsey equation "$r = \rho + \eta g$".

Countering wide-spread belief (e.g. Nordhaus 2007, JEL):

- SCC is (very) sensitive to composition of cons. disc. rate r:
- *not* sensitive to growth term, highly sensitive to p.r.t.p. ρ

Reduce pure time preference from $\rho = 1.75\%$ to $\rho = 0.1\%$

- Normative: Stern Review
- Descriptive: Long-run risk model
- Mix: Generational disentanglement

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}}_{\cancel{1.4}\ 2} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}}_{\cancel{3.7}\ 26} = \cancel{\mathbf{57}}\,\mathbf{660}\ \frac{\$}{tC}$$

Discounting: A Sensitivity Result

Proposition 2:

A carbon cycle whose transition matrix $\mathbf{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \boldsymbol{\frac{1}{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Recall Ramsey equation "$r = \rho + \eta g$".

Countering wide-spread belief (e.g. Nordhaus 2007, JEL):

- SCC is (very) sensitive to composition of cons. disc. rate r:
- *not* sensitive to growth term, highly sensitive to p.r.t.p. ρ

Reduce pure time preference from $\rho = 1.75\%$ to $\rho = 0.1\%$

- Normative: Stern Review
- Descriptive: Long-run risk model
- Mix: Generational disentanglement

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \underbrace{\left[(\mathbf{1}-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}}_{\cancel{1.4}\ 2} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1}-\beta\mathbf{\Phi})^{-1}\right]_{1,1}}_{\cancel{3.7}\ 26} = \cancel{\mathbf{57}}\,\mathbf{660}\ \frac{\$}{tC}$$

Discounting: A Sensitivity Result

Proposition 2:

A carbon cycle whose transition matrix $\mathbf{\Phi}$ satisfies **mass conservation** of carbon **implies a factor** $(1-\beta)^{-1} \approx \boldsymbol{\frac{1}{\rho}}$ in the closed form solution of the **optimal carbon tax**.

Recall Ramsey equation "$r = \rho + \eta g$".
Countering wide-spread belief (e.g. Nordhaus 2007, JEL):

- SCC is (very) sensitive to composition of cons. disc. rate r:
- *not* sensitive to growth term, highly sensitive to p.r.t.p. ρ

Reduce pure time preference from $\rho = 1.75\%$ to $\rho = 0.1\%$

- Normative: Stern Review
- Descriptive: Long-run risk model
- Mix: Generational disentanglement

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1}}_{\cancel{1.4}\ 2} \underbrace{\sigma^{forc}}_{0.42} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1}}_{\cancel{3.7}\ 26} = \cancel{57}\ \mathbf{660}\ \frac{\$}{tC}$$

Uncertainty

Evaluating Uncertainty

1. Logarithmic utility is
   - Reasonable estimate for intertemporal substitution
   - Miserable estimate for risk aversion
2. Expected utility model is
   - unable to match high observed risk premia together with
   - low observed risk-free discount rate

Solution:

- Epstein-Zin-Weil preferences
- I show that closed-form solution of non-linear Bellman for
  - IES=1 (logarithmic), deterministic tradeoffs
  - General CRRA risk attitude
- Observed Arrow-Pratt RRA$\in [6, 9.5]$ translates into intertemporal risk aversion coeff in formulas of $-\alpha \in [1, 1.5]$

details

Uncertainty

Evaluating Uncertainty

1. Logarithmic utility is
   - Reasonable estimate for intertemporal substitution
   - Miserable estimate for risk aversion
2. Expected utility model is
   - unable to match high observed risk premia together with
   - low observed risk-free discount rate

Solution:

- Epstein-Zin-Weil preferences
- I show that closed-form solution of non-linear Bellman for
  - IES=1 (logarithmic), deterministic tradeoffs
  - General CRRA risk attitude
- Observed Arrow-Pratt RRA$\in [6, 9.5]$ translates into intertemporal risk aversion coeff in formulas of $-\alpha \in [1, 1.5]$

details

Carbon Sink Uncertainty

Issue(s):

- About 10-20% of CO_2 released to atm "goes missing"
- How will carbon sinks respond to climate change?

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi M}_t + \boldsymbol{e}_1(\sum_{i=1}^{I^d} E_{i,t}) + \epsilon_t(1,-1,0,...,0)^\top$$

where ϵ_t characterizes uncertain carbon flow between atmosphere and upper-ocean-biosphere reservoir

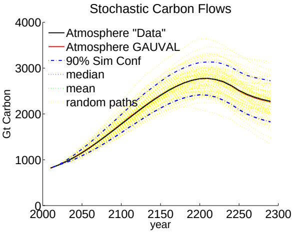
Model I: Unforeseen changes in sink uptake

iid. shocks χ_t moving VAR carbon flows: $\epsilon_{t+1} = \gamma\epsilon_t + \chi_t$

Calibration to scientific model-comparison study (Joos et al. 2013)

$\gamma = 0.997$, and guesstimate $\sigma_\chi \approx 20$Gt/decade

Illustration along DICE BAU



Carbon Sink Uncertainty

Issue(s):

- About 10-20% of CO_2 released to atm "goes missing"
- How will carbon sinks respond to climate change?

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi M}_t + \boldsymbol{e}_1(\textstyle\sum_{i=1}^{I^d} E_{i,t}) + \epsilon_t(1, -1, 0, ..., 0)^\top$$

where ϵ_t characterizes uncertain carbon flow between atmosphere and upper-ocean-biosphere reservoir

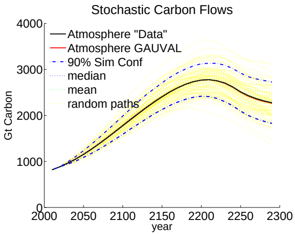
Model I: Unforeseen changes in sink uptake

iid. shocks χ_t moving VAR carbon flows: $\epsilon_{t+1} = \gamma\epsilon_t + \chi_t$

Calibration to scientific model-comparison study (Joos et al. 2013)

$\gamma = 0.997$, and guesstimate $\sigma_\chi \approx 20$Gt/decade

Illustration along DICE BAU



Carbon Sinks: VAR Shocks

back

Welfare loss in the vector auto-regressive shock model (VAR)

$$\Delta W^{VAR} = \frac{1}{\alpha}\frac{\beta}{1-\beta}\log\left[\mathbf{E}\exp\left[\alpha\varphi_\epsilon\chi\right]\right]$$
$$= \frac{1}{\alpha}\underbrace{\frac{\beta}{1-\beta}}_{\text{time}}\left[\sum_{i=1}^{\infty}\underbrace{\kappa_i}_{\text{cumulants}}\underbrace{\frac{(\alpha\varphi_\epsilon)^i}{i!}}_{\text{econ}}\right].$$

- "time": "sums" over discounted loss from all future shocks
- "cumulants": $\kappa_i \approx$ moments
  κ_1: mean $=0$ κ_2: variance κ_3: skewness
- "econ": powers of risk aversion α weighted shadow value of the carbon flow: $\varphi_\epsilon = \frac{\beta}{1-\gamma\beta}[\varphi_{M_1} - \varphi_{M_2}]$
  - Persistence γ & discount factor β weighted difference in
  - shadow value of M_1 in the atmosphere and
  - shadow value of M_2 in the shallow ocean & biosphere

Epistemological Uncertainty

Model II: Bayesian uncertainty & anticipated learning (normal)

Model III: Joint VAR-epistemological, non-Bayesian learning

- general distributions (needed for temperature uncertainty)
- tracking epistemological uncertainty by cumulant expansion

Analytic insights comparing the models

- VAR-shocks (Model I):
  - shocks build up slowly over time
- Learning implies:
  - anticipated updating similar to VAR shocks
  - Uncertainty is prior + stochasticity and falls over time
  - Initially learning acts like fully persistent shocks to mean
  - As decision maker learns, model converges to iid model (model with zero persistence, not sensitive to time pref.)

details

Epistemological Uncertainty

Model II: Bayesian uncertainty & anticipated learning (normal)

Model III: Joint VAR-epistemological, non-Bayesian learning

- general distributions (needed for temperature uncertainty)
- tracking epistemological uncertainty by cumulant expansion

Analytic insights comparing the models

- VAR-shocks (Model I):
  - shocks build up slowly over time
- Learning implies:
  - anticipated updating similar to VAR shocks
  - Uncertainty is prior + stochasticity and falls over time
  - Initially learning acts like fully persistent shocks to mean (very sensitive to time preference)
  - As decision maker learns, model converges to iid model (model with zero persistence, not sensitive to time pref.)

details

Carbon Sinks

Welfare loss along DICE 2013 BAU scenario

Carbon cycle uncertainty for $\rho = 1.75\%$, best guess $\rho = \cancel{1.75\%}\, 0.1\%$,

- VAR: 28 billion
- Bayes: 29 billion

$\approx$ 1.5-2 years of NASA budget

- VAR: 500 billion
- Bayes: 60 trillion

$\approx$ 73% world output

goto: willingness to pay

Temperature uncertainty :

Based on 20 science estimates of climate sensitivity (Meinshausen09)

Welfare loss for $\rho = 1.75\%$ (left) and $\rho = 0.1\%$ (right), lower bound

$\approx$ 20-25% of world output

$\approx 10 \times$ world output

half from present epistem. uncertainty

95% from "future shocks"

Carbon Sinks

Welfare loss along DICE 2013 BAU scenario

Carbon cycle uncertainty for $\rho = \cancel{1.75\%}\, 0.1\%$, best guess

- VAR: 28 billion
- Bayes: 29 billion

$\approx$ 1.5-2 years of NASA budget

- VAR: 500 billion
- Bayes: 60 trillion

$\approx 73\%$ world output

goto: willingness to pay

Temperature uncertainty :

Based on 20 science estimates of climate sensitivity (Meinshausen09)

Welfare loss for $\rho = 1.75\%$ (left) and $\rho = 0.1\%$ (right), lower bound

$\approx$ 20-25% of world output

half from present epistem. uncertainty

$\approx 10 \times$ world output

95% from "future shocks"

Carbon Sinks

Welfare loss along DICE 2013 BAU scenario

Carbon cycle uncertainty for $\rho = \cancel{1.75\%}\, 0.1\%$, best guess

- VAR: 28 billion
- Bayes: 29 billion

$\approx$ 1.5-2 years of NASA budget

- VAR: 500 billion
- Bayes: 60 trillion

$\approx 73\%$ world output

goto: willingness to pay

Temperature uncertainty :

Based on 20 science estimates of climate sensitivity (Meinshausen09)

Welfare loss for $\rho = 1.75\%$ (left) and $\rho = 0.1\%$ (right), lower bound

$\approx$ 20-25% of world output

half from present epistem. uncertainty

$\approx 10 \times$ world output

95% from "future shocks"

The Optimal Carbon Tax - It's quite Independent

Remark: The shadow value of carbon

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} = 56.5 \, \frac{\$}{tC} \; ,$$

is **independent of absolute temperature andb carbon levels!**

Implications:

- That is why SCC=optimal tax
- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live forever with consequences of over-emitting today

Intuition: "Saturation" of atmospheric CO_2 and damage convexity "approximately offset each other"

The Optimal Carbon Tax - It's quite Independent

Remark: The shadow value of carbon

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1} = 56.5 \frac{\$}{tC} \; ,$$

is **independent of absolute temperature andb carbon levels!**

Implications:

- That is why SCC=optimal tax
- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live forever with consequences of over-emitting today

Intuition: "Saturation" of atmospheric CO_2 and damage convexity "approximately offset each other"

The Optimal Carbon Tax - It's quite Independent

Remark: The shadow value of carbon

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1} = 56.5 \, \frac{\$}{tC} \, ,$$

is **independent of absolute temperature andb carbon levels!**

Implications:

- That is why SCC=optimal tax
- Optimal mitigation effort is independent of past emissions!

$\hookrightarrow$ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

$\hookrightarrow$ Live forever with consequences of over-emitting today

Intuition: "Saturation" of atmospheric CO_2 and damage convexity "approximately offset each other"

The Social Cost of Carbon: Formula

The optimal carbon tax:

$$SCC_t = \frac{\beta Y_t}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \sigma^{forc} \underbrace{\left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1}}_{\text{carbon dynamics}}$$

Interpretation of $(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}$:

- Neumann series: $(1 - \beta \boldsymbol{\Phi})^{-1} = \sum_{i=0}^{\infty} \beta^i \boldsymbol{\Phi}^i$. (3)
- E.g. second order contribution $[\beta^2 \Phi^2]_{1,1} = \beta^2 \sum_j \Phi_{1j} \Phi_{j1}$

  is carbon flow (or "heat" flow for $\boldsymbol{\sigma}$) that
  - starts out in layer 1 (atmosphere) and
  - is back in layer 1 after two periods
  - valued after two periods with β^2.

$\hookrightarrow$ **Discounted sum of future carbon in the atmosphere resulting from a ton released today**

Other Quantitative Results

Some net present value calculations:

- The cost of present atmospheric warming (and only that)

$$\Delta W^{Temp}_{USD\ 2015}(T \approx 0.77C) = Y\xi_0 \left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} (\exp(\xi_1 T) - 1)$$
$$\approx \$5 \text{ trillion}$$

- The cost of the present atmospheric CO_2 level

$$\Delta W^{CO_2}_{USD\ 2015}(M_1 \approx 397ppm) = SCC\ (M - M_{pre}) \approx \$14 \text{ trillion}$$

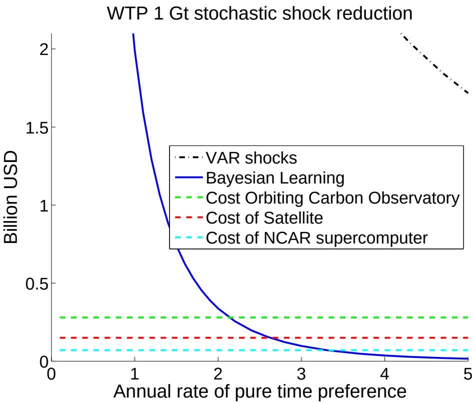
  These add up (+ Warming of and CO_2 in oceans)

- Similarly to atmospheric carbon tax can calculate value of carbon in deep and shallow ocean

↪ Benefit of sequestering carbon into shallow ocean $\approx 41\frac{\$}{tC}$
(Though: Should use better than DICE carbon cycle & ocean damages to quantify value of sequestering to ocean level or ecosystem)

Carbon Sinks: Results

Willingness to pay for a risk reduction back

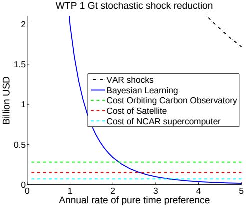


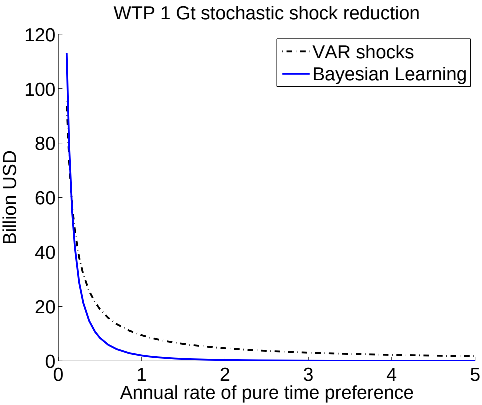
Bayes: Better measurement and faster learning

VAR: Less emissions $\rightarrow$ less risk

Carbon Sinks: Results

Willingness to pay for a risk reduction back





Initial sensitivity (updates as if full shock persistence) in Bayesian learning case dominates

Temperature Uncertainty: Tails

Assume:

- Normal distribution on $T_{1,t}$

Issue:

- Implies log-normal distribution on $\tau_{1,t} = \exp(\xi_1 T_{1,t})$

$\hookrightarrow$ moment generating function of log-normal for welfare loss

$\hookrightarrow$ Infinite welfare loss! "Weiztman-style" dismal result

Interpretation:

- Results very sensitive to temperature uncertainty
- No IAM is built to evaluate $T_{1,t} \to \infty$
- Expected utility model not build for it either
  (rational preferences require some form of boundedness)

Temperature Uncertainty: Tails

Assume:

- Normal distribution on $T_{1,t}$

Issue:

- Implies log-normal distribution on $\tau_{1,t} = \exp(\xi_1 T_{1,t})$

$\hookrightarrow$ moment generating function of log-normal for welfare loss

$\hookrightarrow$ Infinite welfare loss! “Weiztman-style” dismal result

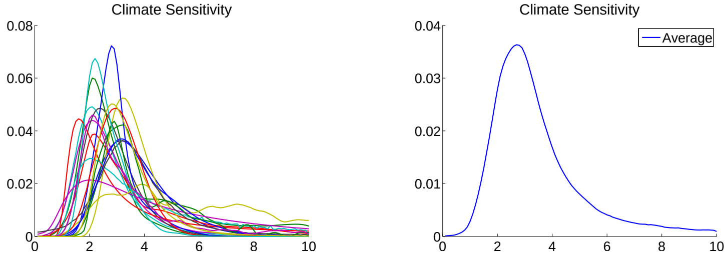
Interpretation:

- Results very sensitive to temperature uncertainty
- No IAM is built to evaluate $T_{1,t} \to \infty$
- Expected utility model not build for it either
  (rational preferences require some form of boundedness)

Temperature Uncertainty: Lower Bound

Approach:

- Model is reasonable for perhaps 10-15C warming
- Meinshausen et al. (2009) offer survey of probability distributions of temperature increase with doubling of CO_2



- I derive lower bound on welfare loss conditional on temperature increase with CO_2 doubling less than 10C

Temperature Uncertainty: Model & Result

Adjusted equation of motion temperature

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t} + G_t}{M_{pre}}\boldsymbol{e}_1 + \epsilon_t^{\tau}\boldsymbol{e}_1 \ .$$

- ϵ_t^{τ} captures epistemological uncert. & stochastic changes
- ϵ_t^{τ} characterized through its cumulants $\kappa_{i,t}, i \in \mathbb{N}$ with equations of motion

$$\kappa_{i,t+1} = \gamma^i \kappa_{i,t} + \chi_{i,t}^{\tau} \ ,$$

- γ captures persistence of
  - epistemological uncertainty
  - shocks to the distribution

Quantification: Lower bound present value welfare loss

$\gamma = 0.6$: 21 billion (26% world output)

$\gamma = 0.9$: 16 billion (20% world output)

$\rho = 0.1\%$ & $\gamma = 0.6$: 13 times world output

$\rho = 0.1\%$ & $\gamma = 0.9$: 9 times world output

Temperature Uncertainty: Model & Result

Adjusted equation of motion temperature

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t} + G_t}{M_{pre}}\boldsymbol{e}_1 + \epsilon_t^\tau \boldsymbol{e}_1 \ .$$

- ϵ_t^τ captures epistemological uncert. & stochastic changes
- ϵ_t^τ characterized through its cumulants $\kappa_{i,t}, i \in \mathbb{N}$ with equations of motion

$$\kappa_{i,t+1} = \gamma^i \kappa_{i,t} + \chi_{i,t}^\tau \ ,$$

- γ captures persistence of
  - epistemological uncertainty
  - shocks to the distribution

Quantification: Lower bound present value welfare loss

$\gamma = 0.6$: 21 billion (26% world output)

$\gamma = 0.9$: 16 billion (20% world output)

$\rho = 0.1\%$ & $\gamma = 0.6$: 13 times world output

$\rho = 0.1\%$ & $\gamma = 0.9$: 9 times world output

Temperature Uncertainty: Model & Result

Adjusted equation of motion temperature

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t} + G_t}{M_{pre}}\boldsymbol{e}_1 + \epsilon_t^{\tau}\boldsymbol{e}_1 \ .$$

- ϵ_t^{τ} captures epistemological uncert. & stochastic changes
- ϵ_t^{τ} characterized through its cumulants $\kappa_{i,t}, i \in \mathbb{N}$ with equations of motion

$$\kappa_{i,t+1} = \gamma^i \kappa_{i,t} + \chi_{i,t}^{\tau} \ ,$$

- γ captures persistence of
  - epistemological uncertainty
  - shocks to the distribution

Quantification: Lower bound present value welfare loss

$\gamma = 0.6$: 21 billion (26% world output)

$\gamma = 0.9$: 16 billion (20% world output)

$\rho = 0.1\%$ & $\gamma = 0.6$: 13 times world output

$\rho = 0.1\%$ & $\gamma = 0.9$: 9 times world output

Temperature Uncertainty: Model & Result

Adjusted equation of motion temperature

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1 + \epsilon_t^{\tau}\boldsymbol{e}_1 \ .$$

- ϵ_t^{τ} captures epistemological uncert. & stochastic changes
- ϵ_t^{τ} characterized through its cumulants $\kappa_{i,t}, i \in \mathbb{N}$ with equations of motion

$$\kappa_{i,t+1} = \gamma^i \kappa_{i,t} + \chi_{i,t}^{\tau} \ ,$$

- γ captures persistence of
  - epistemological uncertainty
  - shocks to the distribution

Quantification: Lower bound present value welfare loss

$\gamma = 0.6$: 21 billion (26% world output)

$\gamma = 0.9$: 16 billion (20% world output)

$\rho = 0.1\%$ & $\gamma = 0.6$: 13 times world output

$\rho = 0.1\%$ & $\gamma = 0.9$: 9 times world output

Integrated Assessment Models & Contribution I

Types of IAMs: Contribution

- **Highly stylized analytic** models (e.g. "prices vs quantities")

- **Golosov et al.** (2014, Econometrica): Analytic model
  Gerlagh & Liski (2012): Added lag in emission impacts
  Climate: Historic emissions affect production
  (Impulse response model)

- **This paper**: Analytic model
  Economy & Energy: general(ized) Golosov et al.
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System

- **Complex numeric** models
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System

Integrated Assessment Models & Contribution I

Types of IAMs: Contribution

- **Highly stylized analytic** models (e.g. "prices vs quantities")
- **Golosov et al.** (2014, Econometrica): Analytic model
  Gerlagh & Liski (2012): Added lag in emission impacts
  Climate: Historic emissions affect production
  (Impulse response model)
- **This paper**: Analytic model
  Economy & Energy: general(ized) Golosov et al.
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System
- **Complex numeric** models
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System

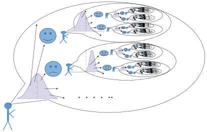
Integrated Assessment Models & Contribution I

Types of IAMs: Contribution

- **Highly stylized analytic** models (e.g. "prices vs quantities")
- **Golosov et al.** (2014, Econometrica): Analytic model
  Gerlagh & Liski (2012): Added lag in emission impacts
  Climate: Historic emissions affect production
  (Impulse response model)
- **This paper**: Analytic model
  Economy & Energy: general(ized) Golosov et al.
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System
- **Complex numeric** models
  Climate: Carbon Cycle, Radiative Forcing,
  Temperature of Atmosphere-Ocean System

Uncertainty in Integrated Assessment & Contribution II

Issue with full uncertainty integration in numeric IAMs



rational decision making & climate states $\rightarrow$ numeric curse

Modeling contribution: Uncertainty

- "Computationally" tractable many states model
- with non-logarithmic risk attitude
- that separates risk premia from risk-free discount rate
- Closed-form solution for welfare loss from uncertainty

Temperature Dynamics

If forcing F_{eq} constant, atmospheric temperature increase

$$T_{1,t} \to T_{1,eq} = \frac{s}{\eta} F_{eq} \tag{4}$$

But: Takes decades to centuries & usually F_t not constant.

↪ Need a model of Temperature Dynamics

☹ Standard models defy analytic traction

My approach:

- Formalize general properties of dynamics:
  - Track temperature of atmosphere & several ocean layers
  - Next period temperature is general mean of temperatures in adjacent layers
  - Correct for asymmetry in atmosphere vs ocean warming
- Derive embedded class of tractable models (Proposition 1)
- Calibrate to see whether any good

Temperature Dynamics

If forcing F_{eq} constant, atmospheric temperature increase

$$T_{1,t} \to T_{1,eq} = \frac{s}{\eta} F_{eq} \tag{4}$$

But: Takes decades to centuries & usually F_t not constant.

$\hookrightarrow$ Need a model of Temperature Dynamics
☹ Standard models defy analytic traction

My approach:

- Formalize general properties of dynamics:
  - Track temperature of atmosphere & several ocean layers
  - Next period temperature is general mean of temperatures in adjacent layers
  - Correct for asymmetry in atmosphere vs ocean warming
- Derive embedded class of tractable models (Proposition 1)
- Calibrate to see whether any good

Temperature Dynamics

If forcing F_{eq} constant, atmospheric temperature increase

$$T_{1,t} \to T_{1,eq} = \frac{s}{\eta} F_{eq} \tag{4}$$

But: Takes decades to centuries & usually F_t not constant.

$\hookrightarrow$ Need a model of Temperature Dynamics
☹ Standard models defy analytic traction

My approach:

- Formalize general properties of dynamics:
  - Track temperature of atmosphere & several ocean layers
  - Next period temperature is general mean of temperatures in adjacent layers
  - Correct for asymmetry in atmosphere vs ocean warming
- Derive embedded class of tractable models (Proposition 1)
- Calibrate to see whether any good

How to solve the model

Solve:

- Reformulate equations of motions in terms of k_t and $\tau_{i,t}$
- Reformulate Bellman equation using consumption *rate* x_t

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \max_{x_t, \boldsymbol{N_t}} \log x_t + a_t + \kappa k_t + (1-\kappa-\nu) \log N_{0,t}$$
$$+ \nu \log E_t - \xi_0 \tau_t + \xi_0 + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, t+1)$$

- Use affine trial solution for value function
- Solve r.h.s. max for labor inputs and consumption rate

$\hookrightarrow$ controls are functions of unknown shadow values (labor input also function of energy sector specification)

- Match coefficients in Bellman equation

$\hookrightarrow$ delivers shadow values

Standard Part of the Calibration

Calibrate Economy:

- Standard (and DICE) capital share of 0.3
- Annual rate of pure time preference of 1.75% calibrated to match IMF's 2015 consumption rate forecast of 75%
- Output is IMF's 2015 forecast of 81.5 trillion USD

Carbon Cycle:

- Take DICE 2013 carbon cycle
- 10 year (instead of 5 year) time step
- Rescaling of transition coefficients
  $\rightarrow$ perfect match of DICE's carbon dynamics

Policy Impact of Uncertainty: General Remarks

Uncertainty affects

- *welfare* through the curvature of the value function
  VAR setting evaluates general scenarios
- *choice variables* by shifting their marginal value
  Additive separable uncertainty no effect at all

$\hookrightarrow$ Cannot use linear-in-state-model for policy analysis

Need to model how uncertainty

- scales with the states

Introduce such a non-linear in state model where

- shocks scale in square root of states
- quadratic equations for shadow values
- generally solves in closed form

Policy Impact of Uncertainty: General Remarks

Uncertainty affects

- *welfare* through the curvature of the value function
  VAR setting evaluates general scenarios
- *choice variables* by shifting their marginal value
  Additive separable uncertainty no effect at all

↪ Cannot use linear-in-state-model for policy analysis

Need to model how uncertainty

- scales with the states

Introduce such a non-linear in state model where

- shocks scale in square root of states
- quadratic equations for shadow values
- generally solves in closed form

Policy Uncertainty: Carbon Sinks

Model of carbon cycle uncertainty:

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi} \boldsymbol{M}_t + \boldsymbol{e}_1 (\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}) + \epsilon_t (1, -1, 0, ..., 0)^\top$$

now with

$$\epsilon_{t+1} = \gamma \epsilon_t + \sqrt{M_{1,t}}\, \chi_t \text{ with } \chi_t \sim N(0, \sigma^2)$$

Result: $tax^{unc} = tax^{det} \left(1 + \theta + 2\theta^2 + 5\theta^3 + O(\theta^4)\right)$

with θ proportional to

- deterministic tax
- Variance of shock χ_t
- $\frac{1}{1-\beta\gamma}$ (shock persistence)
- risk attitude α
- $\left(\left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} - \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{2,2}\right)^2$

Policy Uncertainty: Carbon Sinks

Model of carbon cycle uncertainty:

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi M}_t + \boldsymbol{e}_1\left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\right) + \epsilon_t(1, -1, 0, ..., 0)^\top$$

now with

$\epsilon_{t+1} = \gamma\epsilon_t + \sqrt{M_{1,t}}\,\chi_t$ with $\chi_t \sim N(0, \sigma^2)$

Result: $tax^{unc} = tax^{det}\left(1 + \theta + 2\theta^2 + 5\theta^3 + O(\theta^4)\right)$

with θ proportional to

- deterministic tax
- Variance of shock χ_t
- $\frac{1}{1-\beta\gamma}$ (shock persistence)
- risk attitude α
- $\left(\left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1} - \left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{2,2}\right)^2$

Quantitative Policy Impact

Quantification of carbon uncertainty:

- Negligible impact on tax (+1-3%) more

Quantification of damage uncertainty: more

- Stochstic nature of damages: Very small (percentage order)
- Epistemological uncertainty: Big (similar order of deterministic contribution)

Qualification:

- Square-root-scaling of shock impact stacks cards against uncertainty impact for both carbon and damage uncertainty

Quantitative Policy Impact

Quantification of carbon uncertainty:

- Negligible impact on tax (+1-3%) more

Quantification of damage uncertainty: more

- Stochstic nature of damages: Very small (percentage order)
- Epistemological uncertainty: Big (similar order of deterministic contribution)

Qualification:

- Square-root-scaling of shock impact stacks cards against uncertainty impact for both carbon and damage uncertainty

Policy Impact of Uncertainty: Damages

Damage uncertainty

- Make log-technology level endogenous state $a_t = \log A_t$

$$a_{t+1} = a_t + g_t - \theta\tau_{1,t} + \sqrt{\tau_{1,t} - 1}\,\chi_t \ .$$

Uncertain shock scales with (exponential) temperature state

Policy Impact of Uncertainty: Damages

Find $\varphi_a = \frac{1+\beta\varphi_k}{1-\beta}$ and

$$\boldsymbol{\varphi}_\tau = -\left[\xi_0(1+\beta\varphi_k) + \beta\theta\varphi_a - \alpha\beta^2\varphi_a^2\frac{\sigma_z^2}{2}\right]\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\ .$$

Results:

- Relocate damages from Y_t to A_t: set $\theta = \xi_0$ and then $\xi_0 = 0$ (as is the case for the FUND model)

↪ cost difference: $\varphi_a = \frac{1+\beta\varphi_k}{1-\beta}$ versus $1+\beta\varphi_k$:

↪ perfect level persistence increases SCC by factor $\frac{\beta}{1-\beta} \approx 5$.

- Magnitude uncertainty contribution over deterministic contribution to SCC: $\frac{(-\alpha)\beta^2(\varphi_a+\varphi_z)^2\frac{\sigma_z^2}{2}}{\xi_0(1+\beta\varphi_k)}$

↪ For "low scenario": 8%

↪ For "high scenario": 200%

back

Policy Impact of Uncertainty: Damages

Find $\varphi_a = \frac{1+\beta\varphi_k}{1-\beta}$ and

$$\boldsymbol{\varphi}_\tau = -\left[\xi_0(1+\beta\varphi_k) + \beta\theta\varphi_a - \alpha\beta^2\varphi_a^2\frac{\sigma_z^2}{2}\right] \boldsymbol{e}_1^\top (1-\beta\boldsymbol{\sigma})^{-1} .$$

Results:

- Relocate damages from Y_t to A_t: set $\theta = \xi_0$ and then $\xi_0 = 0$ (as is the case for the FUND model)

↪ cost difference: $\varphi_a = \frac{1+\beta\varphi_k}{1-\beta}$ versus $1+\beta\varphi_k$:

↪ perfect level persistence increases SCC by factor $\frac{\beta}{1-\beta} \approx 5$.

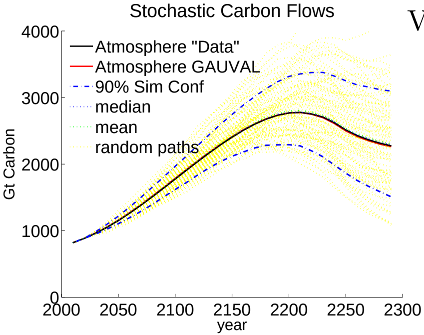
- Magnitude uncertainty contribution over deterministic contribution to SCC: $\frac{(-\alpha)\beta^2(\varphi_a+\varphi_z)^2\frac{\sigma_z^2}{2}}{\xi_0(1+\beta\varphi_k)}$

↪ For "low scenario": 8%

↪ For "high scenario": 200% back

Policy Impact of Uncertainty: Carbon Cycle

Uncertain carbon flow from before now scales with M_t:
$\epsilon_{t+1} = \gamma_M \epsilon_t + \sqrt{M_{1,t}}\ \chi_t$



Value impact proportional to

- SCC difference atmosphere-ocean
- σ of shock
- persistence
- all in higher & coinciding orders

Policy impact:

- Beautiful formula
- Quantitatively irrelevant

back

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
$(\smiley, \frownie, \smiley, \frownie) \sim (\frownie, \smiley, \frownie, \smiley)$

What is your preference in the following choice:

$$(\smiley,\frownie,\smiley,\frownie) \qquad ? \qquad \begin{cases} \frac{1}{2} & (\smiley,\smiley,\smiley,\smiley) \\ \frac{1}{2} & (\frownie,\frownie,\frownie,\frownie). \end{cases}$$

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
(☺, ☹, ☺, ☹) $\sim$ (☹, ☺, ☹, ☺)

What is your preference in the following choice:

(☺,☹,☺,☹) $\prec$ $\frac{1}{2}$ (☺,☺,☺,☺)

$\frac{1}{2}$ (☹,☹,☹,☹).

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
(☺, ☹, ☺, ☹) $\sim$ (☹, ☺, ☹, ☺)

What is your preference in the following choice:

(☺,☹,☺,☹) $\sim$ $\frac{1}{2}$ (☺,☺,☺,☺) ; $\frac{1}{2}$ (☹,☹,☹,☹).

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
(☺, ☹, ☺, ☹) $\sim$ (☹, ☺, ☹, ☺)

What is your preference in the following choice:

$$(☺,☹,☺,☹) \quad \succ \quad \begin{cases} \frac{1}{2} & (☺,☺,☺,☺) \\ \frac{1}{2} & (☹,☹,☹,☹). \end{cases}$$

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
(☺,☹,☺,☹) ∼ (☹,☺,☹,☺)

What is your preference in the following choice:

(☺,☹,☺,☹) ∼
$\frac{1}{2}$ (☺,☺,☺,☺)
$\frac{1}{2}$ (☹,☹,☹,☹).

The only preference that can be represented by the standard discounted expected utility model (intertemporal risk neutral)

Intertemporal risk neutrality

Why does uncertainty have virtually no impact?

Assume you are indifferent in following choice over 4 periods:
(☺,☹,☺,☹) $\sim$ (☹,☺,☹,☺)

What is your preference in the following choice:

$\frac{1}{2}$ (☺,☺,☺,☺)

(☺,☹,☺,☹) $\succ$

$\frac{1}{2}$ (☹,☹,☹,☹).

Intertemporal risk averse
(found in asset pricing observations)

Solving the Model

The equivalent "linear-in-state" system

- Replace control consumption by *consumption rate*
- Define

$$x_t = \frac{C_t}{Y_t[1 - D_t(T_t)]} \tag{5}$$

  - $k_t \equiv \log K_t$
  - $\tau_{i,t} \equiv \exp(\xi_i T_{i,t})$ (vector $\boldsymbol{\tau}_t \in \mathbb{R}^O$)
    Then: $\exists$ a linear transition matrix $\boldsymbol{\sigma}$ for τ-states

- Then Bellman equation

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \max_{x_t, \boldsymbol{N_t}} \log x_t + \log Y_t + \log[1 - D_t(T_t)] \\ + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, t+1) \; .$$

To Results

Solving the Model

subject to the linear equations of motion

$$k_{t+1} = a_t + \kappa k_t + (1-\kappa-\nu)\log N_{0,t} + \nu \log E_t \tag{6}$$

$$-\xi_0 \tau_{1,t} + \xi_0 + \log(1-x_t) \tag{7}$$

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi M}_t + \boldsymbol{e}_1\left(\textstyle\sum_{i=1}^{I^d} E_{i,t}\right) + \boldsymbol{e}_1(E_{1,t} + E_{2,t}) \tag{8}$$

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma\tau}_t + \sigma \boldsymbol{e}_1 \frac{M_{1,t} + G_t}{M_{pre}} \tag{9}$$

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d \tag{10}$$

and the constraints

$$E_t = g(\boldsymbol{E}_t(\boldsymbol{A}_t, \boldsymbol{N}_t))$$

$$\textstyle\sum_{i=0}^{I} N_{i,t} = N_t$$

$$\boldsymbol{R}_t \geq 0 \quad \text{and } \boldsymbol{R}_0 \text{ given.}$$

To Results

Solving the Model

Solution "algorithm"

- Trial solution

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t} \boldsymbol{R}_t + \varphi_t^*$$

- Solve r.h.s. FOCs
- Solve and verify solution of (maximized) Bellman by coefficient matching
- Solve for initial resource price using boundary condition

Summary:

We found a system that is

- Linear in the (transformed) states
- Separable in controls and states
- It is solved by an affine value function

Solving the Model

Solution "algorithm"

- Trial solution

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t} \boldsymbol{R}_t + \varphi_t^*$$

- Solve r.h.s. FOCs
- Solve and verify solution of (maximized) Bellman by coefficient matching
- Solve for initial resource price using boundary condition

Summary:

We found a system that is

- Linear in the (transformed) states
- Separable in controls and states
- It is solved by an affine value function

Results

Shadow values

$$\varphi_k = \frac{\kappa}{\mathbf{1} - \beta\kappa} \tag{11}$$

$$\boldsymbol{\varphi}_\tau = -\xi_1 (1 + \beta\varphi_k) \boldsymbol{e}_1^\top (1 - \beta\boldsymbol{\sigma})^{-1} \tag{12}$$

$$\boldsymbol{\varphi}_M = \frac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}} \boldsymbol{e}_1^\top (\mathbf{1} - \beta\boldsymbol{\Phi})^{-1} \tag{13}$$

$$\varphi_{R,t} = \beta^t \varphi_{R,0} \ , \tag{14}$$

where σ^{forc} is weight of atm. temp. on radiative forcing, and $\varphi_{R,t}$ follows Hotelling (boundary cond$\rightarrow \varphi_{R,0}$), and $\boldsymbol{e}_1^\top \boldsymbol{X}$ returns first row of the corresponding matrix $\boldsymbol{X}$.

From shadow values φ in utils to consumption (IMF 2015)

$$dC = 10\, x\, Y_{2015}\, du \approx 610\, du \text{ in trillion 2015 USD.}$$

The Social Cost of Carbon

Shadow value of carbon:

$$\varphi_{M,1} = -\xi_0 (1 + \beta \varphi_k) \left[(1 - \beta \boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta \sigma^{forc}}{M_{pre}} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}\right]_{1,1} .$$

Interpretation of $(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1}$:

Neumann series: $(1 - \beta \boldsymbol{\Phi})^{-1} = \sum_{i=0}^{\infty} \beta^i \Phi^i$. (15)

E.g. second order contribution

$(\beta^2 \Phi^2)_{11} = \beta^2 \sum_j \Phi_{1j} \Phi_{j1}$

is carbon flow (or "heat" flow for $\boldsymbol{\sigma}$) that

- starts out in layer 1 (atmosphere) and
- is back in layer 1 after two periods
- valued after two periods with β^2.

The Social Cost of Carbon

Shadow value of carbon:

$$\varphi_{M,1} = -\xi_0 (1 + \beta\varphi_k) \left[(1 - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta\sigma^{forc}}{M_{pre}} \left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1} .$$

Interpretation of $(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}$:

Neumann series: $(1 - \beta\boldsymbol{\Phi})^{-1} = \sum_{i=0}^{\infty} \beta^i \Phi^i$. (15)

E.g. second order contribution

$(\beta^2\Phi^2)_{11} = \beta^2 \sum_j \Phi_{1j}\Phi_{j1}$

is carbon flow (or “heat” flow for $\boldsymbol{\sigma}$) that

- starts out in layer 1 (atmosphere) and
- is back in layer 1 after two periods
- valued after two periods with β^2.

The Optimal Carbon Tax

Shadow value of carbon:

$$\varphi_{M,1} = -\xi_0(1+\beta\varphi_k)\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\frac{\beta\sigma^{forc}}{M_{pre}}\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1} .$$

Quantitative: The optimal carbon tax is

- 56.5\$/ton carbon or 15.5 \$/tCO_2
- damage parameter variation from Fig 1: $\pm 50\%$

Compare to DICE 2013:

- 2020 SCC: 21\$/tCO2
- at IMF's predicted growth rate of 4%:

$\hookrightarrow$ GAUVAL's 2020 SCC: $15.5 * 1.04^5 \approx 19\$/tCO_2$

The Optimal Carbon Tax

Shadow value of carbon:

$$\varphi_{M,1} = -\xi_0(1 + \beta\varphi_k)\left[(1 - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta\sigma^{forc}}{M_{pre}} \left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1} .$$

Quantitative: The optimal carbon tax is

- 56.5\$/ton carbon or 15.5 \$/tCO_2
- damage parameter variation from Fig 1: ±50%

Compare to DICE 2013:

- 2020 SCC: 21\$/tCO2
- at IMF's predicted growth rate of 4%:

$\hookrightarrow$ GAUVAL's 2020 SCC: $15.5 * 1.04^5 \approx 19$\$/tCO_2

The Optimal Carbon Tax

The shadow value of carbon

$$\varphi_{M,1} = -\xi_0(1+\beta\varphi_k)\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\frac{\beta\sigma^{forc}}{M_{pre}}\left[(\mathbf{1}-\beta\mathbf{\Phi})^{-1}\right]_{1,1}.$$

is **independent of absolute temperature and carbon levels!**

Implications:

- The SCC along the optimal path is the optimal carbon tax

↪ *The* SCC is *the* optimal tax (there is only one)

- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live with consequences of over-emitting today forever

The Optimal Carbon Tax

The shadow value of carbon

$$\varphi_{M,1} = -\xi_0(1+\beta\varphi_k)\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\frac{\beta\sigma^{forc}}{M_{pre}}\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}.$$

is **independent of absolute temperature and carbon levels!**

Implications:

- The SCC along the optimal path is the optimal carbon tax

↪ *The* SCC is *the* optimal tax (there is only one)

- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live with consequences of over-emitting today forever

The Optimal Carbon Tax

The shadow value of carbon

$$\varphi_{M,1} = -\xi_0(1+\beta\varphi_k)\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\frac{\beta\sigma^{forc}}{M_{pre}}\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}.$$

is **independent of absolute temperature and carbon levels!**

Implications:

- The SCC along the optimal path is the optimal carbon tax

↪ *The* SCC is *the* optimal tax (there is only one)

- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live with consequences of over-emitting today forever

The Optimal Carbon Tax

The shadow value of carbon

$$\varphi_{M,1} = -\xi_0(1+\beta\varphi_k)\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\frac{\beta\sigma^{forc}}{M_{pre}}\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}.$$

is **independent of absolute temperature and carbon levels!**

Implications:

- The SCC along the optimal path is the optimal carbon tax

↪ *The* SCC is *the* optimal tax (there is only one)

- Optimal mitigation effort is independent of past emissions!

↪ If we over-emit today (BAU),
future optimal policy does *not* over-compensate

↪ Live with consequences of over-emitting today forever

Uncertainty: The Carbon Sinks

General solution of persistent shock case: Welfare loss

$$\Delta W = \frac{1}{\alpha} \sum_{i=t}^{\infty} \beta^{i-t} \log \left[\mathbf{E} \exp \left[\alpha\beta\varphi_\epsilon \chi_i \right] \right]$$

$$= \frac{1}{\alpha} \frac{1}{1-\beta} \left[\kappa_1 (\alpha\beta\varphi_\epsilon) + \kappa_2 \frac{(\alpha\beta\varphi_\epsilon)^2}{2!} + \kappa_3 \frac{(\alpha\beta\varphi_\epsilon)^3}{3!} + ... \right].$$

- Discounted sum of log of moment generating function of χ_t-shocks
- cumulant weighted order of shadow value φ_ϵ times risk aversion α
    - κ_1: mean
    - κ_2: variance
    - κ_3: skewness

where shadow price $\varphi_\epsilon = \frac{\beta}{1-\gamma\beta} [\varphi_{M_1} - \varphi_{M_2}]$,
persistence weighted cost of carbon switching reservoirs

Learning: The Carbon Sinks

Model II: Bayesian uncertainty & anticipated learning

Prior

$$\epsilon_t \sim N(\mu_t, \sigma^2_{\epsilon,t}) \ , \ \mu_{\epsilon,0} = 0.$$

and stochasticity

$$\nu_t \sim N(0, \sigma^2_{\nu,t})$$

which restricts learning

Here,

- The carbon cycle follows a given though unknown (stochastic) motion
- But we don't know it (slowly learn it)

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

$$\Delta W = \alpha\beta \frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 \left(\varphi_{M_1} - \varphi_{M_2}\right)^2 \frac{\sigma_\chi}{2} \ .$$

2) The Bayesian Learning Model

$$\Delta W = \textstyle\sum_{i=t}^{\infty} \beta^{i-t+2} \frac{\sigma^2_{\epsilon,i}+\sigma^2_{\nu,i}}{2} \alpha \left(\varphi_{M_1} - \varphi_{M_2}\right)^2 \left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma^2_{\epsilon,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{1 \,*\, \text{weight}} + \underbrace{(1-\beta)\frac{\sigma^2_{\nu,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{(1-\beta)\,*\,\text{(1-weight)}} \Bigg)^2 .$$

- Initially weight$\approx$ 1 and acts as perfectly persistent model
- While learning weight$\rightarrow$ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercompu

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

$$\Delta W = \alpha\beta\frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 (\varphi_{M_1}-\varphi_{M_2})^2\frac{\sigma_\chi}{2} \, .$$

2) The Bayesian Learning Model

$$\Delta W = \sum_{i=t}^{\infty}\beta^{i-t+2}\frac{\sigma^2_{\epsilon,i}+\sigma^2_{\nu,i}}{2}\alpha\left(\varphi_{M_1}-\varphi_{M_2}\right)^2\left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma^2_{\epsilon,t}}{\sigma^2_{\nu,t+1}+\sigma^2_{\epsilon,t}}}_{1\,*\,\text{weight}}+\underbrace{(1-\beta)\frac{\sigma^2_{\nu,t}}{\sigma^2_{\nu,t+1}+\sigma^2_{\epsilon,t}}}_{(1-\beta)\,*\,\text{(1-weight)}}\Bigg)^2.$$

- Initially weight$\approx$ 1 and acts as perfectly persistent model
- While learning weight$\to$ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercompu

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

$$\Delta W = \alpha\beta\frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 (\varphi_{M_1}-\varphi_{M_2})^2\frac{\sigma_\chi}{2} \quad .$$

2) The Bayesian Learning Model

$$\Delta W = \sum_{i=t}^{\infty}\beta^{i-t+2}\frac{\sigma^2_{\epsilon,i}+\sigma^2_{\nu,i}}{2}\alpha\left(\varphi_{M_1}-\varphi_{M_2}\right)^2\left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma^2_{\epsilon,t}}{\sigma^2_{\nu,t+1}+\sigma^2_{\epsilon,t}}}_{1\,*\,\text{weight}} + \underbrace{(1-\beta)\frac{\sigma^2_{\nu,t}}{\sigma^2_{\nu,t+1}+\sigma^2_{\epsilon,t}}}_{(1-\beta)\,*\,\text{(1-weight)}}\Bigg)^2 .$$

- Initially weight$\approx$ 1 and acts as perfectly persistent model
- While learning weight$\rightarrow$ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercompu

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

return

$$\Delta W = \alpha\beta \frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 (\varphi_{M_1} - \varphi_{M_2})^2 \frac{\sigma_\chi}{2} \; .$$

2) The Bayesian Learning Model

$$\Delta W = \sum_{i=t}^{\infty} \beta^{i-t+2} \frac{\sigma^2_{\epsilon,i} + \sigma^2_{\nu,i}}{2} \alpha \left(\varphi_{M_1} - \varphi_{M_2}\right)^2 \left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma^2_{\epsilon,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{1\,*\,\text{weight}} + \underbrace{(1-\beta)\frac{\sigma^2_{\nu,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{(1-\beta)\,*\,\text{(1-weight)}}\Bigg)^2 .$$

- Initially weight$\approx$ 1 and acts as perfectly persistent model
- While learning weight$\rightarrow$ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercomputi

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

return

$$\Delta W = \alpha\beta \frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 (\varphi_{M_1} - \varphi_{M_2})^2 \frac{\sigma_\chi}{2} \quad .$$

2) The Bayesian Learning Model

$$\Delta W = \sum_{i=t}^{\infty} \beta^{i-t+2} \frac{\sigma_{\epsilon,i}^2 + \sigma_{\nu,i}^2}{2} \alpha \left(\varphi_{M_1} - \varphi_{M_2}\right)^2 \left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma_{\epsilon,t}^2}{\sigma_{\nu,t+1}^2 + \sigma_{\epsilon,t}^2}}_{1\,*\,\text{weight}} + \underbrace{(1-\beta)\frac{\sigma_{\nu,t}^2}{\sigma_{\nu,t+1}^2 + \sigma_{\epsilon,t}^2}}_{(1-\beta)\,*\,\text{(1-weight)}}\Bigg)^2 .$$

- Initially weight≈ 1 and acts as perfectly persistent model
- While learning weight→ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercompu

Uncertainty vs Learning: The Carbon Sinks

Welfare loss for normally distributed, stationary models:

1) VAR(1) Uncertainty model:

return

$$\Delta W = \alpha\beta \frac{\beta}{1-\beta}\left(\frac{\beta}{1-\gamma\beta}\right)^2 (\varphi_{M_1} - \varphi_{M_2})^2 \frac{\sigma_\chi}{2} \ .$$

2) The Bayesian Learning Model

$$\Delta W = \sum_{i=t}^{\infty} \beta^{i-t+2} \frac{\sigma^2_{\epsilon,i} + \sigma^2_{\nu,i}}{2} \alpha \left(\varphi_{M_1} - \varphi_{M_2}\right)^2 \left(\frac{\beta}{1-\beta}\right)^2$$

$$\Bigg(\underbrace{\frac{\sigma^2_{\epsilon,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{1 \, * \, \text{weight}} + \underbrace{(1-\beta)\frac{\sigma^2_{\nu,t}}{\sigma^2_{\nu,t+1} + \sigma^2_{\epsilon,t}}}_{(1-\beta) \, * \, \text{(1-weight)}} \Bigg)^2 .$$

- Initially weight$\approx$ 1 and acts as perfectly persistent model
- While learning weight$\to$ 0 and acts as iid model

Magnitude: Trill. USD. Enough to pay NASA's budget & supercompu

Uncertainty: Risk Attitude

Recursive preferences change Bellman equation to return

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) =$$
$$\max_{x_t, \boldsymbol{N}} \frac{1}{\alpha} \log \Big(\mathbf{E}_t \exp \big[\alpha \big(\log c_t + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, t) \big) \big] \Big) .$$

where

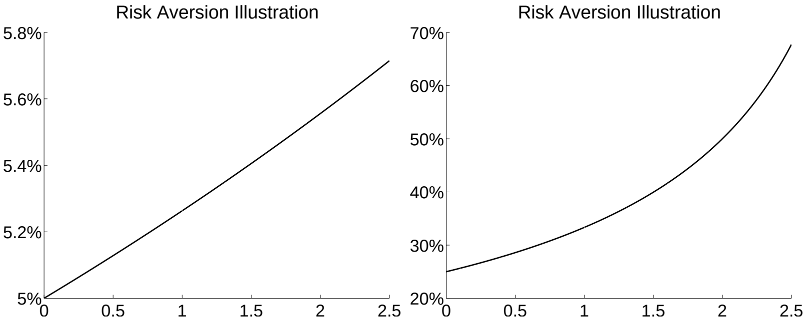
- Non-linear uncertainty aggregator, a generalized mean
  $f^{-1}\mathbf{E}_t f$ with $f(\cdot) = \exp[\alpha \cdot]$
  replaces usual linear uncertainty aggregation $\mathbf{E}_t$
- RRA$=1-\alpha^* = 1 - \frac{\alpha}{(1-\beta)}$: Epstein-Zin's coefficient of relative risk aversion
  Long-run risk literature: RRA$\in [6, 9.5] \to \alpha \in [-1.5, -1]$
- Expected value operator at beginning of current period allows absolute consumption to be uncertain (x_t fix)

Calibrating Risk Aversion

What is your risk aversion $RRA = 1 - \alpha$?

- .5 probability: consumption loss of 5% (left) or 25% (right)
- .5 probability: consumption gain of X% (y-axis)

that leaves you indifferent to original position



It's consumption (!) loss or gain during one decadal period