Particules Élémentaires, Gravitation et Cosmologie
Année 2009-'10

Théorie des Cordes: une Introduction

Cours XIII: 26 mars 2010

D-cordes, D-branes

- T-duality for open strings, D-strings
- D-branes as end-points of D-strings
- D-branes as classical solutions, DBI action

Several compact dimensions (Narain)

Consider the case of d > 1 (but still toroidal) compact coordinates. Both the internal momenta and the windings become d-dimensional vectors. The analogues of the constraints we had for 1 compact (closed string) coordinate:

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \ ; \ N - \tilde{N} + nw = 0$$

now become (the suffix 0 indicates the zero-mode part):

$$M^2 = p_L^2 + p_R^2 + \frac{4}{\alpha'}(N + \tilde{N} - 2) \ ; \ 0 = p_L^2 - p_R^2 + \frac{4}{\alpha'}(N - \tilde{N})$$

where: $\vec{p}_{L,R} = \pi(\vec{P} \pm T\vec{X}')_0 = \pi T(\dot{\vec{X}} \pm \vec{X}')_0$

The second constraint is invariant under an O(d,d) non-compact group of rotations in 2d-dimensions, while the first is only invariant under an O(d)xO(d) subgroup. The coset O(d,d)/[O(d)xO(d)] labels the inequivalent compactifications.

$O(d,d)/[O(d)xO(d)]$

The number of parameters needed to specify a given toroidal compactification is thus $2d(2d-1)/2 - d(d-1) = d^2$.

This is precisely the number of "internal" G_{ij} and B_{ij} backgrounds which, indeed, can be used to specify the compactification while keeping the d-coordinates simply periodic with the same period $2\pi R$.

However, as for d=1, this is not the full story: there are discrete duality transformations (of the $R \to 1/R$ type) that make apparently different compactifications actually equivalent.

They form a discrete $O(d,d;Z)$ group. Thus the true moduli space of toroidal compactifications is:

$$O(d,d)/[O(d)xO(d)x\ O(d,d;Z)].$$

Constraints on compactifications

Consider the dimensionless left and right-moving momenta:

$$k_{L,R} = \frac{\alpha'}{l_s} p_{L,R} \sim \sqrt{\frac{\alpha'}{2}} p_{L,R}$$ and rewrite one of the constraints:

$$p_L^2 - p_R^2 + \frac{4}{\alpha'}(N - \tilde{N}) = 0 \Rightarrow k_L^2 - k_R^2 + 2(N - \tilde{N}) = 0$$

Left + right momenta belong to a 2d-dimensional lattice Γ.

Defining a Lorentzian scalar product (with d +signs and d -signs) we conclude that Γ must be an even lattice (k^2 = even).

This is sufficient for the modular subgroup τ --> τ +1.

However, invariance under τ --> -1/ τ is more restrictive and requires Γ to be a self-dual lattice: $\Gamma = \Gamma^*$, where Γ^* consists of all the points having integer scalar product with those of Γ.

In conclusion: modular invariance (and thus Green-Schwarz anomaly cancellation) restricts Γ to be even and self-dual.

T-duality and the dilaton

There is a subtle point about T-duality. It can be appreciated by looking at the effective action:

$$\Gamma_{eff} = -\left(\frac{1}{l_s}\right)^{D-2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{4(D-D_c)}{3 l_s^2} + R(G) - 4\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{12} H^2 + \dots\right]$$

When one dimension is compactified on a circle, physics in the remaining D-1 dimensions depends on a rescaled dilaton:

$$g_s^{-2} = e^{-2\Phi} \to e^{-2\Phi} \int dy_5 \sqrt{g_{55}} = e^{-2\Phi} 2\pi R = g_{eff}^{-2}$$

As it turns out T-duality has to be accompanied by a transformation of Φ such that the effective coupling in the non-compact dimensions remains the same. In general:

$$g_{eff}^{-2} = e^{-2\Phi} \int dy^i \sqrt{g_{ij}} = e^{-2\Phi} V_c \to g_{eff}^{-2}$$

A cosmological variant of T-duality?

In our description of toroidal compactifications and of T-duality all the "internal" backgrounds G_{ij} and B_{ij} were constant. For certain properties it is sufficient that they are independent of just the "internal" coordinates themselves.

A physically interesting case it that of an homogeneous cosmology with G_{ij} and B_{ij} just functions of cosmic time. In that case we can still perform CT mixing P_i and X_i' and find out what transformations they induce on $G_{ij}(t)$ and $B_{ij}(t)$. In analogy with Narain's case these transformations, if applied to a cosmological solution, lead, in general, to other inequivalent cosmological solutions. They form, again, an O(d,d) group (involving also a change of the dilaton).

An interesting example is scale-factor-duality whereby the scale factor $a(t)$ of FRW cosmology goes to $a^{-1}(\pm t)$ (it can connect a decelerating expansion to an accelerating one driven by a growing dilaton) => a non-singular string cosmology?

T-duality for open strings

We have seen that, for closed strings moving in a space with a compact dimension, there is an interesting duality changing R into $l_s^2/2R$ and swapping momentum and winding.

This result looks so peculiar to closed strings that, for many years, no one thought that anything similar could apply to open strings since they cannot wind.

On the other hand, open strings can evolve into closed strings and back (in fact they cannot exist in isolation!) and closed strings can wind. Something looks wrong (or rather looked wrong to J. Polchinski in 1995).

It was the start of the so-called 2nd revolution (after the 1984 GS revolution)!

The key to solving this puzzle is in the boundary conditions for open strings:

$$X'_\mu \delta X^\mu(\sigma = 0) = X'_\mu \delta X^\mu(\sigma = \pi) \quad ; \quad (\text{no sum over } \mu)$$

We have seen that T-duality corresponds to a canonical transformation exchanging TX' with P.

Neumann boundary conditions correspond to setting X'=0 at the ends of the open string, while Dirichlet boundary conditions mean δ X=0, which amounts to setting P = 0.

It looks therefore highly reasonable that, for open strings, T-duality simply changes their boundary conditions from N to D, and vice versa.

Unlike closed strings, open string are not "self T-dual": they come in two kinds which are T-dual to each other!

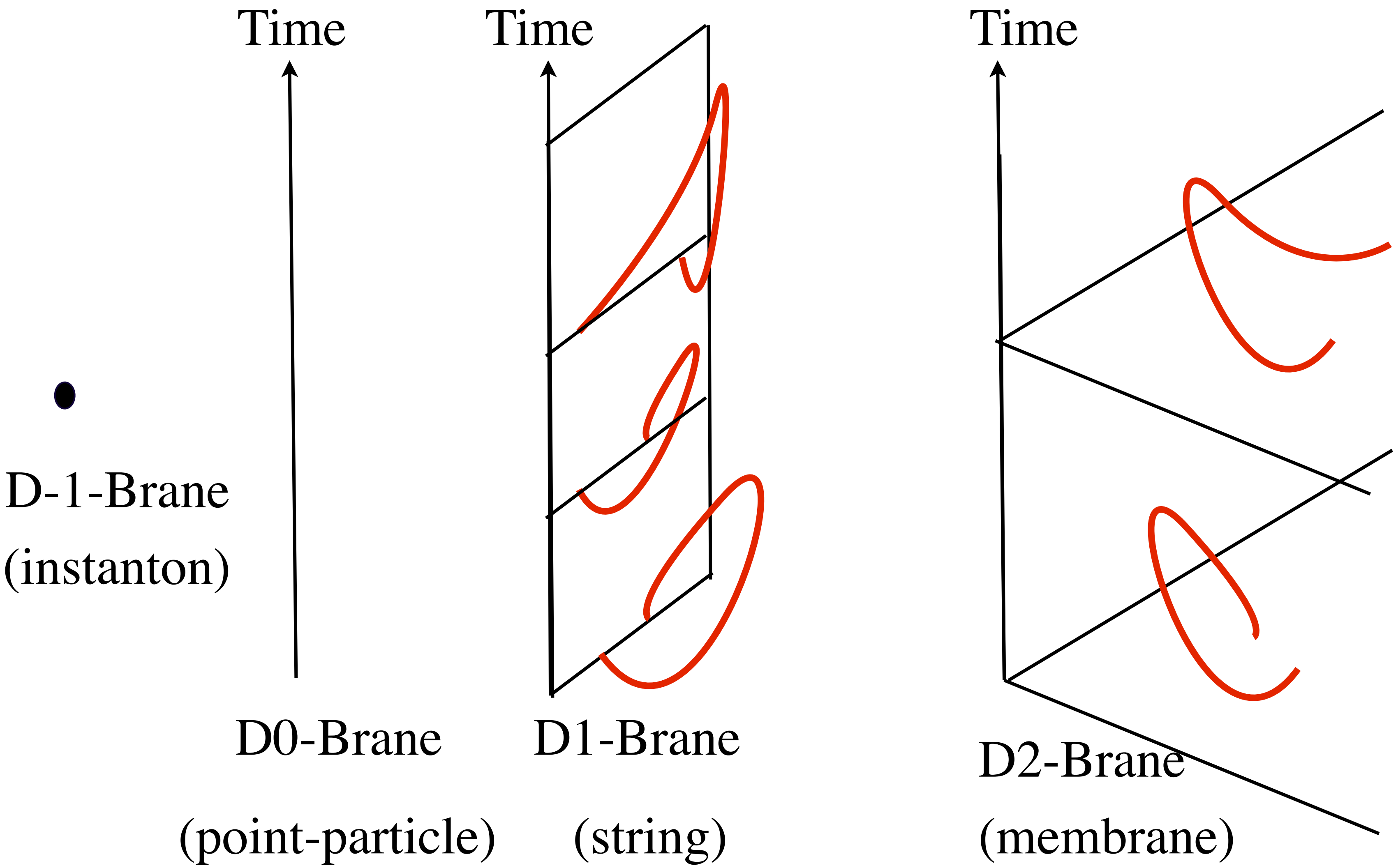
Recall that we can choose N or D boundary conditions independently for each string coordinate.

If we set Dirichlet BC for a certain number n of spatial directions, the ends of such strings are only free to move in the remaining (D-n) directions. These span (D-n)-dimensional hyperplanes immersed in the full spacetime.

Such hyperplanes are called D-branes (D for Dirichlet) or, more precisely, Dp-branes, where p = (D-n-1). p is the number of spatial dimensions of the hyperplane to which one should add time in order to arrive at p+1 = D-n.

Time is usually assumed to satisfy NBC, otherwise it does not flow for the ends of the string (however, see below).

Summarizing: the hypersurface on which the ends of our D-strings can move is (p+1)-dimensional. Such open strings have (p+1) Neumann and n Dirichlet directions. Examples:


Time
Time
Time
D-1-Brane
(instanton)
D0-Brane
(point-particle)
D1-Brane
(string)
D2-Brane
(membrane)

Let's now look at how this works when we compactify one dimension, x_5. The X_5 coordinate of an N-string is given by:

$$X_5^{(N)}(\sigma,\tau) = q_5 + 2n\alpha'\frac{\hbar}{R}\tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,5}}{\sqrt{n}}e^{-in\tau} - \frac{a_{n,5}^{\dagger}}{\sqrt{n}}e^{in\tau}\right]cos(n\sigma)$$

We now use T-duality i.e. interchange P_5 and TX'_5. It is easy to see that the result is simply:

$$X_5^{(D)}(\sigma,\tau) = q_5 + 2w\tilde{R}\sigma + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,5}}{\sqrt{n}}e^{-in\tau} - \frac{a_{n,5}^{\dagger}}{\sqrt{n}}e^{in\tau}\right]sin(n\sigma)$$

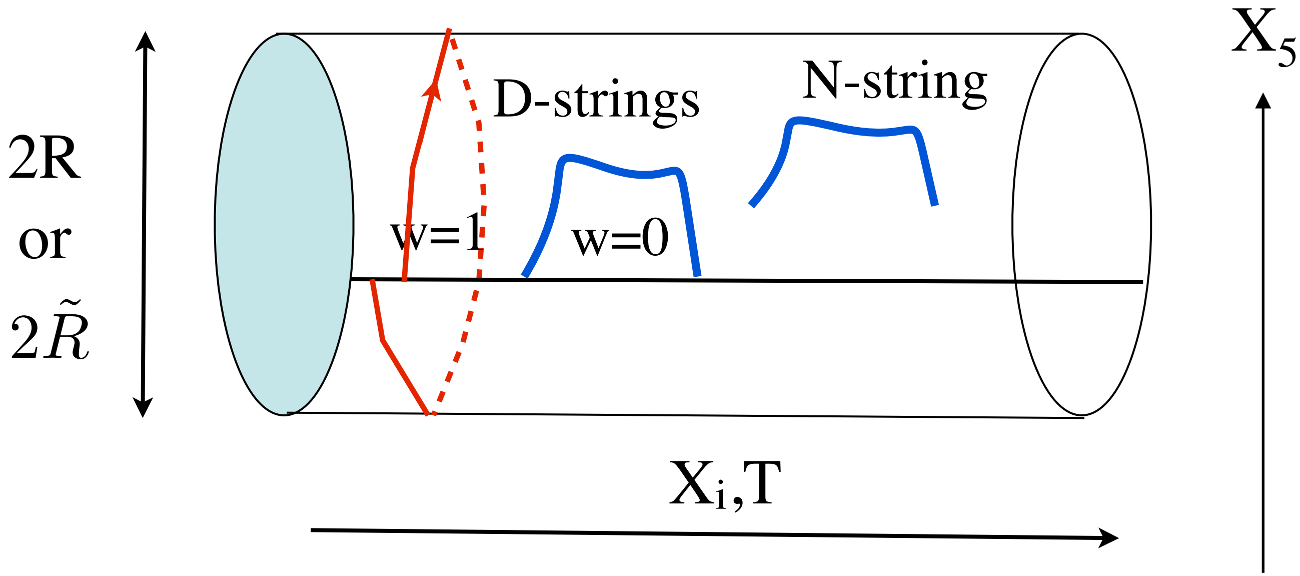
Indeed:

$$\begin{aligned}\frac{1}{T}P_5^{(N)} &= \dot{X}_5^{(N)} = 2n\alpha'\frac{\hbar}{R} + \cdots = n\frac{l_s^2}{R} + \dots \\ X_5^{'(D)} &= 2w\tilde{R} + \dots \; ; \; \text{OK if } \; \tilde{R} = \frac{l_s^2}{2R}\end{aligned}$$

But then the D-string winds around the dual circle w-times!

D-strings can wind!!

NB: T-dual N and D-strings move/wind around dual circles!

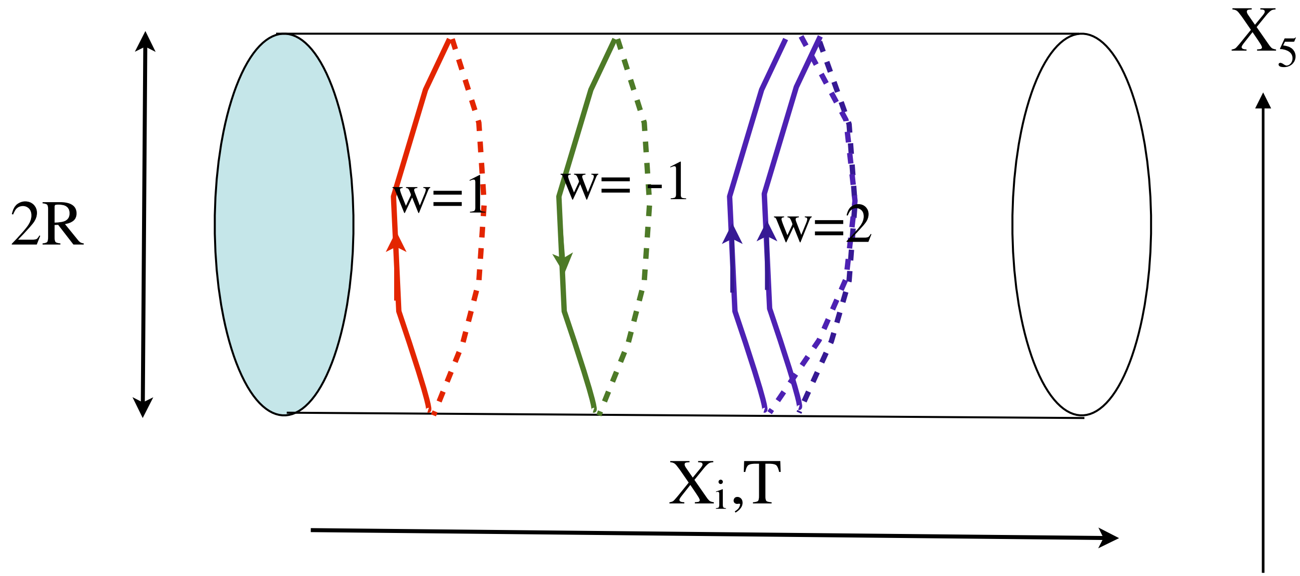


$$X_5^{(N)}(\sigma,\tau) = q_5 + 2n\alpha' \frac{\hbar}{R}\tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,5}}{\sqrt{n}}e^{-in\tau} - \frac{a_{n,5}^{\dagger}}{\sqrt{n}}e^{in\tau}\right] cos(n\sigma)$$

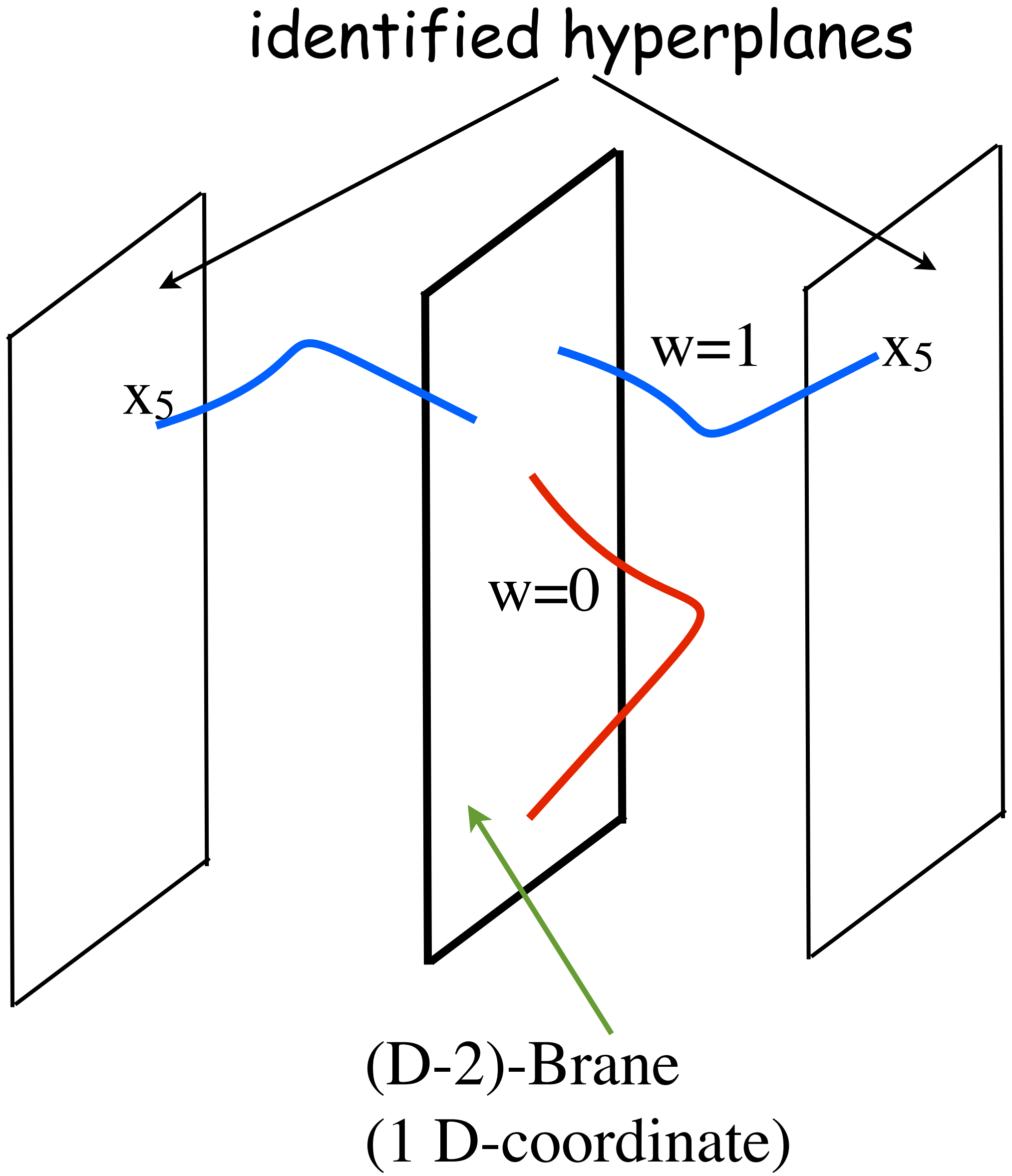
$$X_5^{(D)}(\sigma,\tau) = q_5 + 2w\tilde{R}\sigma + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,5}}{\sqrt{n}}e^{-in\tau} - \frac{a_{n,5}^{\dagger}}{\sqrt{n}}e^{in\tau}\right] sin(n\sigma)$$

$$X_5^{(D)}(\sigma=\pi,\tau) = X_5^{(D)}(\sigma=0,\tau) + 2\pi\tilde{R}\ w$$

to be compared with the closed string case:



$$
\begin{aligned}
X_5(\sigma,\tau) &= q_5 + 2n\alpha'\frac{\hbar}{R}\tau + 2wR\sigma \\
&+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,5}}{\sqrt{n}}e^{-2in(\tau-\sigma)} - \frac{a^{\dagger}_{n,5}}{\sqrt{n}}e^{2in(\tau-\sigma)}\right] \\
&+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\tilde{a}_{n,5}}{\sqrt{n}}e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^{\dagger}_{n,5}}{\sqrt{n}}e^{2in(\tau+\sigma)}\right]
\end{aligned}
$$


identified hyperplanes
x5
w=1
x5
w=0
(D-2)-Brane
(1 D-coordinate)

For the open bosonic string the mass shell condition reads:

$$L_0 = 1 \Rightarrow M^2 = \frac{\hbar^2 n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{1}{\alpha'}(N-1)$$

where w=0 for the N-case and n=0 for D. For generic R the massless states are given by n=w=0, N=1, i.e. by the states $a^\dagger_{1\mu}|0\rangle$. Let us concentrate on the Dirichlet case.

If the index of the oscillator is not 5 this is a gauge boson stuck on the brane (in (D-1)-dimensions with (D-3) physical components); if the index is 5 it's a massless scalar also confined to the brane. What's the meaning of this scalar?

The answer is quite simple and elegant. The presence of the brane clearly breaks (spontaneously) translation invariance in the 5th direction. The massless scalar is the Nambu-Goldstone boson of that broken symmetry and describes the possible local deformations of the brane itself!

Turning on Wilson lines

In QFT there are interesting gauge-invariant non-local operators called Wilson loops (cf. confinement criteria):

$$W_C^R = TrTexp\left(iq\int_C dx^\mu A_\mu^a(x)T_R^a\right)$$

where C is a closed loop in spacetime. When one dimension of space is compactified one can consider a (topologically non-trivial) C that wraps around the compact dimension. Even if the gauge field is trivial (a pure gauge) such a Wilson loop, called a Wilson line, can be non-trivial. Take indeed:

$$qA_5^a(x)T_R^a = \ -\frac{1}{2\pi R}\mathrm{diag}\ (\theta_1, \theta_2, \ldots \theta_N) \qquad W = \sum_{i=1}^N e^{-i\theta_i}$$

Where the θ_i are constants (hence A is a pure gauge). It is known that non-trivial Wilson lines break spontaneously the gauge symmetry.

In the presence of a U(1) gauge potential, we need to make the usual (minimal) substitution:

$$p_5 \to p_5 - \hbar q A_5 = n\frac{\hbar}{R} + \frac{\hbar\theta}{2\pi R}$$

In our case, consider the open string sector with Chan-Paton quantum numbers (ij). It has charge +q with respect to the ith gauge boson and charge -q wrt the jth. Hence, for such a (ij) string, minimal substitution gives:

$$p_5^{ij} \to n\frac{\hbar}{R} + \frac{\hbar(\theta_i - \theta_j)}{2\pi R}$$ and the mass formulae become:

$$M_{ij}^2 = \left(\frac{n\hbar}{R} + \frac{(\theta_i - \theta_j)\hbar}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N-1) \quad \text{for NBC}$$

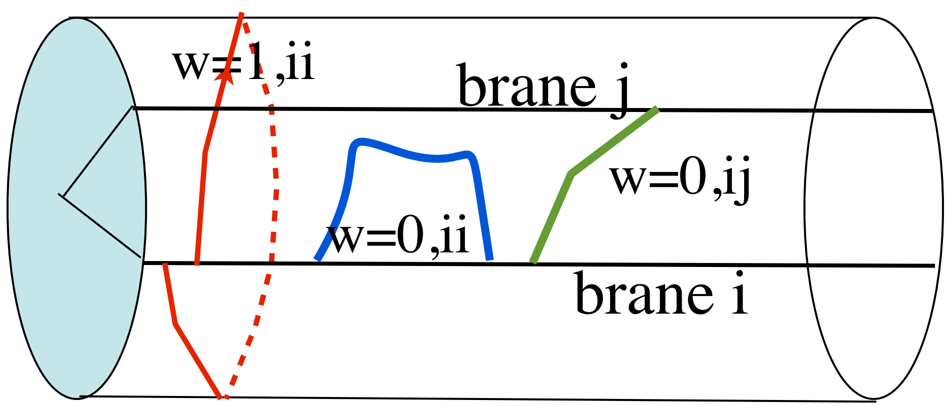
Its T-dual is again obtained by P-->X' and R--> $l_s^2/2R$. It has:

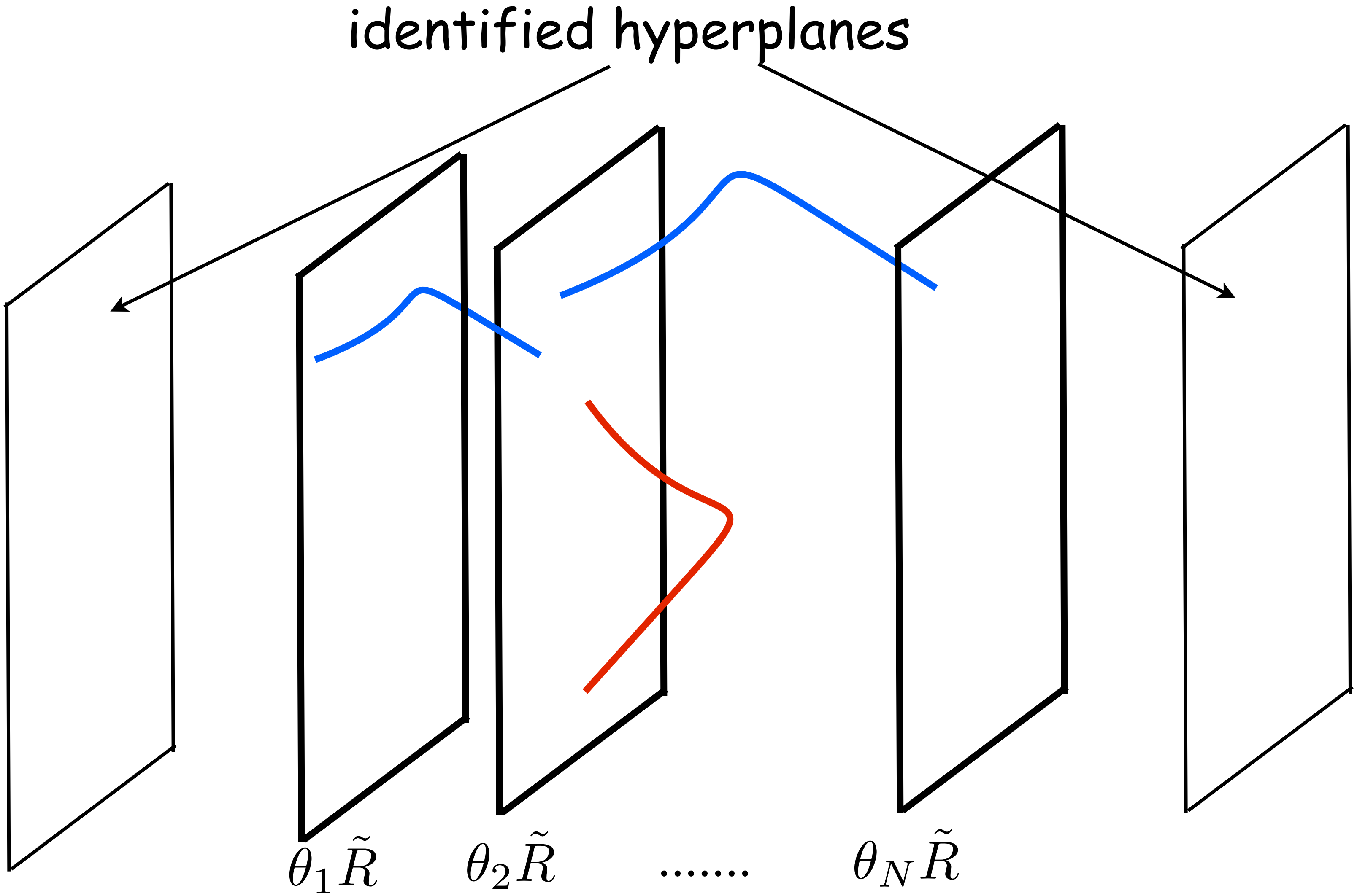
$$X_5^{ij} \to \left(2w + \frac{(\theta_i - \theta_j)}{\pi}\right)\tilde{R}\,\sigma$$

i.e. the string spans an angle $2w\pi + (\theta_i - \theta_j)$ around the circle.

Also: $M_{ij}^2 = \left(\frac{w\tilde{R}}{\alpha'} + \frac{(\theta_i - \theta_j)\tilde{R}}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1)$ for DBC

We thus see that, in the T-dual (Dirichlet) description, the presence of Wilson lines has added a contribution to the mass of the gauge bosons equal to T times the distance between two branes placed at angles θ_i and θ_j along the (dual) circle. A D-string with quantum numbers ij has now one end on the i^{th} brane and the other on the j^{th} brane. While the (ii) & (jj) strings can be massless, the (ij) string cannot. This is how one sees the breaking of the U(N) gauge symmetry in the dual picture!





In general the symmetry is broken down to $U(1)^N$. If some branes overlap it is broken to $U(1)^{N1}\,U(1)^{N2}$ ..... $U(1)^{Np}$

Note that in the generic case each "diagonal" massless vector is accompanied by a massless scalar interpreted as the field describing the transverse fluctuations of that brane.

When n branes overlap (i.e. when some of the angles coincide) we get n^2 massless gauge bosons since each end can be on any of the n coincident branes without generating mass.

Also, one finds the same number n^2 of massless scalars whose meaning is not entirely clear (I think).

These scalar fields are themselves matrices and are like non-commuting coordinates of a many-brane system.

The Dp-brane action

Can we find a low-energy effective action that describes the dynamics of a single D-brane?

The answer is yes and quite simple. It goes back to an action invented long ago by Born and Infeld (for completely different and unsuccessful purposes), used by Dirac, and called the DBI (Dirac-Born-Infeld) action.

$$\frac{1}{\hbar} S_p = -c_p l_s^{-(d+1)} \int d^{p+1}\xi e^{-\Phi} \left[-\det\left(G_{ab} + B_{ab} + \pi l_s^2 F_{ab}\right)\right]^{1/2}$$

where c_p is a known p-dependent number and G_{ab} , B_{ab} , are the induced metric and B on the brane's world sheet:

$$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)) \qquad B_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi))$$

while F_{ab} is the gauge field strength of its associated U(1).

Note that this action also includes the scalar fields X^{μ}, describing the transverse fluctuations of the brane.

The physical scalar degrees of freedom are only (D-p-1), i.e. exactly what we have after using invariance under reparametrization of the (p+1) ξ^{a} coordinates.

Another observation concerns the dependence of the brane action from the dilaton. Instead of the overall $\exp(-2\Phi)$ of the effective closed-string action we get here a factor $\exp(-\Phi)$. This is exactly what we should expect since the open-string coupling is $\exp(-\Phi)$ while the closed-string coupling is $\exp(-2\Phi)$.

Let us finally mention the idea of a brane Universe. Since the ends of open strings are stuck on the brane, if all the SM particles are open strings we may be living on a 3-brane and only gravity and other gravitational-like forces would feel the full dimensionality of spacetime. In this brane-Universe scenario gravity is typically modified at short distances without contradicting, so far, any experimental facts.

D-branes as classical solutions

We have described D-branes from an open-string viewpoint (hypersurfaces on which open strings end) but actually D-branes also emerge as classical solutions of the string effective action when we add all the massless bosonic fields contained in Type IIa or IIb superstring theories. Of crucial importance are the RR forms present in such theories since, as it turns out, D-brane are "charged" under those fields (i.e. they are sources for the forms). A p-brane couples naturally to a (p+1)-form potential. Thus, Type IIa, having odd forms, gives rise to even-p-branes, the opposite being the case for Type IIb.

The solutions are relatively simple: metric, dilaton and the relevant forms only depend on the transverse distance from the brane. If wrapped they can give rise to "black-branes".

We will not need these for this year's course (but will be important for Black-Holes, AdS/CFT etc.)