Particules Élémentaires, Gravitation et Cosmologie
Année 2009-'10

Théorie des Cordes: une Introduction

Cours X: 12 mars 2010

Boucles en QFT et QST

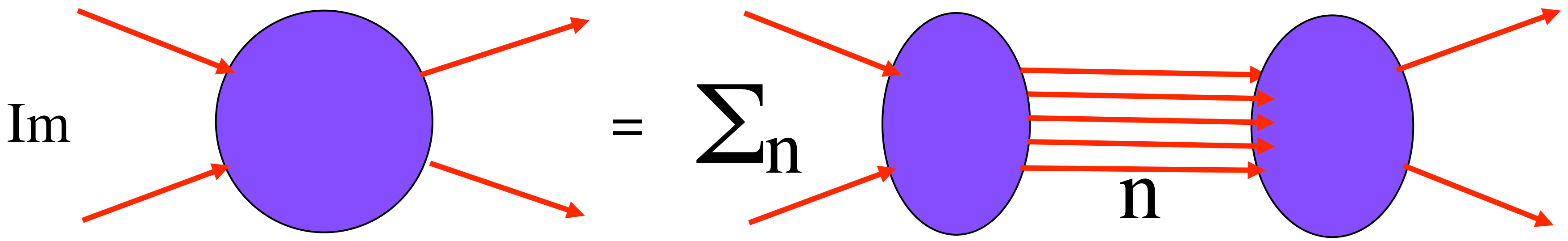
- Loops in QFT
- Loops in QST
- Modular invariance

Loops in QFT

In QFT loops come out naturally from its formalism (Wick theorem etc.). Physically, loops are needed to ensure unitarity of the S-matrix. Writing S=1 +iT unitarity gives:

$$i(T^\dagger - T) = 2ImT = T^\dagger T$$

In pictures:



Even if the blobs are tree diagrams the rhs of this equation gives loop diagrams. Unitarity is implemented order by order in perturbation theory through Cutkowski's cutting rules for Feynman's diagrams.

Loops also follow, of course, from Feynman's path integral formalism. Schematically, if ϕ_{cl} is a classical solution of the field equations,

$$
\begin{aligned}
&\int d[\phi(x)] exp(-\frac{1}{\hbar} S(\phi)) \sim \\
&exp(-\frac{1}{\hbar} S(\phi_{cl})) \int d[\phi(x) - \phi_{cl}(x)] exp\left(-\frac{1}{2\hbar} S''(\phi_{cl})(\phi - \phi_{cl})^2\right) \\
= \quad &exp(-\frac{1}{\hbar} S(\phi_{cl})) \left(det S''(\phi_{cl})\right)^{-1/2} = exp\left(-\frac{1}{\hbar} S(\phi_{cl}) - \frac{1}{2} tr[log S''(\phi_{cl})]\right)
\end{aligned}
$$

The trlog(...) is h-independent and represents a one-loop correction to the semiclassical approximation.

How do loops appear in string theory? In the DRM loops were first constructed by hand (sewing trees) but what is the analogue of Feynman's path integral in ST? The quantum fields are NOT some spacetime fields in D= 10 but the string coordinates X^{μ}, ψ^{μ} and the 2D metric $\gamma_{\alpha\beta}$.

So far we have been working in what is usually referred to as 1st quantization. In QFT books it is explained that, in order to go to a relativistic quantum theory where real and virtual particle production is allowed, we have to abandon 1st quantization techniques and go over to a so-called 2nd quantization (the wave-function itself has to be quantized). The coordinates x^{μ} become c-numbers while the fields $\phi(x^{\mu})$ become operators.

If we try to do the same in QST we end up with what is called String Field Theory which is a QFT involving an infinite number of spacetime fields, one for each state of the string.

There have been attempts to construct such a theory (in particular by Witten on open strings) with some interesting conceptual results but also with a lot of technical complications.

Fortunately, it turns out that in QST, at least in perturbation theory, one can introduce the equivalent of QFT's loops while staying all the time within 1st quantization.

This amounts to working with a finite number of quantum fields in D=2, an immense simplification (also, D=2 QFTs have nice UV properties).

But how can loops emerge for 1st quantization? This looks impossible at first sight.

Consider a Feynman path integral approach to string quantization starting from a Polyakov-like action:

$$Z \sim \int .. \int [d\gamma_{\alpha\beta}(\xi)][dX^\mu(\xi)][d\psi^\mu(\xi)]exp(-S_P)$$

$$S_P = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{\alpha\beta}(\xi)\partial_\alpha X^\mu(\xi)\partial_\beta X^\nu(\xi)G_{\mu\nu}(X(\xi)) + \ldots$$

We will see next week how the integrals over the bosonic and fermionic string coordinates produce effects proportional to α' that are absent in QFT. Let us concentrate instead now on the integral over the 2-metric $\gamma_{\alpha\beta}$.

At first sight such integral should be trivial since 2D reparametrization plus Weyl invariance should allow to gauge-fix completely $\gamma_{\alpha\beta}$. This statement is certainly true locally but there is a "global obstruction".

A well-known theorem states that :

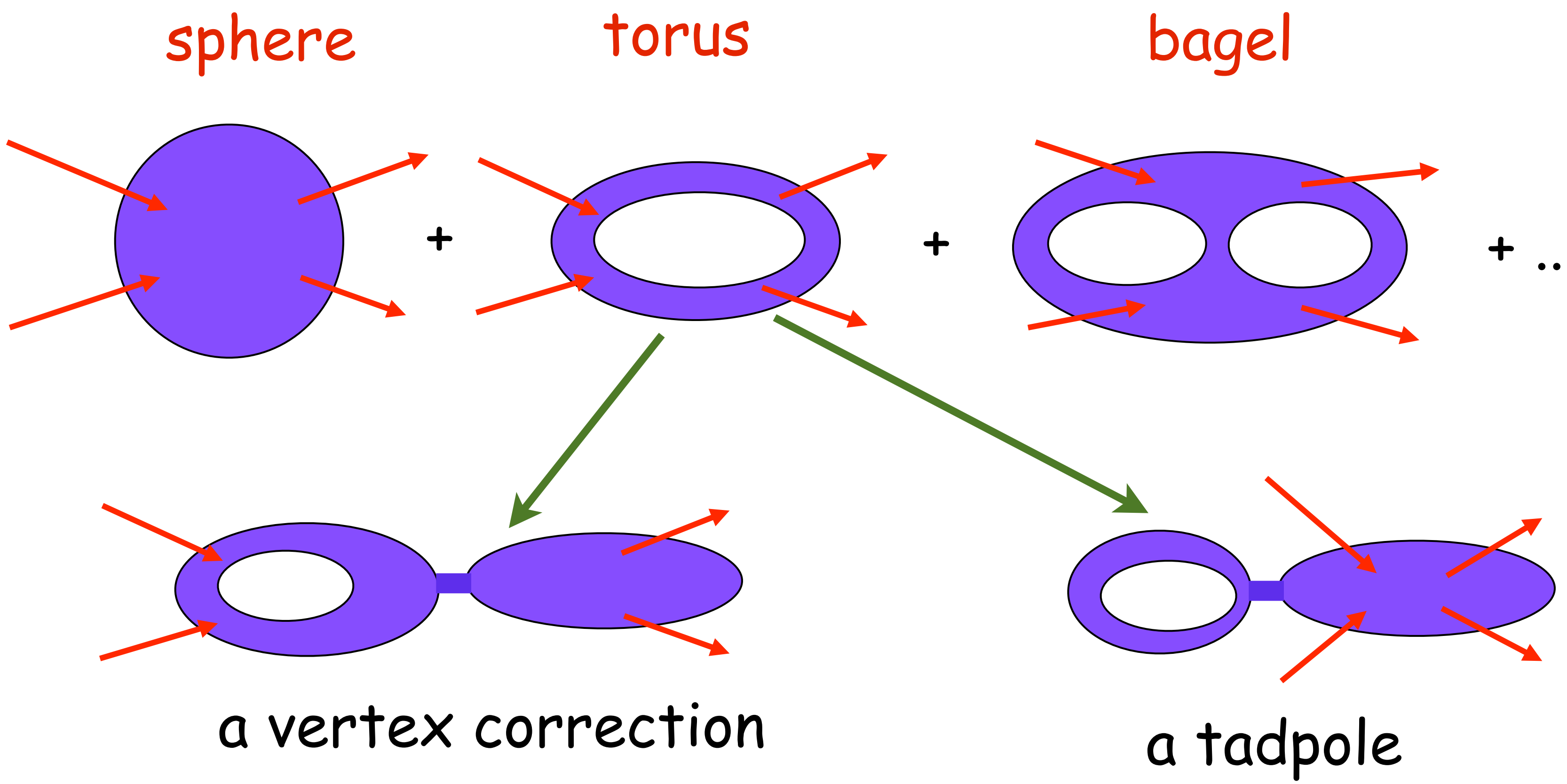
$$\frac{1}{4\pi}\int d^2\xi\sqrt{-\gamma}R(\gamma) = 2(1-g)$$

where g is the genus of the 2D Riemann surface (g=0 for the sphere, g =1 for the torus, etc.) whose geometry is given by $\gamma_{\alpha\beta}$. Fixing globally $\gamma_{\alpha\beta}$ would mean fixing g!

But why should one fix g rather than summing over it? In other words, the functional integral over the 2D metric naturally splits into a sum of functional integrals each representing Riemann surfaces of a given genus g. Precisely this sum over g corresponds to the loop expansion in QFT! QST has managed to introduce QFT's loops without invoking any 2nd quantization!

There is even an extra bonus: while in QFT the number of diagrams grows like a factorial of the loop order, here there is just one diagram at each loop order. It is again DHS duality at work...

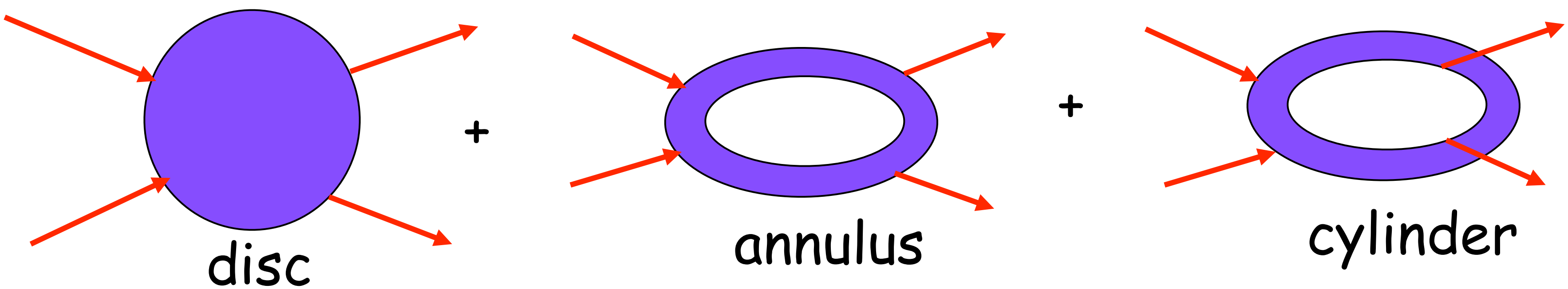
Loop expansion for closed string collisions



Closed strings attach at points on the Riemann surface. These are just our good old Koba-Nielsen variables z_i (complex numbers for closed strings) on which one has to integrate.

Open strings instead attach to boundaries of the Riemann surface, the analogue of quark loops in QCD. The sum over topologies is also a sum over different "boundaries", their total number, which have strings attached to them and which do not, etc.

The tree level corresponds now to the disc. At one loop we find the annulus, the cylinder, the Moebius strip. One can then also add "handles" (increasing the genus) as for closed strings.



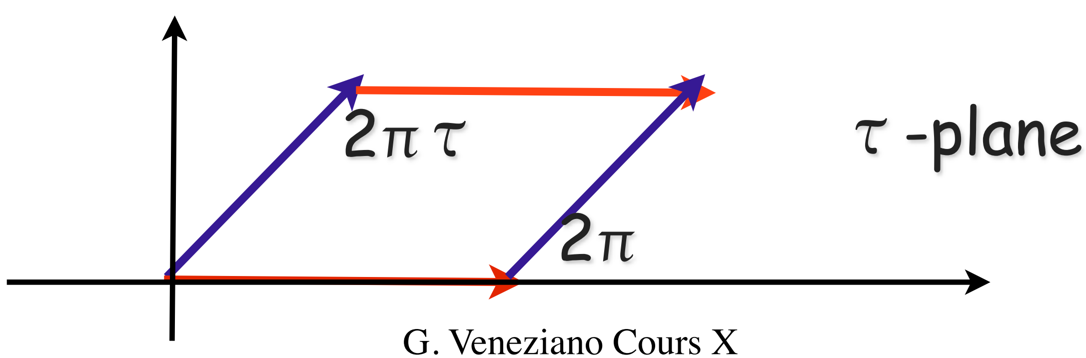
The positions at which the open strings are attached are real, ordered Koba-Nielsen variables on which one has to integrate.

Modular Invariance

Things are even more complicated. For a given topology of the Riemann surface, one has to find out exactly what the integration variables are after gauge fixing. The result is that:

1. For the sphere (and the disc for open strings) there is no integral over the size of the sphere and, furthermore, there is a residual invariance under projective O(2,1) transformations that allows to fix 3 KN coordinates (exactly what we had in the DRM!).

2. For g=1 (torus) there is still an integration over the complex parameter τ that characterizes each torus.

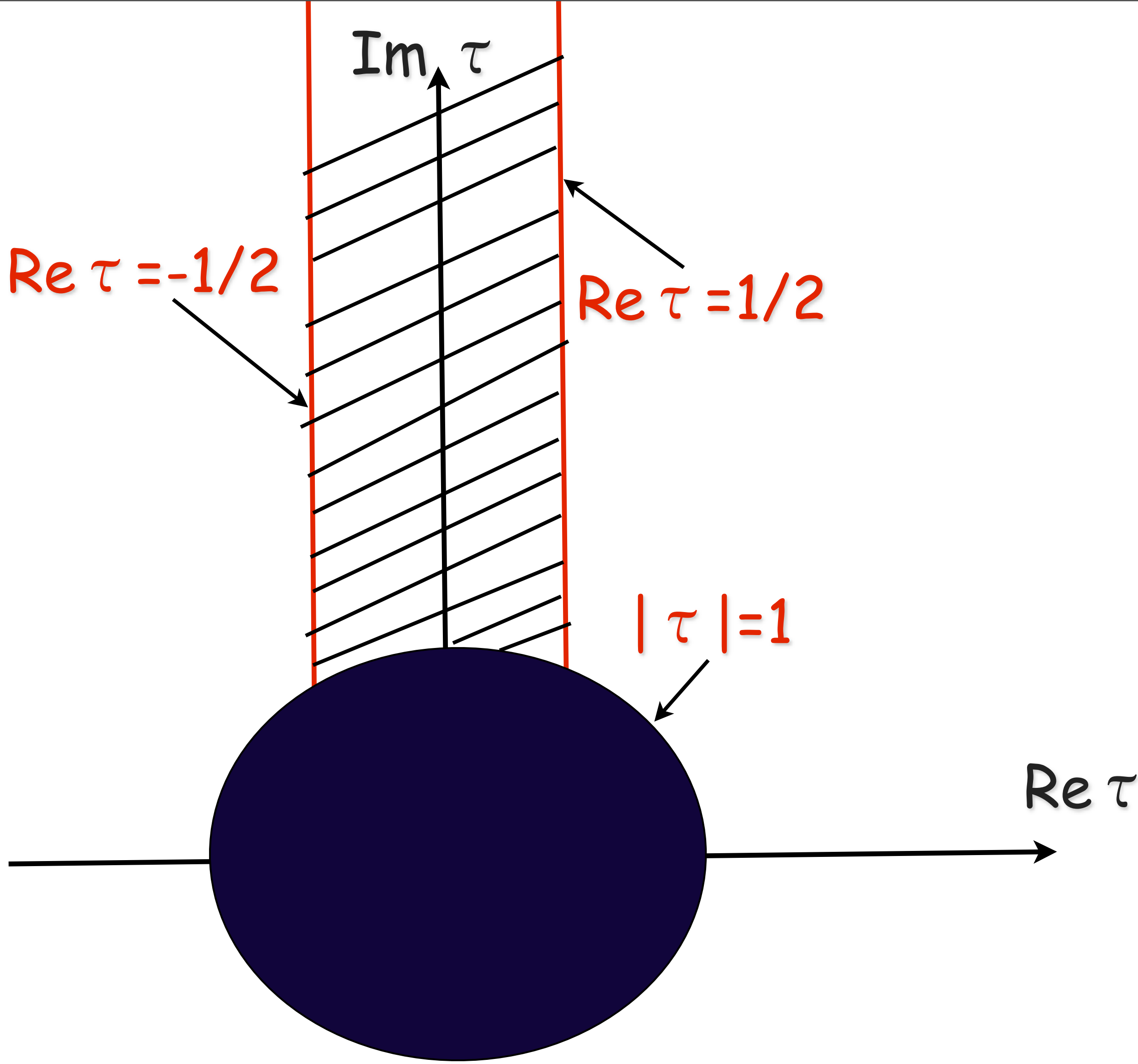


3. For $g > 1$ there is an integration over $3(g-1)$ complex parameters that characterize the Riemann surface.

Coming back to the torus, there is still a discrete group of transformations that leaves the torus invariant. This is the group of modular transformations:

$$\tau \to \frac{p\,\tau + q}{r\,\tau + s} \quad ; \quad p, q, r, s \in Z \quad ; \quad ps - qr = 1$$

Such a transformation maps the same torus in the complex τ plane an infinite number of times leading (again!) to an infinite result if we were to integrate over the whole complex plane. We should only take one region e.g. the so-called fundamental region. This region nicely avoids the point $\tau = 0$ that turns out to be associated (in a naive QFT limit) to the UV region. This is how string theory avoids UV infinities!



Fundamental region for the torus (shaded)

Modular invariance is as essential for the consistency of string theory as Weyl and reparametrization invariance (they are all parts of the gauge invariances of ST). As it turns out, imposing modular invariance at the one-loop level eliminates the gauge and gravitational anomalies (also one-loop effects!) that the GS mechanism cancels by a brute-force calculation (see seminar #4).

The search for consistent QSTs is therefore reduced to the problem of finding theories that respect modular invariance (and have no tadpoles).

This is how the two consistent heterotic string theories were found!