Particules Élémentaires, Gravitation et Cosmologie
Année 2010-'11

Théorie des cordes: quelques applications

Cours II: 4 février 2011

Résumé des cours 2009-'10: deuxième partie

From the bosonic string to the superstring

Birth of string theory: NG action, LC quantization	Polyakov's CFT approach
Neveu-Schwarz and Ramond generalizations: WS-SUSY	GSO projection: target-space SUSY

After recalling how several hints of a string underlying DRM were missed, we introduced the bosonic string.

We started from the construction of a geometric action (the Nambu-Goto action) that generalizes the action of a relativistic particle of mass m. In string theory, the role of m is replaced by the string tension T, a quantity with dimensions of energy/length (c=1). Its inverse $\alpha' = 1/(2\pi T)$ has dimensions of J/M^2 (the α' of the DRM!).

$$S_{NG} = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}}$$

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

We compared the constraints for points and strings:

$$p_\mu(\tau)p_\nu(\tau)g^{\mu\nu}(x(\tau)) = -m^2$$

$$P_\mu(\xi)X'^\mu(\xi) = 0$$

$$P_\mu(\xi)P_\nu(\xi)G^{\mu\nu}(X(\xi)) + T^2 X'^\mu(\xi)X'^\nu(\xi)G_{\mu\nu}(X(\xi)) = 0$$

$$\dot{X}^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^0} \; , \quad X'^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^1}$$

Both come from the invariance of the action under transformations of the world line/sheet coordinates.

In Minkowski spacetime and in the so-called orthonormal (or conformal) gauge the eom of the string and the constraints simplify:

$\ddot{X}_\mu = X''_\mu$ w/ solution: $X_\mu(\sigma,\tau) = F_\mu(\tau-\sigma) + G_\mu(\tau+\sigma)$

$$(\dot{X} \pm X')^2 = 0$$

Next we discussed boundary conditions for open strings. They gave rise to two options (for each direction of space): Neumann (N) or Dirichlet (D)

N $\quad X'_\mu = 0 \quad , \quad \sigma = 0, \pi$

D $\quad \dot{X}_\mu = 0 \quad , \quad \sigma = 0, \pi$

The possibility of having Dirichlet b.c. is at the origin of the "brane" (or second string) revolution!

Leaving D-strings for later we concentrated on N-strings and wrote down solutions to the eom and to the boundary conditions.

General solution of $\ddot{X}_\mu = X''_\mu$ and of b.c.

Open (Neumann) strings ($X'_\mu(\sigma=0, \pi) = 0$)

$$X_\mu(\sigma,\tau) = q_\mu + 2\alpha' p_\mu \tau + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[\frac{a_{n,\mu}}{\sqrt{n}} e^{-in\tau} - \frac{a^*_{n,\mu}}{\sqrt{n}} e^{in\tau} \right] cos(n\sigma)$$

Closed strings $X_\mu(\sigma=0) = X_\mu(\sigma=\pi)$

$$\begin{aligned} X_\mu(\sigma,\tau) &= q_\mu + 2\alpha' p_\mu \tau + \frac{i}{2}\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[\frac{a_{n,\mu}}{\sqrt{n}} e^{-2in(\tau-\sigma)} - \frac{a^*_{n,\mu}}{\sqrt{n}} e^{2in(\tau-\sigma)} \right] \\ &+ \frac{i}{2}\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[\frac{\tilde{a}_{n,\mu}}{\sqrt{n}} e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^*_{n,\mu}}{\sqrt{n}} e^{2in(\tau+\sigma)} \right] \end{aligned}$$

to be added

$$\begin{aligned} \dot{X} \cdot X' &= 0 \\ \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \end{aligned}$$

We then proceeded to the quantum theory

In the course we did that by going to the light-cone gauge a further specification of the conformal gauge (which still leaves a residual gauge freedom under separate reparametrizations of $\tau \pm \sigma$ = conformal transformations).

$$X^+(\sigma,\tau) \equiv \frac{X^0 + X^{D-1}}{\sqrt{2}} = 2\alpha' p^+ \tau \qquad\qquad (\dot{X} \pm X')^2 = 0$$

In the L.C. gauge we can easily express all the oscillators in terms of the transverse ones (with D-2 components, Cf. DDF states of the DRM). The Hilbert space has positive norm but manifest Lorentz invariance is lost. Imposing the Lorentz algebra gives back the two DRM conditions:

$$D=26\ ,\ \alpha_0=1$$

In the seminar string theory was reformulated starting from the so-called Polyakov action allowing a more elegant method of quantization, which keeps manifest Lorentz invariance at the price of introducing Fadeev-Popov ghosts and a corresponding BRST nilpotent operator Q.

At the end of some non-trivial calculations one arrives at the conclusion that, for consistency ($Q^2=0$ at quantum level), one has to impose: $D=26$ and $\alpha_0=1$.

In that case, at the level of the spectrum, DRM and string theory are equivalent provided $D=26$ and $\alpha_0=1$ (DDF states vs. transverse oscillations of a string!)

Full equivalence, including the scattering amplitudes, can be established with quite some extra work.

Polyakov formulation of bosonic string

Polyakov, following several other authors, has given an alternative formulation of string theory with the action:

$$S_P = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)$$

After expressing $\gamma_{\alpha\beta}$ in terms of X^μ and $G_{\mu\nu}$ using its eom one gets back the NG action (where $\gamma_{\alpha\beta}$ was NOT an independent field, see page 3).

Polyakov's action is useful for generalizations to fermions and/or to background fields other than the metric $G_{\mu\nu}$. Looks like a GR in 2 dimensions with X^μ as scalar fields. World-sheet-reparametrizations become GCT in 2 D.

It is also often referred to as the non-linear σ-model formulation of string theory.

Adding Fermions

We introduced fermionic dof (anticommuting coordinates) i.e. 2-component spinors on the world sheet, yet D-component vectors in spacetime. In Polyakov-style form the action has local 2D supersymmetry and is quite complicated:

$$
\begin{aligned}
S &= S^{\rm b} + S^{\rm f} \; ; \; S^{\rm b} = -\frac{T}{2}\int d^2\xi \sqrt{-g}\; g^{\alpha\beta}\; \partial_\alpha X^\mu \cdot \partial_\beta X_\mu \\
S^{\rm f} &= -\frac{T}{2}\int d^2\xi \sqrt{-g}\left[i\bar{\psi}^\mu \gamma^\alpha \cdot \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \gamma^\beta \partial_\beta X^\mu \cdot \gamma^\alpha \psi_\mu \right. \\
&- \left. 1/4(\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu) \cdot (\bar{\chi}_\beta \psi_\mu) \right]
\end{aligned}
$$

(NB:to avoid confusion, we denoted by $g_{\alpha\beta}$ the 2D-metric $\gamma_{\alpha\beta}$)

In a "superconformal" gauge S simplifies drastically:

$$
\begin{aligned}
S^{SCG} &= -\frac{T}{2}\int d^2\xi\,\left[\ \partial_\alpha X^\mu\cdot\partial^\alpha X_\mu + i\bar{\psi}^\mu\gamma^\alpha\cdot\partial_\alpha\psi_\mu\right] \\
&= T\int d^2\xi\,\left[\ 2\partial_+ X^\mu\cdot\partial_- X_\mu + \psi_-^\mu\partial_+\psi_{-\mu} + \psi_+^\mu\partial_-\psi_{+\mu}\right]
\end{aligned}
$$

leading to very simple solutions

$$
\begin{aligned}
X_\mu(\sigma,\tau) &= F_\mu(\tau-\sigma)+G_\mu(\tau+\sigma) \\
\psi_-^\mu(\sigma,\tau) &= \psi_-^\mu(\tau-\sigma)\ ;\ \psi_+^\mu(\sigma,\tau)=\psi_+^\mu(\tau+\sigma)
\end{aligned}
$$

The fermionic boundary conditions

$$\left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = 0) = \left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = \pi)$$

and give, for open strings,

$$\begin{aligned} \psi_+^\mu(\sigma = 0) &= \pm\psi_-^\mu(\sigma = 0) \\ \psi_+^\mu(\sigma = \pi) &= \pm\psi_-^\mu(\sigma = \pi) \end{aligned}$$

R: $\psi_+^\mu(\sigma = 0) = +\psi_-^\mu(\sigma = 0)$ and $\psi_+^\mu(\sigma = \pi) = +\psi_-^\mu(\sigma = \pi)$

NS: $\psi_+^\mu(\sigma = 0) = +\psi_-^\mu(\sigma = 0)$ and $\psi_+^\mu(\sigma = \pi) = -\psi_-^\mu(\sigma = \pi)$

This leads to the mode expansions:

$$\psi^{\mu}_{\pm}(\sigma,\tau) = \sqrt{\alpha'} \sum_{n\in Z} d^{\mu}_{n} e^{-in(\tau\pm\sigma)}$$ R

$$\psi^{\mu}_{\pm}(\sigma,\tau) = \sqrt{\alpha'} \sum_{r\in Z+1/2} b^{\mu}_{r} e^{-ir(\tau\pm\sigma)}$$ NS

For closed strings we impose $\psi^\mu_\pm \delta\psi^\mu_\pm(\sigma=0) = \psi^\mu_\pm \delta\psi^\mu_\pm(\sigma=\pi)$

giving the choices:

$$\begin{aligned} \psi^\mu_+(\sigma) &= \pm\psi^\mu_+(\sigma+\pi) \quad \text{R, NS} \\ \psi^\mu_-(\sigma) &= \pm\psi^\mu_-(\sigma+\pi) \quad \text{R, NS} \end{aligned}$$

and we get 4 kinds of states (physical closed-string states combine left and right movers)

NS-NS: $\psi^\mu_+(\sigma) = -\psi^\mu_+(\sigma+\pi)\ ,\ \psi^\mu_-(\sigma) = -\psi^\mu_-(\sigma+\pi)$

NS-R: $\psi^\mu_+(\sigma) = -\psi^\mu_+(\sigma+\pi)\ ,\ \psi^\mu_-(\sigma) = +\psi^\mu_-(\sigma+\pi)$

R-NS: $\psi^\mu_+(\sigma) = +\psi^\mu_+(\sigma+\pi)\ ,\ \psi^\mu_-(\sigma) = -\psi^\mu_-(\sigma+\pi)$

R-R: $\psi^\mu_+(\sigma) = +\psi^\mu_+(\sigma+\pi)\ ,\ \psi^\mu_-(\sigma) = +\psi^\mu_-(\sigma+\pi)$

The canonical anticommutation relations for $\psi^\mu_a(\xi)$ lead to

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s} \;;\; \{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n+m}$$

In the R sector, the vacuum cannot satisfy: $d_0^\mu|0\rangle = 0$

since $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$

In the R sector the vacuum state is a representation of the Dirac algebra and is, indeed, a spacetime fermion.

Since the other states of the R sector are obtained by applying spacetime vector creation operators, all the states of the R-sector are fermions.

Similarly, all the states in the NS sector are bosons.

For closed strings NS-NS and R-R are bosons while NS-R and R-NS are fermions.

Proceeding as in the bosonic case with quantization in the light cone gauge:

$$X^+(\sigma,\tau) \equiv \frac{X^0 + X^{D-1}}{\sqrt{2}} = 2\alpha' p^+ \tau \ ; \ \psi^+(\sigma,\tau) \equiv \frac{\psi^0 + \psi^{D-1}}{\sqrt{2}} = 0$$

and imposing the constraints

$$\begin{aligned} \partial_\pm X \cdot \partial_\pm X \quad &+ \quad \frac{i}{2}\psi \cdot \partial_\pm \psi = 0 \\ \psi \cdot \partial_\pm X \quad &= \quad 0 \end{aligned}$$

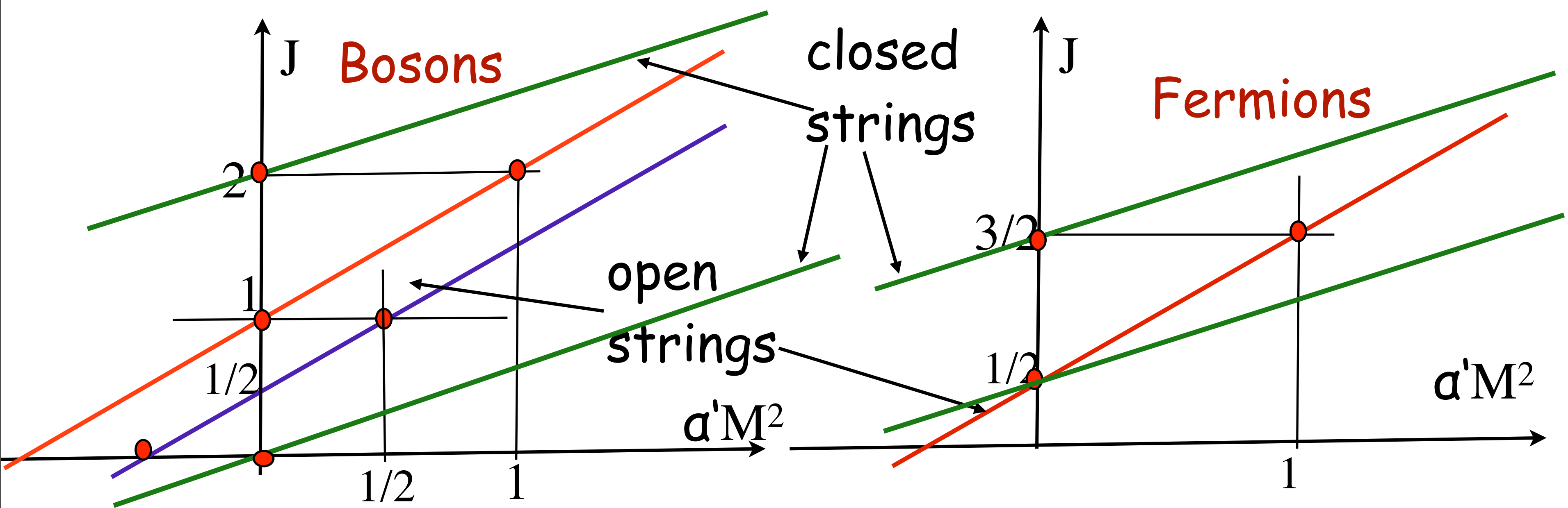
we can solve for X^- and ψ^- in terms of the transverse operators. Imposing the validity of the Lorentz algebra forces the conditions D=10 and α_0 = 1/2, 1.

These results are confirmed by the covariant (BRST) quantization procedure.

Conclusions on NSR string

The NSR model is much richer than the bosonic string: it has also fermions (with no tachyon) and a bosonic trajectory with intercept 1/2 (with a tachyon on it).

It also has an amusing (though only partial) degeneracy between the bosonic and fermionic spectra.



GSO projection and spacetime SUSY

In 1976, Gliozzi, Scherk and Olive found a smart way to eliminate the tachyon. They introduced, in the NSR model, a WS fermion "parity" P_F.

This is defined in different ways for the NS and R sectors.

For NS one defines:

$$P_F^{NS} = (-1)^{F_{NS}+1} \ ; \ F_{NS} = \sum_{r=+1/2}^{\infty} b_{-r} b_r$$

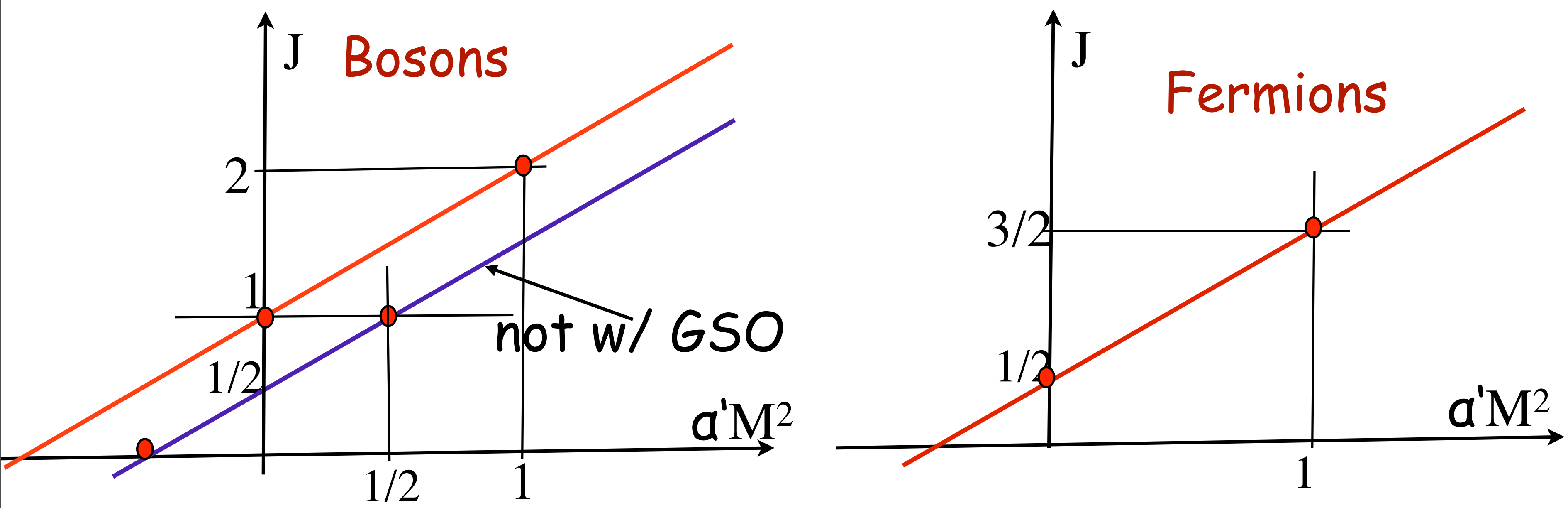
while for R:

$$P_F^{R} = \gamma_{11} (-1)^{F_R} \ ; \ F_R = \sum_{n=+1}^{\infty} d_{-n} d_n$$

NB: the tachyon has P_F=-1, the massless vector P_F=+1

GSO then proved that P_F is conserved in the NSR model so that a projection on states with $P_F=+1$ is consistent with factorization ($P_F=-1$ states do not appear as intermediate states). The tachyon is eliminated. Also, half of the fermions in the fermionic sectors are projected out.

The fermionic ground state is a Majorana-Weyl spinor in D=10. It has 8 components (just like a massless vector) and is chiral in the 10-dimensional sense. Open string spectra:



A counting of states shows that, not only the massless spectrum, but also the excited states contain the same number of bosons and fermions.

This (spacetime) supersymmetry of the spectrum can be generalized to interactions and is a true symmetry of string theory after the GSO projection.

This is the so-called Type I superstring, a theory of open strings. It automatically generates also a closed string sector at the non-planar-loop level.

At tree level its massless states are a vector (in the adjoint representation of an SO(N) or Sp(N) group) and a Weyl-Majorana spinor. It is a chiral theory in D=10 as a consequence of the GSO projection.

Changing γ_{11} into $-\gamma_{11}$ does not lead to a different theory (there is only one Type I theory!).

For closed strings one applies GSO separately to left- and right-movers. However, we can use either the same or an opposite γ_{11} projection for left and right-movers. In the former case we have a chiral theory, called Type IIB, in the latter a non-chiral theory, called Type IIA.

Up to the 1984 paper by Green and Schwarz, one knew about 3 fully consistent (no ghost, no tachyon) string theories: Type I, IIA, IIB...

NB: SUSY looks like a common ingredient to all known consistent string theories!

Strings need SUSY, QFT can do without it!