Particules Élémentaires, Gravitation et Cosmologie
Année 2009-'10

Théorie des Cordes: une Introduction
Cours XI: 19 mars 2010

Cordes dans un fond non banal et action efficace

- Strings in non-trivial backgrounds
- Imposing local WS symmetries
- The effective action and its 2 meanings
- The effective action and its 2 expansions

Bosonic strings in non-trivial backgrounds

We have already seen how to generalize the NG action to a non-trivial space-time background $G_{\mu\nu}(x)$.

To extend this construction to other backgrounds it is easier to start from Polyakov's formulation.

For a closed string in a pure metric background we have:

$$S_G = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{\alpha\beta}(\xi)\partial_\alpha X^\mu(\xi)\partial_\beta X^\nu(\xi)G_{\mu\nu}(X(\xi))$$

Note that, in the quantum action S/h, only the combination $G_{\mu\nu}/l_s^2$ appears (it has dimensions length^{-2}). This will provide useful checks later. The local D=2 symmetries are present for any $G_{\mu\nu}$. Which other backgrounds can we add? All we have to require is to preserve the local WS symmetries at the quantum level. Let us proceed by analogy with the point-particle case.

A charged point-particle couples naturally to a vector potential (a 1-form) without even invoking a 1D-metric:

$$S_A^{point} = q \int d\tau \; \dot{x}^\mu(\tau) \; A_\mu(x(\tau)) = q \int dx^\mu(\tau) \; A_\mu(x(\tau))$$

This action is invariant under the gauge transformation A--> A + dΛ .

In perfect analogy, a string naturally couples to a 2-form $B_{\mu\nu}=-B_{\nu\mu}$ without invoking a 2D-metric:

$$S_B = -\frac{T}{2} \int d^2\xi \epsilon^{\alpha\beta} \partial_\alpha X^\mu(\xi) \partial_\beta X^\nu(\xi) B_{\mu\nu}(X(\xi))$$

with $\varepsilon^{\alpha\beta}$ the Levi-Civita symbol in D=2. This action is invariant under B --> B + dΛ where Λ is a one-form.

This can be easily generalized to p-branes...

Can we write anything else that satisfies classically the 2D local symmetries, and in particular Weyl invariance? The only possibility appears to be:

$$S_\Phi = \frac{1}{4\pi}\int d^2\xi \sqrt{-\gamma}R(\gamma)\Phi(X(\xi))$$

but only if the field $\Phi(x)$, called the dilaton, is a constant.

In that case, the integral is proportional to the Einstein-Hilbert action, which, in D=2, has a topological meaning and thus is clearly Weyl-invariant. As already discussed:

$$\frac{1}{4\pi}\int d^2\xi \sqrt{-\gamma}R(\gamma) = 2(1-g)$$

It is related to the genus g of the Riemann surface described by the metric $\gamma_{\alpha\beta}$. Thus, if Φ is constant, $S_\Phi = 2\Phi(1-g)$; if it isn't, S_Φ is non-trivial and classically not Weyl-invariant.

Let's put anyway all 3 terms together and write the action for a string in a metric, antisymmetric and dilaton background as:

$$S = -\frac{T}{2}\int d^2\xi\sqrt{-\gamma}\left[\partial_\alpha X^\mu \partial_\beta X^\nu\left(\gamma^{\alpha\beta}G_{\mu\nu} + \frac{\epsilon^{\alpha\beta}}{\sqrt{-\gamma}}B_{\mu\nu}\right) - \frac{1}{2\pi T}R(\gamma)\Phi\right]$$

Under what conditions for the background fields can we statisfy the conditions of 2D-rep. and Weyl invariance at the quantum level?

This is, in general, a highly non trivial problem. We know one solution: Minkowski spacetime, vanishing B, and constant Φ, provided that D takes a critical value (D=26, 10).

This is the string we have been discussing so far with just one small additional point.

When the above action is inserted in the (Euclidean) path integral it will weight the contribution of each Riemann surface of genus g with a factor $\exp(-2\Phi(1-g))$ hence with an extra factor $\exp(2\Phi)$ for each extra string loop.

Therefore $\exp(2\Phi)$ plays, in QST, the same role that α plays in QED (or in the SM). It is the loop-counting parameter.

In QED a square-root of α also appears in the scattering amplitude for each emitted (or incoming) photon.

As we shall see, a factor $\exp(\Phi)$ will be associated with each external closed string.

In order to look for more general solutions we have to resort to some kind of perturbation theory around the "trivial" backgrounds.

We can do that by expanding $G(X)$ and $B(X)$ around a particular point x. This generates terms in the action that are cubic, quartic and so on in the string coordinates.

(For a $\Phi(x)$ which is at most quadratic in x, the action remains quadratic, see an example below).

From the point of view of a 2-dimensional field theory we go from a free theory to an interacting one where the effective coupling is l_s/L, with L the typical length scale of the geometry (scale over which the backgrounds change by $O(1)$). New contributions to the anomaly (or to the anomaly-cancellation conditions) will come as a power expansion in $(l_s/L)^2 \sim \alpha'$.

This method is referred to as the α' expansion.

The possible breaking of Weyl invariance can be formulated in terms of 2D β-functions in analogy with what we do when we describe the breaking of scale invariance in QFT in terms of some functions (called β-functions) of its various couplings.

Since in QST the backgrounds, G, B etc. play the role of couplings there is a β-function associated with each background field.

Setting all β-functions to 0 will give the conditions to be satisfied by the backgrounds in order to preserve all the crucial local 2D symmetries.

In each one of these backgrounds string quantization should be free of pathologies (although it could be quite non-trivial).

A non trivial calculation leads to the following equations to leading non-trivial order in α' (l_s^2):

$$\beta^{\Phi} = \frac{D - D_c}{3} + l_s^2 \left(\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} D_\mu D^\mu \Phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) + O(l_s^4) = 0$$

$$\beta^{G}_{\mu\nu} = l_s^2 \left(R_{\mu\nu} + \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\ \rho\sigma} - 2 D_\mu D_\nu \Phi \right) + O(l_s^4) = 0$$

$$\beta^{B}_{\mu\nu} = l_s^2 \left(D^\rho H_{\mu\nu\rho} - 2 \partial^\rho \Phi H_{\mu\nu\rho} \right) + O(l_s^4) = 0 \quad ; \quad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic}$$

We can now reinterpret in a more satisfactory way the meaning of $D=D_c$. If $D \neq D_c$, there is no solution to the above equations with nearly constant backgrounds. All solutions will have necessarily some fields whose space-time variations are so large to compensate for the extra factor l_s^2. However, in that case, we are not allowed, in principle, to neglect the higher-order corrections (e.g. $O(l_s^4)$) and we cannot be sure, in general, that we do have a solution. There are fortunately exceptions.

The linear-dilaton case

Take a background in which the metric is Minkowskian (but with D≠Dc), B=0, and $\Phi = Q_\mu X^\mu$ where Q_μ is a constant vector. At the order we are computing we find that all the β-functions are zero provided we take:

$$Q_\mu Q^\mu = \frac{D_c - D}{3 l_s^2}$$

This shows how a classically non-Weyl invariant term in the action can be used to give back WI at the quantum level! In fact this solution turns out to be exact (at g=0 level) since a linear dilaton keeps the action quadratic. Necessarily, however, the effective coupling of string theory grows large (either at space- or at time-like infinity) and one has to worry about loop corrections. The WS theory of this background is the so-called Liouville theory and has been studied in great detail (e.g by Gervais).

The effective action of QST

A very interesting property of our β-function equations is that they define what is called a "gradient flow": the β-functions are derivatives of a function(al).

That means that they correspond to the eom that follow from an effective action (i.e. by setting to zero its variation wrt the various fields). Up to the order we have considered the effective action reads:

$$\Gamma_{eff} = -\left(\frac{1}{l_s}\right)^{D-2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{4(D-D_c)}{3l_s^2} + R(G) - 4\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{12} H^2 + \dots \right]$$

1. For the dots, see below.
2. The dilaton appears with the "wrong" sign, but there is nothing wrong with this, see also below.

Some interesting properties of Γ_{eff}

1. The dilaton appears through an overall factor multiplying something that can only depend on its derivatives. This is as expected since, if Φ is constant, the only dependence on Φ must be an overall factor exp $(-2\Phi(1-g))$.

2. Γ_{eff} contains no arbitrary dimensionless parameters and just one dimensionful one, l_s. Actually, even l_s can be eliminated if one uses, instead of G, B, Φ, the rescaled fields $l_s^{-2}\,G$, $l_s^{-2}\,B$, Φ. Again, this is as expected.

3. Γ_{eff} is general covariant. It is also invariant under $B \to B + d\Lambda$. Indeed, B only enters through its field strength $H = dB$. We will come back to the symmetries of Γ_{eff}.

The two meanings of Γ_{eff}

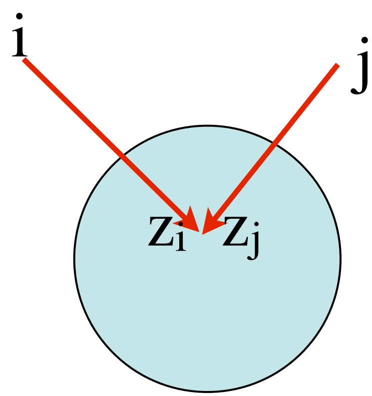
The effective action actually has two distinct meanings. The first is the one we have just said: it generates (as eom) the conditions to be satisfied by the background fields in order to preserve the 2D local symmetries of string theory.

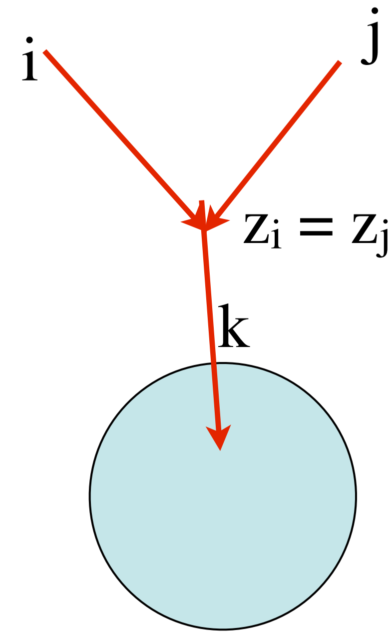
The second meaning is a more familiar one for an effective action: Γ_{eff} can be used to compute the couplings of various massless particles and their scattering amplitudes as an expansion in powers of energy (Cf. zero-slope limit).

It is amazing that these two concepts get related in string theory.

There is an intuitive reason (Polyakov): the breaking of scale-invariance on the WS comes from short-distances on the WS, say $z_i \dashrightarrow z_j$ for two external particles. But this is precisely the region that produces poles in the S-matrix due to the coupling of particles i and j to a third one, k.

This is precisely the coupling one can compute from the effective action.





Where is the extra factor exp(Φ) for each external closed string? This is not hard to get. We have to remember the the single-particle states are associated with properly normalized fields, fields that appear in the action with canonical kinetic terms.

This is not the case for the original background fields because of the overall factor exp(-2Φ). In order to define canonical fields we have to expand the action to quadratic order in the fluctuations and normalize the fluctuation by absorbing in it a factor exp(-Φ). If we now rewrite a generic interaction term coming from Γ_{eff} in terms of the normalized fields we get the claimed result.

$$e^{-2\Phi}(\delta\phi)^n = e^{(n-2)\Phi}(\delta\phi_{can})^n \equiv g_n(\delta\phi_{can})^n$$

A theory of gravity but not Einstein's!

In D dimensions, the analogue of the Einstein-Hilbert action takes the form:

$$\frac{1}{\hbar} S_{EH} = \left(\frac{1}{l_P}\right)^{D-2} \int d^D x \sqrt{-g(x)} \left(\Lambda - \frac{1}{2} R(g)\right) \quad ; \quad 8\pi G_N \hbar \equiv l_P^{D-2}$$

while in QST we found:

$$\Gamma_{eff} = -\left(\frac{1}{l_s}\right)^{D-2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{4(D-D_c)}{3 l_s^2} + R(G) - 4\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{12} H^2 + \dots\right]$$

Are they equivalent up to some field redefinition? The answer is obviously no, even if we set H=0. QST gives a scalar-tensor theory of a Jordan-Brans-Dicke kind!

$$\frac{1}{\hbar} S_{EH} = \left(\frac{1}{l_P}\right)^{D-2} \int d^D x \sqrt{-g(x)} \left(\Lambda - \frac{1}{2} R(g)\right) \quad ; \quad 8\pi G_N \hbar \equiv l_P^{D-2}$$

$$\Gamma_{eff} = -\left(\frac{1}{l_s}\right)^{D-2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{4(D-D_c)}{3 l_s^2} + R(G) - 4\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{12} H^2 + \dots\right]$$

For a constant Φ we can identify l_P^{D-2} with $\exp(2\Phi)\, l_s^{D-2}$ but

a massless dilaton still produces long-range interactions that violate the equivalence principle: the dilaton, having spin zero, couples (non universally!) to mass rather than to energy and produces violations of UFF (T. Taylor and GV).

This is a real threat to QST, making it vulnerable even to long-distance/low-energy experiments. In fact, at tree-level, string theory is already ruled out by present precision tests of the EP (reviewed in last year's seminar by T. Damour).

The two expansions of Γ_{eff}

We have (roughly) seen how quantization of (integrating over) the string coordinates produces potential anomalies that have a natural expansion in powers of l_s.

We have also seen that integrating over the 2D metric produces another expansion in powers of $\exp(2\Phi)$.

Therefore Γ_{eff} has a double perturbative expansion:

$$\begin{aligned}\Gamma_{eff} &= -\left(\frac{1}{l_s}\right)^{D-2}\int d^Dx\sqrt{-G}e^{-2\Phi}\left[\frac{4(D-D_c)}{3l_s^2} + R(G) - 4\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{12}H^2 + O(l_s^2)\right]\\ &+ \left(\frac{1}{l_s}\right)^{D-2}\int d^Dx\sqrt{-G}\,[\ldots] + O(e^{2\Phi})\end{aligned}$$

One expansion has a QFT analogue. The other does not and has the virtue of making the former much better!

This effective action modifies gravity at large distances (which is dangerous but hopefully cured by loop and non-perturbative corrections) and also, of course, at short distances $O(l_s)$.

These latter modifications make loop corrections well defined in the UV. Indeed, one gets their correct order of magnitude, $\exp(2\Phi)$, by computing loops as in a QFT but with a short distance cutoff given by the string length.

Here is an example of a quantum-gravity loop correction:

$$\left(\frac{\text{loop}}{\text{tree}}\right) \sim G_N \Lambda_{UV}^{D-2} \to \left(\frac{l_P}{l_s}\right)^{D-2} = exp(2\Phi)$$

which is roughly of the same order as a gauge-loop correction.