GSO projection and target space supersymmetry

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Plan of the talk

Introduction

- The bosonic string has no ghosts, but its lowest state is a tachyon.
- Furthermore, if it has to describe hadrons, we expect the lowest state to be an almost massless pion (not a tachyon) and the next state to be a spin 1 massive ρ-meson (not a massless photon).
- How can one shift the spectrum of the bosonic string to achieve this?
- The most important result of these attempts was the so called Lovelace-Shapiro amplitude for $\pi\pi$ scattering:

$$A(s,t) = \beta \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_t-\alpha_s)} \quad ; \quad \alpha_s = \alpha_0 + \alpha' s$$

- It contains three parameters: the intercept of the ρ Regge trajectory α_0, the Regge slope α' and the overall constant β.

- The ρ Regge trajectory must give the spin 1 of the ρ meson when $s = m_\rho^2$:

$$\alpha(m_\rho^2) = 1 = \alpha_0 + \alpha' m_\rho^2 \qquad (1)$$

- Impose the Adler's self-consistency condition, that requires the vanishing of the amplitude when $s = t = u = m_\pi^2$ and one of the pions is massless:

$$1 - 2\alpha_{m_\pi^2} = 0 \Longrightarrow 1 - 2\alpha_0 - 2\alpha' m_\pi^2 = 0 \qquad (2)$$

- Eqs. (1) and (2) give the following Regge trajectory for the ρ-meson:

$$\alpha_s = \frac{1}{2}\left[1 + \frac{s - m_\pi^2}{m_\rho^2 - m_{\pi^2}}\right] = 0.48 + 0.885s$$

- The model predicts the masses and the couplings of the resonances that decay in $\pi\pi$ in terms of a unique parameter β.

- The values obtained were in reasonable agreement with the experiments.
- Moreover one could compute the $\pi\pi$ scattering lengths:

$$a_0 = 0.395\beta \qquad a_2 = -0.103\beta$$

  and one found that their ratio is within 10% of the current algebra ratio given by $a_0/a_2 = -7/2$.
- At this point the obvious thing to do was to try to generalize the Lovelace-Shapiro model to the scattering of many pions in order to extract the spectrum of hadrons.
- But nobody has been able to do this with the intercept of the ρ Regge trajectory $\alpha_0 \sim \frac{1}{2}$.
- The generalization of the previous amplitude has been done by Neveu and Schwarz, obtaining the Neveu-Schwarz model, but in this case the intercept of the leading Regge trajectory is $\alpha_0 = 1$ and not $\alpha_0 = \frac{1}{2}$.
- Therefore a consistent extension of the LS model with $\alpha_0 = \frac{1}{2}$ is still lacking and may be impossible to realize.

- From all the attempts to construct more realistic models the most important result is the construction of the Neveu-Schwarz model for bosons and of the Ramond model for fermions.
- Only later on, it was realized that they are part of the same model, called today the Neveu-Schwarz-Ramond model (NSR).
- Both in the NS and R model, together with the string coordinate $x^\mu(\tau, \sigma)$, one introduces a world-sheet Majorana fermion $\psi^\mu(\tau, \sigma)$ that is a vector as x^μ in the target Minkowski space-time and is a spinor in the two-dimensional world-sheet.
- Another important result of this period is the construction of models with world-sheet fermions $\psi^i(\tau, \sigma)$ where i is an index of an internal symmetry.
- In the conformal gauge they contain not only the Virasoro algebra, but also an affine Kac-Moody Lie algebra [Bardacki and Halpern, 1970].
- Strong connection with developments in mathematics.

The Neveu-Schwarz-Ramond (NSR) model

- In the NSR model one introduces together with the bosonic coordinate $x^\mu(\tau, \sigma)$ also a fermionic one $\psi^\mu(\tau, \sigma)$.
- Both coordinates have a vector index μ of the d-dimensional target Minkowski space-time.
- The coordinate $x^\mu(\tau, \sigma)$ is a world-sheet scalar.
- $\psi^\mu(\tau, \sigma)$ is a world-sheet Majorana fermion describing spin degrees of freedom along the string.
- In the previous lectures we have seen that the invariance under reparametrization of the world-sheet coordinates was essential to eliminate from the physical subspace the states with negative and zero norm.
- Since here also the fermionic coordinate has a Lorentz vector index, in order to cancel the negative norm states, we need an additional fermionic symmetry.
- This fermionic symmetry has been called supersymmetry.

- As in the case of the bosonic string, also the NSR model was fully developed before its Lagrangian was constructed and even before it was known that a string theory was its underlying theory.
- The Lagrangian describing the NSR model was constructed only in 1976 just after the construction of the first supergravity.
- It is given by:

$$L = T\sqrt{-g}\left[-\frac{1}{2}\partial_\alpha x\cdot\partial_\beta x\, g^{\alpha\beta} - \frac{i}{2}\bar{\psi}\gamma^\alpha\cdot\partial_\alpha\psi + \frac{i}{2}\bar{\chi}_\alpha\gamma^\beta\partial_\beta x\cdot\gamma^\alpha\psi + \frac{1}{8}(\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\psi)\cdot(\bar{\chi}_\beta\psi)\right] \quad ; \quad g_{\alpha\beta} = e^a{}_\alpha e^b{}_\beta \eta_{ab} \;\; ; \;\; \gamma_\alpha = \gamma^a e^b{}_\alpha \eta_{ab}$$

[Brink, DV, Howe and Deser and Zumino, 1976]

- Together with the bosonic and fermionic string coordinates x^μ and ψ^μ, it contains the world-sheet metric $g_{\alpha\beta}$ and the gravitino χ_α.
- If we want to include fermions in the Polyakov action we have to work with the vierbein $e^a{}_\alpha$ (or rather zweibein) that has a curved index α and a flat index a rather than with the metric tensor $g_{\alpha\beta}$.

- The metric tensor is constructed from the vierbein that is also necessary to transform the flat index of the Dirac γ matrices into a curved index.
- L is invariant under an arbitrary reparametrization of the world-sheet coordinates and under an arbitrary local world-sheet supersymmetry transformation.
- As in the bosonic string one can choose a gauge, called superconformal gauge:

$$g_{\alpha\beta} = \rho(\xi)\eta_{\alpha\beta} \;\; ; \;\; \chi_\alpha = \gamma_\alpha \chi(\xi)$$

- In this gauge, the Lagrangian becomes that of free bosons and fermions:

$$L = -\frac{T}{2}\left[\partial_\alpha x \cdot \partial^\alpha x + i\bar{\psi}\rho^\alpha \partial_\alpha \cdot \psi\right] \;\; ; \;\; \{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$$

- To avoid confusion with the target space γ-matrices, we call ρ^α the two-dimensional world-sheet γ-matrices.

- It is invariant under the following world-sheet supersymmetry transformations:

$$\delta X^{\mu} = \bar{\alpha}\psi^{\mu} \quad , \quad \delta\psi^{\mu} = i\rho^{\alpha}\partial_{\alpha}X^{\mu}\alpha \quad , \quad \delta\bar{\psi} = -i\bar{\alpha}\rho^{\alpha}\partial_{\alpha}X^{\mu}$$

  if α satisfies the following equation:

$$\rho^{\beta}\rho^{\alpha}\partial_{\beta}\alpha = 0 \Longrightarrow \alpha_{+}(\tau+\sigma) \quad ; \quad \alpha_{-}(\tau-\sigma) \quad ; \quad \alpha_{\pm} = \frac{1 \pm \rho^{3}}{2}\alpha$$

- The fermionic Noether current associated with this symmetry is given by:

$$j^{\alpha} = \rho^{\beta}\rho^{\alpha}\partial_{\beta}x \cdot \psi$$

- If the Eqs. of motion for x^{μ} and ψ^{μ} are satisfied:

$$\eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}x = 0 \quad ; \quad \rho^{\alpha}\partial_{\alpha}\psi(\tau,\sigma) = 0$$

  the fermionic Nöther current is conserved:

$$\partial_{\alpha}j^{\alpha} = 0$$

- Then we can proceed as in the case of the bosonic string.
- In this covariant gauge the Eqs. of motion and the boundary conditions for the bosonic coordinate are the same as before.
- The fermionic coordinate instead satisfies the two-dimensional Dirac equation:

$$\rho^\alpha \partial_\alpha \psi(\tau, \sigma) = 0$$

- In the case of an open string, one gets two possible boundary conditions:

$$\psi_-(\tau, 0) = \psi_+(\tau, 0) \quad ; \quad \psi_-(\tau, \pi) = \eta \psi_+(\tau, \pi)$$

  where $\eta = \pm 1$ and $\psi_\pm = \frac{1 \mp \rho^3}{2} \psi$ with $\rho^3 \equiv \rho^0 \rho^1$.
- The model has two sectors:
  the NS sector for $\eta = -1$ and the R sector for $\eta = +1$.
- The NS sector contains space-time bosons, while the R sector contains space-time fermions.

- The inclusion of fermionic coordinates reduces the space-time dimension from $d = 26$ to $d = 10$.
- The mass spectrum of the NS sector is given by:

$$\alpha' m^2 = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n + \sum_{r=\frac{1}{2}}^{\infty} r \psi_r^\dagger \cdot \psi_r - \frac{1}{2}$$

- The oscillators satisfy the algebra:

$$[a_n^\mu, a_m^{\dagger\nu}] = \delta_{nm}\eta^{\mu\nu} \quad ; \quad \{\psi_r^\mu, \psi_s^{\dagger\nu}\} = \delta_{rs}\eta^{\mu\nu}$$

- The physical states are defined as those satisfying the conditions:

$$L_n|Phys.\rangle = G_r|Phys.\rangle = 0 \quad ; \quad n = 1, 2 \ldots \quad ; \quad r = \frac{1}{2}, \frac{3}{2} \ldots$$

- The lowest state is given by the oscillator vacuum and is again a tachyon:

$$|0, P\rangle \quad ; \quad -\alpha' P^2 = \alpha' m^2 = -\frac{1}{2}$$

- The next state is a massless vector:

$$\psi^{\dagger\mu}_{\frac{1}{2}}|0, P\rangle \quad ; \quad k^2 = 0$$

- We can proceed in the same way with the higher mass levels.
- In particular, the states with an odd number of world-sheet fermions have an integer value of $\alpha' m^2$
- Those with an even number of world-sheet fermions have a half-integer value of $\alpha' m^2$.
- One can define a world-sheet fermion number:

$$(-1)^F \quad \text{where} \quad F = \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \psi_r - 1$$

- The states with an odd (even) number of world-sheet fermions are even (odd) under the action of $(-1)^F$.

- In particular, the tachyon is odd under $(-1)^F$, while the gauge boson is even.
- Let us turn now to the R sector.
- In the R sector the fermionic oscillators ψ_n^μ have, unlike the NS sector, an integer index n and satisfy the algebra:

$$\{\psi_n^\mu, \psi_m^{\dagger\nu}\} = \delta_{nm}\eta^{\mu\nu}$$

- In particular, there is a zero mode that satisfies the same anti-commutation relations as the Dirac γ-matrices;

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \Leftrightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

  apart from an overall normalization (factor 2).
- This means that the ground vacuum state $|0, A>$ has a ten-dimensional Dirac spinor index A.

- Therefore, it is a space-time fermion.
- The mass spectrum in the R sector is given by:

$$\alpha' m^2 = \sum_{n=1}^{\infty} a_n^\dagger \cdot a_n + \sum_{n=1}^{\infty} n \psi_n^\dagger \cdot \psi_n \Longrightarrow L_0|\psi\rangle = 0$$

- The physical states must satisfy the conditions:

$$L_n|Phys.\rangle = F_n|Phys.\rangle = 0 \quad ; \quad n = 1, 2 \ldots$$

- They must also satisfy the additional on-shell condition:

$$F_0|\psi\rangle = 0 \quad ; \quad L_0 = F_0^2$$

- It is the string extension of the Dirac equation.
- It is also the extension of the fact that the Klein-Gordon operator is the square of the Dirac operator:

$$(\gamma^\mu \partial_\mu)^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

- The lowest state of the R sector is a massless spinor given by:

$$|0, A> \qquad m^2 = 0$$

- Both this state and all excited states are ten-dimensional spinors.
- But what kind of spinors: Dirac, Majorana or Weyl spinors?
- This question remained unanswered for some time and we will come back to it.
- As in NS sector the spectrum consists of states that are even and states that are odd under the action of the fermion number operator.
- Also in the R sector one can introduce a fermion number operator given by:

$$(-1)^F = \gamma_{11}(-1)^{F_R} \quad \text{where} \quad F_R = \sum_{n=1}^{\infty} \psi_{-n} \cdot \psi_n$$

$$\gamma_{11} \equiv 2^5 \psi_0^0 \psi_0^1 \dots \psi_0^9 = \gamma^0 \gamma^1 \dots \gamma^9$$

where γ_{11} is the chirality operator in ten dimensions,

Historical intermezzo

- After the construction of the NSR model it was clear that in this model the conformal algebra of the dual resonance model was extended with fermionic operators.
- Today this extension is called super-conformal algebra.
- This gave hope that also in this model the ghosts could be eliminated from the physical spectrum.
- It became then clear that this invariance was the invariance of a free theory involving a set of equal number of scalar and fermion fields in two dimensions [Gervais and Sakita, 1971].
- This symmetry was generalized to a four-dimensional space-time by Wess and Zumino in 1973.
- It was then extended to four-dimensional gauge theories by Ferrara and Zumino in 1974.
- Supersymmetry has been advocated to solve the hierarchy problem and a supersymmetric extension of the Standard model has been constructed.

- Supersymmetric particles may be observed at the Large Hadron Collider (LHC) in Geneva.
- Supergravity that is a supersymmetric extension of Einstein's gravity was constructed in 1976 independently by Ferrara, Friedman, Van Niewenhuizen and by Deser and Zumino.
- Immediately after, in 1976, the correspondent of the Polyakov action for the NSR model was constructed.

The Gliozzi-Scherk-Olive (GSO) projection

- At this stage the NSR model was better than the bosonic string because included not only bosons but also space-time fermions.
- But it was not much more realistic than the bosonic string for describing the hadrons.
- It still contained massless vectors and spinors that do not appear in the hadronic spectrum.
- It contained also a tachyon as the bosonic string.
- It became more and more clear that string theory could not be the theory for hadrons.
- It was proposed by Scherk and Schwarz in 1974 that it could be instead a theory unifying all interactions.
- But in 1976 there was still a tachyon in the spectrum.

- In 1976 GSO proposed (here in Paris) to truncate the spectrum of the NSR model keeping only the states $|\psi\rangle$ that are even under the action of the fermion number operator:

$$(-1)^F|\psi\rangle = |\psi\rangle$$

- This is called GSO projection.
- For counting the states it is convenient to work in the light-cone gauge.
- In the NS sector, the spectrum is given the spectrum of physical states is given by:

$$\alpha' M^2 = \sum_{i=1}^{8}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r\,\psi^i_{-r}\psi^i_r\right) - \frac{1}{2} \equiv N - \frac{1}{2}$$

- The degeneracy of states at a certain mass level (after GSO projection) can be obtained from the partition function:

$$
\begin{aligned}
Z_{NS}^{GSO} &= \mathrm{Tr}\,(q^{N-1/2}\frac{1+(-1)^F}{2}) \\
&= \frac{1}{2}q^{-1/2}\left[\prod_{n=1}^{\infty}\left(\frac{1+q^{n-1/2}}{1-q^n}\right)^8 - \prod_{n=1}^{\infty}\left(\frac{1-q^{n-1/2}}{1-q^n}\right)^8\right] \\
&= \sum_{n=0}^{\infty} c_n q^n = 8 + c_1 q + c_2 q^2 + \ldots
\end{aligned}
$$

- c_n gives the degeneracy of states at the level with $\alpha' m^2 = n$.
- The GSO projection eliminates the states of the NS sector having half-integer values of $\alpha' m^2$ keeping only those with integer values of $\alpha' m^2$.
- In particular, the tachyon is odd under $(-1)^F$ and is therefore eliminated by the GSO projection!!

Computation of the partition function

- The bosonic part of the partition function is given by:

$$Z_B(q) \equiv \prod_{n=1}^{\infty} \prod_{i=1}^{8} \mathrm{Tr} \left(q^{n a^{\dagger}_{ni} a_{ni}} \right) = \prod_{n=1}^{\infty} \prod_{i=1}^{8} \sum_{m=0}^{\infty} \langle m | q^{n a^{\dagger}_{ni} a_{ni}} | m \rangle$$

- The state $|m\rangle$ is an eigenstate of the number operator with eigenvalue equal to m:

$$a^{\dagger}_{ni} a_{ni} |m\rangle = m|m\rangle \quad ; \quad \langle m|m\rangle = 1$$

- Then we get:

$$Z_B(q) = \prod_{n=1}^{\infty} \prod_{i=1}^{8} \sum_{m=0}^{\infty} \langle m | q^{nm} | m \rangle = \prod_{n=1}^{\infty} \prod_{i=1}^{8} \sum_{m=0}^{\infty} q^{nm}$$

$$= \prod_{n=1}^{\infty} \prod_{i=1}^{8} \frac{1}{1 - q^n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^8}$$

- The fermionic partition function is given by:

$$
\begin{aligned}
Z_F(q) &\equiv \prod_{n=1}^{\infty} \prod_{i=1}^{8} \mathrm{Tr}\left(q^{(n-\frac{1}{2})\psi^{\dagger}_{n-\frac{1}{2},i}\psi_{n-\frac{1}{2},i}} \right) \\
&= \prod_{n=1}^{\infty} \prod_{i=1}^{8} \sum_{m=0}^{1} \langle m | q^{(n-\frac{1}{2})m} | m \rangle = \prod_{n=1}^{\infty} \prod_{i=1}^{8} (1 + q^{n-\frac{1}{2}}) \\
&= \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^8
\end{aligned}
$$

- In this case $|m\rangle$ satisfies the eigenvalue equation:

$$
\psi^{\dagger}_{n-\frac{1}{2},i}\psi_{n-\frac{1}{2},i}|m\rangle = m|m\rangle
$$

- Let us now turn our attention to the R sector.
- The mass spectrum in the R sector in the light-cone gauge is given by:

$$\alpha' M^2 = \sum_{i=1}^{8} \left(\sum_{n=1}^{\infty} n a_n^{\dagger i} a_n^i + \sum_{n=1}^{\infty} n \psi_{-n}^i \psi_n^i \right) \equiv N_R$$

- In this sector each state is a ten-dimensional spinor.
- Since the fermionic coordinate is real we expect the spinors to be Majorana spinors.
- A Dirac spinor in ten dimensions has $2^5 = 32$ physical degrees of freedom, while a Majorana or a Weyl spinor have only 16 physical components.
- In $d = 10$ it is possible to have Weyl-Majorana spinors that have only 8 degrees of freedom.

A detour on spinors

- A Dirac fermion in four dimensions has four complex components corresponding to 8 real degrees of freedom.
- When we impose the Dirac equation we are left with 4 real degrees of freedom.
- They are the physical degrees of freedom of a Dirac fermion.
- They correspond to the two states of electron and positron and each of them can have helicity $\pm\frac{1}{2}$.
- In Dirac theory, it exists a charge conjugation operator that, acting on a spinor ψ, gives its charge conjugate ψ^{c}:

$$\psi_D^c = C\bar{\psi}_D^T \quad ; \quad C\gamma_\mu^T C^{-1} = -\gamma_\mu$$

  T means the transposed matrix.
- C connects the field of an electron to that of a positron.

- A Majorana spinor satisfies the property of self-conjugation:

$$\psi_M^c = \psi_M$$

- It has only half of the degrees of freedom of a Dirac spinor.
- It has only two on-shell degrees of freedom corresponding to the two helicities $\pm\frac{1}{2}$.
- A left-handed Weyl spinor is obtained from a Dirac spinor by:

$$\psi_W = \frac{1-\gamma_5}{2}\psi_D \quad ; \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

- It has two on-shell degrees of freedom corresponding to the two helicities $\pm\frac{1}{2}$.
- They can all be expressed in terms of two-dimensional Weyl spinors:

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \; ; \; \psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \; ; \; \psi_W = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}$$

- The previous considerations can be generalized to a d-dimensional space-time with d even.
- An on-shell Dirac spinor has $2^{d/2}$ physical degrees of freedom.
- On-shell Majorana and a Weyl spinors have $\frac{2^{d/2}}{2}$ physical degrees of freedom.
- There are certain dimensions where we can impose both the Weyl and the Majorana conditions.
- This happens for $d = 2, 10, 18 \ldots$.
- In particular, in ten dimensions we can have Weyl-Majorana spinors that have only 8 on shell degrees of freedom.

- If we perform also in the R sector the GSO projection keeping only those states that are even under $(-1)^F$, the lowest massless state becomes a Weyl-Majorana spinor.
- Actually the two theories that one obtains choosing the ground state to be a left- or a right-handed Weyl spinor are equivalent.
- Then the partition function in the R sector becomes:

$$Z_R^{GSO} = \mathrm{Tr}\,(q^{N_R}\frac{1+(-)^F}{2}) = 8\prod_{m=1}^{\infty}\left(\frac{1+q^m}{1-q^m}\right)^8$$

where the term with $(-1)^F$ gives no contribution.

- It turns out that

$$Z_{NS}^{GSO} = Z_R^{GSO}$$

- This follows from the "aequatio identica satis abstrusa" (Jacobi):

$$\frac{1}{2}q^{-1/2}\left[\prod_{m=1}^{\infty}\left(\frac{1+q^{m-1/2}}{1-q^m}\right)^8-\prod_{m=1}^{\infty}\left(\frac{1-q^{m-1/2}}{1-q^m}\right)^8\right]$$

$$=8\prod_{m=1}^{\infty}\left(\frac{1+q^m}{1-q^m}\right)^8$$

- Actually the proof of the previous relation is given as an exercise in the book by Whittaker and Watson that the authors of GSO found in the library of École Normale Supérieure in Rue Lhomond.
- It implies that, at each mass level, we have the same number of bosonic and fermionic physical degrees of freedom.
- This is a necessary condition for supersymmetry.
- It can be shown that with the GSO projection the NSR model is supersymmetric at the string level.
- For the first time one has a string model without a tachyon in the physical spectrum!!

Type IIA and IIB superstrings

- Up to now we have considered the GSO projection in the open superstring theory that is also called type I superstring.
- In the following we want to discuss two closed string theories, called type IIA and IIB.
- In the case of a closed string the equations of motion of the fermionic coordinate are:

$$(\partial_\tau + \partial_\sigma)\,\psi^\mu_-(\tau,\sigma) = 0 \quad ; \quad (\partial_\tau - \partial_\sigma)\,\psi^\mu_+(\tau,\sigma) = 0$$

where

$$\psi^\mu_\pm = \frac{1 \mp \rho^3}{2}\psi^\mu \quad \text{with} \;\; \rho^3 \equiv \rho^0 \rho^1$$

- In a closed string the fermionic coordinates $\psi_\pm$ are independent from each other.
- They can be either periodic or anti-periodic.
- This amounts to impose the following conditions:

$$\psi^\mu_-(0,\tau) = \eta_3 \psi^\mu_-(\pi,\tau) \qquad \psi^\mu_+(0,\tau) = \eta_4 \psi^\mu_+(\pi,\tau)$$

- Therefore, we have four different sectors according to the two values that η_3 and η_4 take

$$\begin{cases} \eta_3 = \eta_4 = 1 \Rightarrow (\mathrm{R} - \mathrm{R}) \\ \eta_3 = \eta_4 = -1 \Rightarrow (\mathrm{NS} - \mathrm{NS}) \\ \eta_3 = -\eta_4 = 1 \Rightarrow (\mathrm{R} - \mathrm{NS}) \\ \eta_3 = -\eta_4 = -1 \Rightarrow (\mathrm{NS} - \mathrm{R}) \end{cases} .$$

- The NS-NS and R-R sectors consist of space-time bosons, while the R-NS and NS-R sectors contain fermions.
- The mass spectrum of these sectors is given by:

$$\frac{\alpha'}{2} m^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{t>0} t\psi_{-t} \cdot \psi_t - a_0$$

$$+ \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n + \sum_{t>0} t\widetilde{\psi}_{-t} \cdot \widetilde{\psi}_t - \tilde{a}_0$$

- where t is half-integer (integer) in the NS (R) sector and

$$a_0 = \frac{1}{2} \quad \text{for the NS sector} \;\; ; \quad a_0 = 0 \quad \text{for the R sector}$$

- We must also impose the level matching condition:

$$\left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{t>0} t\psi_{-t} \cdot \psi_t - a_0 \right.$$

$$\left. - \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n + \sum_{t>0} t\widetilde{\psi}_{-t} \cdot \widetilde{\psi}_t + \tilde{a}_0 \right) |\psi\rangle = 0$$

- In each of the sectors we perform the GSO projection as before.
- There is no tachyon in any NS sector and the ground state fermion in any R sector is a Weyl-Majorana spinor with only 8 physical degrees of freedom.
- The massless states of the NS-NS sectors are a graviton, a two-index antisymmetric tensor $B_{\mu\nu}$ and a scalar dilaton.
- This sector coincides with the closed string sector of the bosonic string, apart from a different space-time dimension ($d = 10$ instead of $d = 26$)
- The massless states of the NS-R sector are a gravitino with spin $\frac{3}{2}$ and a dilatino with spin $\frac{1}{2}$.

- Also the massless states of the R-NS sector are a gravitino and a dilatino.
- However the two gravitinos can have the same or opposite chirality.
- This depends on if we impose in the two R sectors the same GSO or opposite GSO projections.
- In the case of the same GSO projection we obtain a chiral theory called type IIB superstring.
- With opposite projection we obtain a non-chiral theory called type IIA superstring.
- Finally we have the R-R sector.
- In type IIB the R-R sector contains a scalar field C_0, a two-index antisymmetric potential C_2 and a four-index antisymmetric potential C_4 with self-dual field strength $\tilde{F}_5 = F_5$.
- In type IIA we have a gauge vector C_1 and a three-index antisymmetric potential C_3.

The closed string sector of type I superstring

- In the previous section we have determined the massless spectrum of two type II closed superstring theories.
- Previously, by means of the GSO projection on the NSR open string model, we have obtained the massless spectrum of what is called the open string sector of type I superstring.
- As we have seen already in the bosonic string, the non-planar loop introduces in the open string a closed string sector.
- One can have a theory of only closed strings, but not a theory of only open strings!
- Therefore, besides the open string sector, type I superstring has also a closed string sector.
- It can be obtained from that of type IIB by what is called an "orientifold" projection.
- This projection consists in truncating the spectrum of type IIB eliminating some of the states.

- I will not describe this truncation in detail here, but I will only give the results.
- In particular, in the NS-NS sector the graviton and the dilaton are kept, while the two-index antisymmetric tensor $B_{\mu\nu}$ is eliminated.
- In the NS-R and R-NS we keep only one of the two gravitinos and dilatinos.
- Finally in the R-R sector we keep the two-index antisymmetric tensor C_2, while C_0 and C_4 are projected out.

A non-abelian gauge symmetry

- Let us go back to the open superstring and discuss how to introduce a non-abelian gauge symmetry.
- It is introduced by requiring that the string states do not only have the degrees of freedom corresponding to the harmonic oscillators and the momentum

$$|\alpha, P\rangle \Longrightarrow |\alpha, P\rangle \ \lambda_{a\bar{b}}$$

but also have two additional indices $(a, \bar{b})$, one transforming as the N and the other transforming as the $\bar{N}$ fundamental representations of $U(N)$:

$$\lambda_{a\bar{b}} \to \sum_{c,\bar{c}=1}^{N} U_{ac} \lambda_{c\bar{c}} U^{\dagger}_{\bar{c}\bar{b}}$$

- λ transforms according to the adjoint representation of $U(N)$.
- One index corresponds to the N degrees of freedom of a "quark" and the other to the $\bar{N}$ of the "anti-quark" located at the two end-points of the string as represented in the duality diagram.
- λ is a hermitian matrix and therefore, it can always be written as a linear combination of the generators of $U(N)$ in the fundamental representation:

$$\lambda_{a\bar{b}} = \sum_{A=1}^{N^2} c_A T^A_{a\bar{b}}$$

- In particular, a massless gauge boson is now described by the state:

$$c_A T^A_{a\bar{b}} \; \epsilon^i_\mu (\psi^\dagger_{1/2})^\mu |0, P\rangle$$

- ϵ^i_μ describes the polarization, while c_A describes the gauge degrees of freedom.

- Since the two end-points transform according to two different representations of $U(N)$, the open strings are oriented.
- Open strings with different orientations are two different string states.
- It turns out that the type I is a theory of un-oriented open strings.
- This means that the two end-points must transform according to the same representation of the gauge group.
- In the case of $U(N)$ the product $N \times N$ does not contain the adjoint representation.
- No good for describing gauge bosons.
- If we consider $SO(N)$ or $Sp(N)$, then the product $N \times N$ contains the adjoint but also another representation.

- On the other hand, if we remember that also in this case we have to perform the "orientifold projection", we can see that we are left only with the adjoint representation in both cases.
- Being type I a theory with chiral fermions one has potentially gauge anomalies.
- in the next lecture we will see that only the case of the gauge group $SO(32)$ is anomaly free.

- Type IIB
  NS-NS: $G_{\mu\nu}, \phi, B_{\mu\nu}$; R-R: $C_0, C_{2\mu\nu}, C_{4\mu\nu\rho\sigma}$
  R-NS+NS-R: 2 gravitinos+2 dilatinos
  Two gravitinos and dilatinos have the same chirality, but gravitinos and dilatinos have opposite chirality: chiral theory.
- Type IIA
  NS-NS: $G_{\mu\nu}, \phi, B_{\mu\nu}$; R-R: $C_{1\mu}, C_{3\mu\nu\rho}$
  R-NS+NS-R: 2 gravitinos+2 dilatinos
  Two gravitinos and dilatinos have opposite chirality: non-chiral theory.
- Type I
  Closed string sector: $G_{\mu\nu}, \phi, C_{2\mu\nu}$, 1 gravitino, 1 dilatino.
  Open string sector: Gauge boson+ gaugino with gauge group $SO(32)$: chiral theory.

Conclusions

- In this seminar we have discussed the three superstring theories that were constructed before 1985.
- Only after the first string revolution two other superstring theories were constructed.
- They are called heterotic strings.
- They are closed string theories, but, unlike type II theories, they contain a gauge theory.
- There exist two heterotic string theories, one with gauge group $SO(32)$ and the other with gauge group $E_8 \times E_8$.
- Other choices of gauge symmetries will generated gauge and gravitational anomalies.
- Those five superstring theories are all consistent string theories in ten-dimensional Minkowski space-time.
- They are perturbatively inequivalent and supersymmetric in $d = 10$.

- This has generated a puzzle for many years: If string theory is a theory of everything why do we have 5 theories instead of just 1?
- It turns out that they are actually part of a unique 11-dimensional theory, called M-theory.
- But it is not easy to investigate this theory because an explicit formulation of M theory is lacking.