Particules Élémentaires, Gravitation et Cosmologie
Année 2009-'10

Théorie des Cordes: une Introduction

Cours VII: 26 février 2010

Généralisations de Neveu-Schwarz & Ramond

- Classical vs. quantum strings
- Shortcomings of the bosonic string
- Adding world-sheet fermions: NS vs. R
- Intercepts and critical dimensions in the NSRM

Classical vs. Quantum strings

- We have already seen that there are non-trivial constraints on the dimensionality of space-time in which a quantized string can consistently propagate.
- We have also seen that the spectrum of quantum strings is strongly constrained (α_0=1).
- These are just examples of properties that distinguish classical from quantum strings. We will find several such properties later in the course but, for the moment, let's focus on two crucial ones.

Target-space scale invariance

- It is trivial to check that classical string theory is scale-invariant in Minkowski spacetime.
- Given a solution of the equations of motions and constraints we can generate another solution by multiplying all the string coordinates by the same arbitrary factor k.
- The new solution is a string whose M (J) is k times (k^2 times) larger. The ratio J/M^2 remains the same.
- We can also trivially change T (S_{NG} is rescaled)

- It follows that, starting from any classical solution we can rescale it and make the string arbitrarily light while its angular momentum becomes arbitrarily small.
- Note that such a scale invariance is broken if the string moves in a non-trivial background that contains a length scale L (e.g. a Hubble or a Schwarzschild radius).
- In that case we can only rescale simultaneously the size of the string and L but their ratio is an essential dimensionless parameter: large and small strings (wrt L) behave very differently.

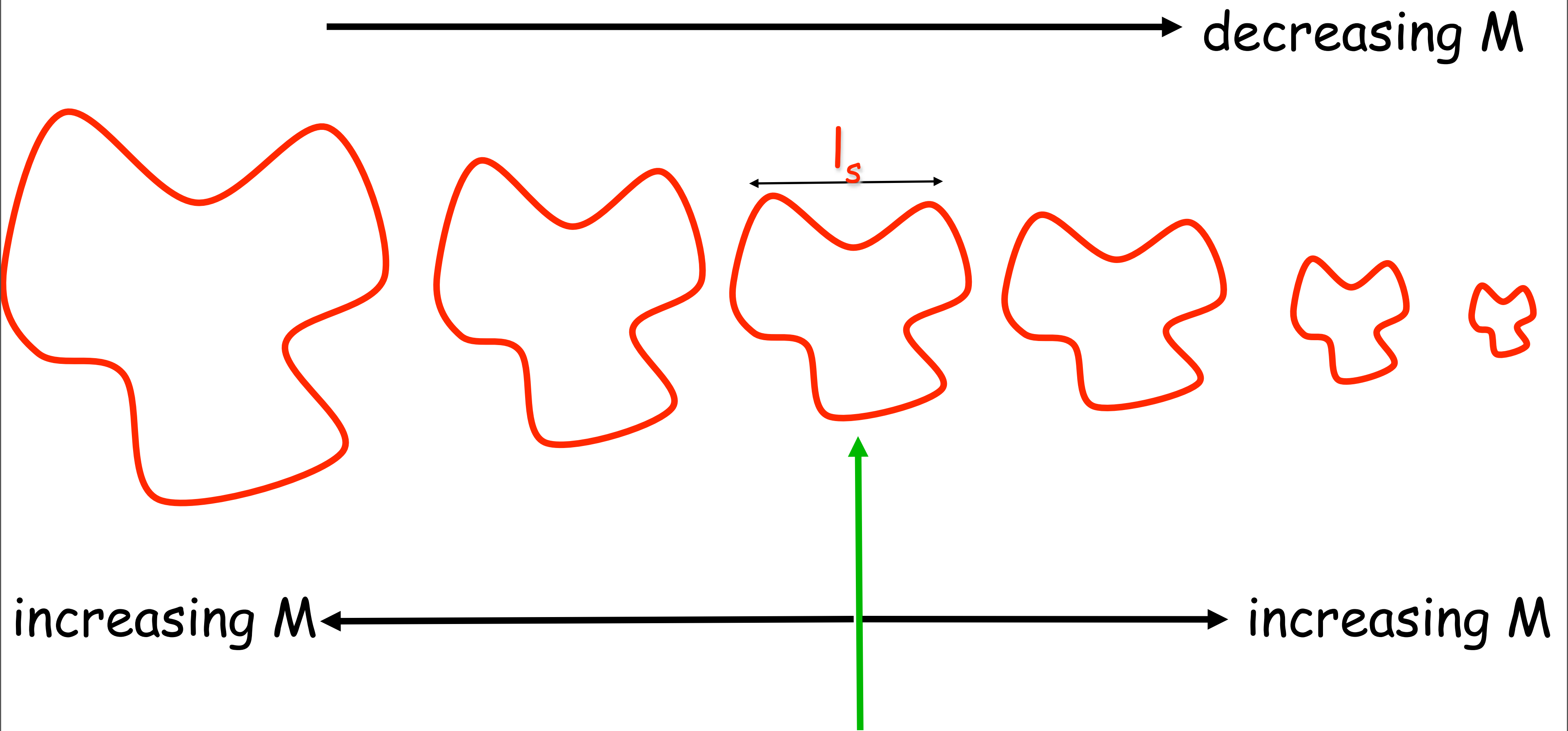
- In the quantum theory the crucial quantity (Cf. quantization by Feynman's path integral) is the dimensionless quantity S/h. Since:

$$\frac{1}{\hbar} S_{NG} = -\frac{T}{\hbar}(\text{Area swept}) \equiv -\frac{1}{\pi l_s^2}(\text{Area swept}) \quad ; \quad l_s^2 \equiv 2\alpha' \hbar$$

quantization has introduced a fundamental length, l_s. The ratio of a string's size and l_s is now a crucial dimensionless parameter (even in Minkowski: in curved spacetime there will be further relevant ratios...).

- This fundamental quantum length enters string theory in many ways. It is the characteristic size of a (minimal-mass) string (Cf. ground st. of h.o.).
- NB: In the string literature l_s^2 and $2\alpha'$ are usually not distinguished. In string units they are both set to 1.

Without QM strings become lighter and lighter as they shrink



With QM strings are lightest when their size is l_s

Angular momentum bound

A classical string cannot have angular momentum without having a finite length, hence a finite mass. The rigid-rod solution maximizes J/M^2. A quantum string, instead, can have up to two units of angular momentum without gaining mass. The fact that this is a quantum effect is clear:

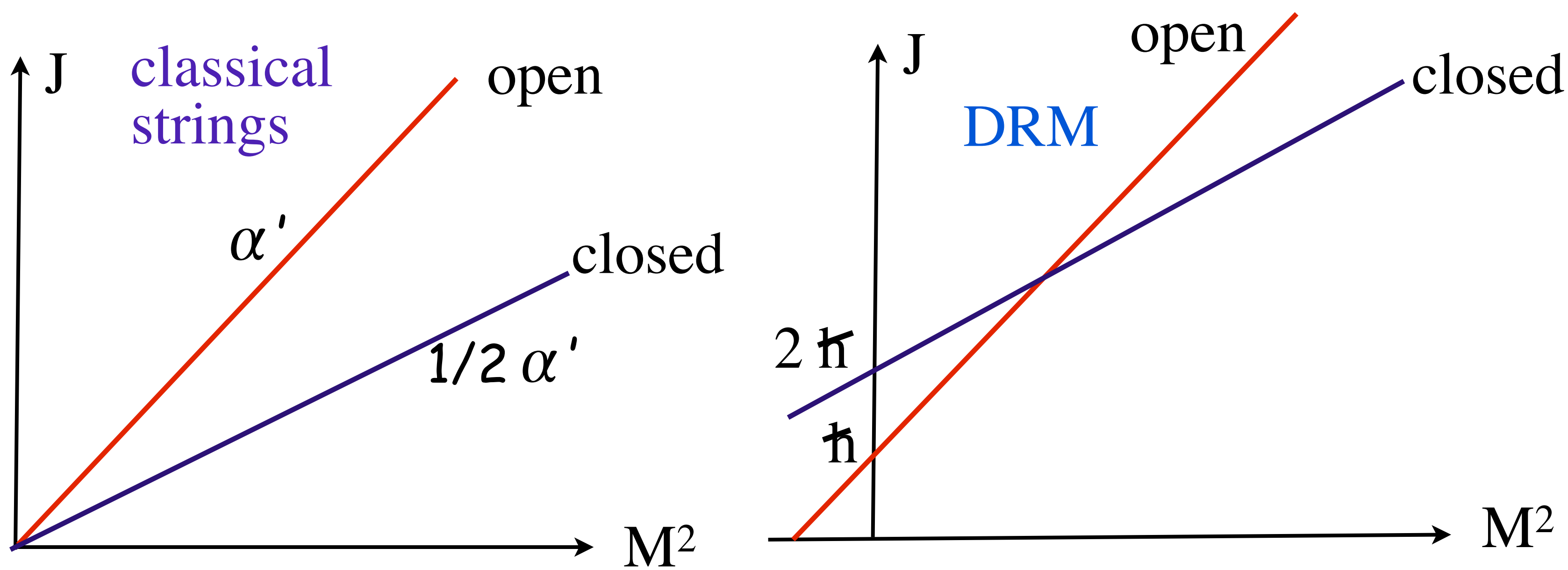
after consistent regularization

$$\frac{M^2}{2\pi T} \geq J + \hbar \sum_1^\infty \frac{n}{2} = J - \alpha_0 \hbar \qquad \alpha_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$$

So far we have seen the examples of α_0 =1 and 2 in the bosonic string. Later today we shall see the case of massless half-integer spin strings.

The classical limit corresponds to taking strings that are large in l_s units. They correspond to large occupation numbers i.e. to heavy strings. This is indeed where the two graphs below agree with each other.

On the other hand for small, light strings the classical picture fails. Only quantum strings can be interesting for a unified description of fundamental forces and particles!



Shortcomings of the bosonic string

1. Presence of a tachyon
2. Absence of fermions
3. $D \neq 4$

Presence of a Tachyon

Tachyons are a problem but not necessarily a killer. In QFT a tachyon is a sign of instability of the (false) vacuum state around which we carry out quantization and perturbation theory.

The Higgs model for the SSB of SU(2)xU(1) is the most famous example of how we can use a tachyon to our own advantage.

Unfortunately is was (and still is) not at all clear how to change the vacuum in the DRM or in String Theory.

This is why people tried to find tachyon-free models with a nice perturbative expansion.

Absence of fermions

This was a very obvious shortcoming of the bosonic DRM. After all one wanted to describe also protons and neutrons besides mesons!

$$D \neq 4$$

This was another big disappointment. People wanted to find out whether by adding different degrees of freedom one could get down to D=4.

Encouraging early signs

For consistency (absence of ghosts) the bosonic DRM had to have the wrong intercept, $\alpha_0 = 1$. On the other hand the Lovelace-Shapiro model for $\pi\,\pi$ scattering was phenomenologically appealing.

It had no tachyon and several other virtues (see also today's seminar) and raised great hopes at the beginning.

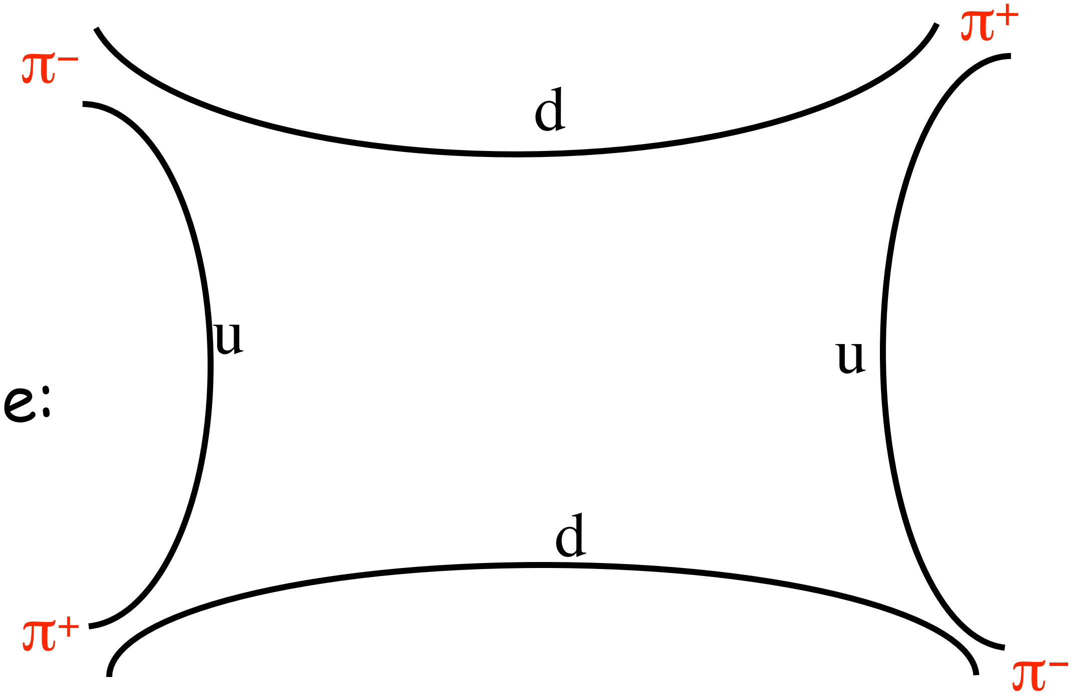
A similar success came from the study of the Dalitz plot for $\omega \dashrightarrow 3\pi$ ($s, t, u > 0$ and $s+t+u = m_\omega^2$)

$$A(\pi_1\pi_2 \to \pi_3\pi_4) \;=\; g^2 \sum_{\text{cyclic perm.}} Tr[\tau_i\tau_j\tau_k\tau_l] \, \frac{\Gamma\left(1-\alpha(s_{ij})\right)\Gamma\left(1-\alpha(s_{jk})\right)}{\Gamma\left(1-\alpha(s_{ij})-\alpha(s_{jk})\right)}$$

$$\alpha(s) \;=\; \alpha_\rho(s) \sim 0.5 + \alpha' s \; ; \; \alpha' \sim 0.9 \ \mathrm{GeV}^{-2}$$

Tr[$\tau_i \tau_j \tau_k \tau_l$] are called Chan-Paton factors. They will be very important for some properties of open string theory

example:



$$A(\pi^+\pi^- \to \pi^+\pi^-) = g^2 \frac{\Gamma\left(1-\alpha(s)\right)\Gamma\left(1-\alpha(t)\right)}{\Gamma\left(1-\alpha(s)-\alpha(t)\right)}$$

Adding world-sheet fermions

Even before the string reinterpretation of the DRM Neveu & Schwarz and Ramond tried to generalize the operator formalism by adding to the bosonic field $Q(z)$ a Grassmann (i.e. anticommuting) field $\psi(z)$. Let us describe it in string theory language.

The "Polyakov" action for the bosonic string can be generalized by adding to the string coordinate $X^{\mu}(\xi)$ a fermionic "coordinate" $\psi^{\mu}{}_{a}(\xi)$ which is a two-component spinor in 2-dimensions (a world-sheet spinor) but, like X^{μ}, a spacetime vector (in D dimensions).

This very nice formulation (Brink-Di Vecchia-Howe; Deser and Zumino, 1976) came much later, after the discovery of supersymmetry and supergravity. The earliest (gauge-fixed) formulation preceeded SUSY and actually led to its discovery in the West (see seminar).

$$
\begin{aligned}
S &= S^{\mathrm{b}} + S^{\mathrm{f}} \ ; \ S^{\mathrm{b}} = -\frac{T}{2}\int d^2\xi\sqrt{-g}\ g^{\alpha\beta}\ \partial_\alpha X^\mu \cdot \partial_\beta X_\mu \\
S^{\mathrm{f}} &= -\frac{T}{2}\int d^2\xi\sqrt{-g}\left[i\bar{\psi}^\mu\gamma^\alpha \cdot \partial_\alpha\psi_\mu - i\bar{\chi}_\alpha\gamma^\beta\partial_\beta X^\mu \cdot \gamma^\alpha\psi_\mu\right. \\
&- \left. 1/4(\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\psi^\mu)\cdot(\bar{\chi}_\beta\psi_\mu)\right]
\end{aligned}
$$

On top of 2-D reparametrization invariance, this action has a a local form of supersymmetry (it's a supergravity in D=2). This leads to additional fermionic constraints and allows to choose an extension of the ON (also called conformal) gauge, called the superconformal gauge, in which S simplifies drastically. This gauge is defined by:

$$g_{\alpha\beta} = \rho\ \eta_{\alpha\beta} \quad ; \quad \chi_\alpha = \gamma_\alpha\ \chi$$

The extra fermionic constraints are of course welcome since, a priori, there are now more ghosts to be killed, those related to the time-components of the WS fermions. In this gauge only the first term in S^{f} survives and we get:

$$S^{SCG} = -\frac{T}{2}\int d^2\xi\,[\,\partial_\alpha X^\mu\cdot\partial^\alpha X_\mu + i\bar{\psi}^\mu\gamma^\alpha\cdot\partial_\alpha\psi_\mu]$$
$$= T\int d^2\xi\,[\,2\partial_+ X^\mu\cdot\partial_- X_\mu + \psi^\mu_-\partial_+\psi_{-\mu} + \psi^\mu_+\partial_-\psi_{+\mu}]$$

leading to very simple decoupled e.o.m.

$$\partial_\alpha\partial^\alpha X^\mu = \partial_+\partial_- X^\mu = 0$$
$$\gamma^\alpha\cdot\partial_\alpha\psi^\mu = \partial_+\psi^\mu_- = \partial_-\psi^\mu_+ = 0$$
$$\psi^\mu_a = (\psi^\mu_-, \psi^\mu_+)$$

solved by

$$X_\mu(\sigma,\tau) = F_\mu(\tau-\sigma) + G_\mu(\tau+\sigma)$$
$$\psi^\mu_-(\sigma,\tau) = \psi^\mu_-(\tau-\sigma)\ ;\ \psi^\mu_+(\sigma,\tau) = \psi^\mu_+(\tau+\sigma)$$

To these we have to add, like in the bosonic case, constraints and boundary conditions

Boundary conditions

For the bosons the boundary conditions are unchanged:

$$[X'^{\mu}\ \delta X^{\mu}](\sigma = 0) = [X'^{\mu}\ \delta X^{\mu}](\sigma = \pi)$$

They lead to $X^{\mu}(0, \tau) = X^{\mu}(\pi, \tau)$ for closed strings and to

either X' (NBC) or $\delta X = 0$ (DBC) for the open strings.

For the fermions the boundary terms one gets from varying the action:

$$S^{SCG,f} = T \int d^2\xi \left[\psi_-^{\mu}\partial_+\psi_{-\mu} + \psi_+^{\mu}\partial_-\psi_{+\mu}\right]$$

lead to:

$$\left[\psi_+^{\mu}\delta\psi_+^{\mu} - \psi_-^{\mu}\delta\psi_-^{\mu}\right](\sigma = 0) - [\ldots](\sigma = \pi) = 0$$

$$\left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = 0) = \left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = \pi)$$

Once more we have to distinguish open and closed strings:

For open strings we have to set to zero each side on the equation since the variations are independent. This still leaves two possibilities (at each end):

$$\begin{aligned} \psi_+^\mu(\sigma = 0) &= \pm\psi_-^\mu(\sigma = 0) \\ \psi_+^\mu(\sigma = \pi) &= \pm\psi_-^\mu(\sigma = \pi) \end{aligned}$$

The four possible cases are pairwise equivalent leading to 2 physically distinct cases: Ramond (R) and Neveu-Schwarz (NS)

R: $\psi_+^\mu(\sigma = 0) = +\psi_-^\mu(\sigma = 0)$ and $\psi_+^\mu(\sigma = \pi) = +\psi_-^\mu(\sigma = \pi)$

NS: $\psi_+^\mu(\sigma = 0) = +\psi_-^\mu(\sigma = 0)$ and $\psi_+^\mu(\sigma = \pi) = -\psi_-^\mu(\sigma = \pi)$

$$\left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = 0) = \left[\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu\right](\sigma = \pi)$$

For closed strings we identify as usual $\sigma = 0$ and $\sigma = \pi$.

However, we can still choose either periodic or antiperiodic b.c. independently for the two components of

$$\begin{aligned} \psi_+^\mu(\sigma) &= \pm\psi_+^\mu(\sigma+\pi) \\ \psi_-^\mu(\sigma) &= \pm\psi_-^\mu(\sigma+\pi) \end{aligned}$$

Physical states must contain both left and right-movers. We thus get four kinds of states:

NS-NS: $\psi_+^\mu(\sigma) = -\psi_+^\mu(\sigma+\pi)\ ,\ \psi_-^\mu(\sigma) = -\psi_-^\mu(\sigma+\pi)$

NS-R: $\psi_+^\mu(\sigma) = -\psi_+^\mu(\sigma+\pi)\ ,\ \psi_-^\mu(\sigma) = +\psi_-^\mu(\sigma+\pi)$

R-NS: $\psi_+^\mu(\sigma) = +\psi_+^\mu(\sigma+\pi)\ ,\ \psi_-^\mu(\sigma) = -\psi_-^\mu(\sigma+\pi)$

R-R: $\psi_+^\mu(\sigma) = +\psi_+^\mu(\sigma+\pi)\ ,\ \psi_-^\mu(\sigma) = +\psi_-^\mu(\sigma+\pi)$

Mode expansions

For open strings we get the mode expansions:

$$\psi^\mu_\pm(\sigma,\tau) = \sqrt{\alpha'} \sum_{n \in Z} d^\mu_n e^{-in(\tau\pm\sigma)}$$ for Ramond

$$\psi^\mu_\pm(\sigma,\tau) = \sqrt{\alpha'} \sum_{n \in Z+1/2} b^\mu_r e^{-ir(\tau\pm\sigma)}$$ for Neveu-Schwarz

For closed strings we get instead:

$$\psi^\mu_\pm(\sigma,\tau) = \sqrt{2\alpha'} \sum_{n \in Z} d^\mu_{\pm,n} e^{-2in(\tau\pm\sigma)}$$ for Ramond

$$\psi^\mu_\pm(\sigma,\tau) = \sqrt{2\alpha'} \sum_{n \in Z+1/2} b^\mu_{\pm,r} e^{-2ir(\tau\pm\sigma)}$$ for Neveu-Schwarz

Constraints

Because of its larger local symmetry the NG-NSR action leads to an enlarged set of constraints. The old bosonic constraints get an additional piece:

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu} = 0 \; ; \; T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu} = 0$$

To these we have to add new "fermionic" constraints

$$J_+ = \psi_+^\mu \partial_+ X_\mu = 0 \quad ; \quad J_- = \psi_-^\mu \partial_- X_\mu = 0$$

The Fourier modes (in σ) of T give the usual L_n while

those of J give fermionic operators (called F_n for R and G_r for NS). Classically the constraints satisfy a superalgebra

Skematically:

$$[L, L] = L \;,\; [L, G] = G \;,\; [L, F] = F \;,\; \{G, G\} = L \;,\; \{F, F\} = L$$

Quantization

The canonical (anti)commutation relations for $X^\mu(\xi)$ and $\psi^\mu{}_a(\xi)$ lead to the usual commutation relations for the a and a oscillators while for the fermionic ones we get:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s} \ ; \ \{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n+m}$$

The lowest state satisfies the obvious conditions:

$$a_n^\mu|0\rangle = b_r^\mu|0\rangle = d_n^\mu|0\rangle = 0 \quad ; \quad n, r > 0$$

but, in the R sector, it cannot satisfy: $d_0^\mu|0\rangle = 0$

since $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$

Modulo a factor 2 this is just the algebra of Dirac's γ^μ-matrices!

The vacuum must be a representation of this algebra. The smallest such representation is a spinor in D-dimensions. Thus in the R sector the vacuum state carries a spinor index and is, indeed, a fermion. Since the other states of the R sector are obtained by applying spacetime vector creation operators, all the states of the R-sector are fermions (with arbitrarily high half-integer spin)

The mass of a generic state is given again by:

$$\begin{aligned}\alpha' M^2 &= N - \alpha_0 \ ; \ N = N_a + N_b \text{ for NS} \ ; \ N = N_a + N_d \text{ for R} \\ N_a &= \sum_{n=1}^{\infty} n a_n^\dagger a_n \ , \ N_b = \sum_{r=1/2}^{\infty} r b_{-r} b_r \ , \ N_d = \sum_{n=1}^{\infty} n d_{-n} d_n\end{aligned}$$

Physical on-shell states must satisfy the equations:

$$(L_0 - \alpha_0)|\text{Phys.}\rangle = L_n|\text{Phys.}\rangle = G_r|\text{Phys.}\rangle = F_n|\text{Phys.}\rangle = 0 \quad ; \quad n, r > 0$$

$$\alpha' M^2 = N - \alpha_0 \; ; \; N = N_a + N_b \text{ for NS} \; ; \; N = N_a + N_d \text{ for R}$$

$$N_a = \sum_{n=1}^{\infty} n a_n^\dagger a_n \; , \; N_b = \sum_{r=1/2}^{\infty} r b_{-r} b_r \; , \; N_d = \sum_{n=1}^{\infty} n d_{-n} d_n$$

$$(L_0 - \alpha_0)|\text{Phys.}\rangle = L_n|\text{Phys.}\rangle = G_r|\text{Phys.}\rangle = F_n|\text{Phys.}\rangle = 0 \quad ; \quad n, r > 0$$

The normal ordering constant α_0 has to be fixed, as usual, by some consistency requirement (Lorentz inv. , Q_{BRST}^2 = 0, ...). It turns out that α_0=1/2 for the NS sector, α_0=0 in the R sector. This is related to the central charges that enter the previous L_n, G_r, F_n commutation relations after quantization. On top, the dimension of spacetime has now to be 10. We will discuss below how to get these results in l.c.q.

The open string spectrum (NS sector)

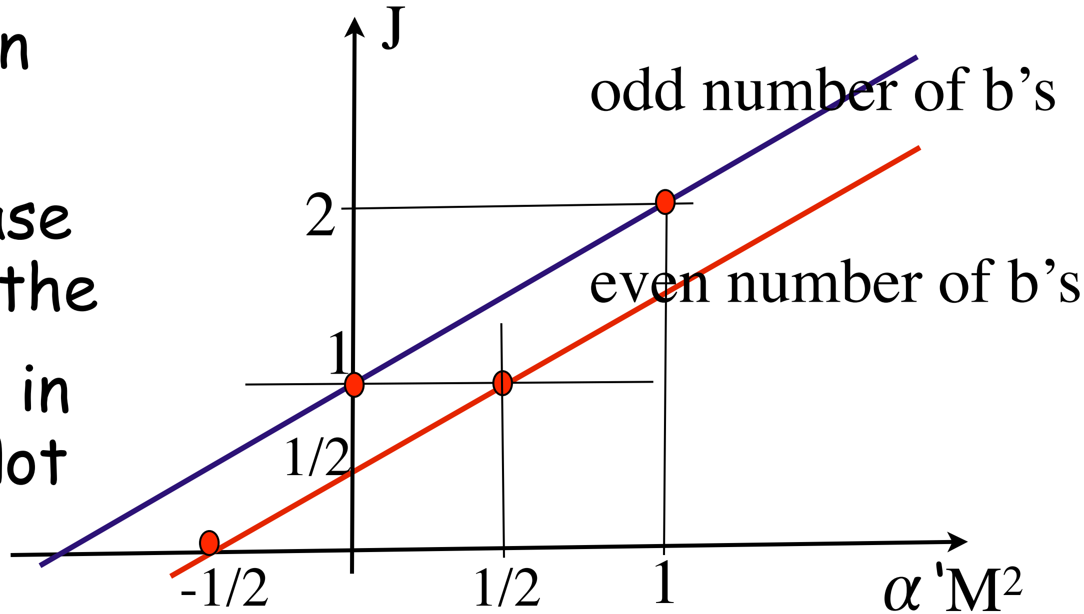
In this (bosonic) sector the mass formula reads:

$$\alpha' M^2 = -1/2 + \sum_{n=1}^{\infty} n a_n^{\dagger\mu} a_{n\mu} + \sum_{r=1/2}^{\infty} r b_{-r\mu} b_r^{\mu}$$

The lowest state |0> is physical but it's (again!) a tachyon! We can excite this state by applying to it either $a^{+}_{1\mu}$ or $b_{-1\mu}$ In

both cases we get a vector. In the first case it has $\alpha' M^2 = +1/2$, in the

latter it has M=0 (like in the bosonic string!). Not much improvement?

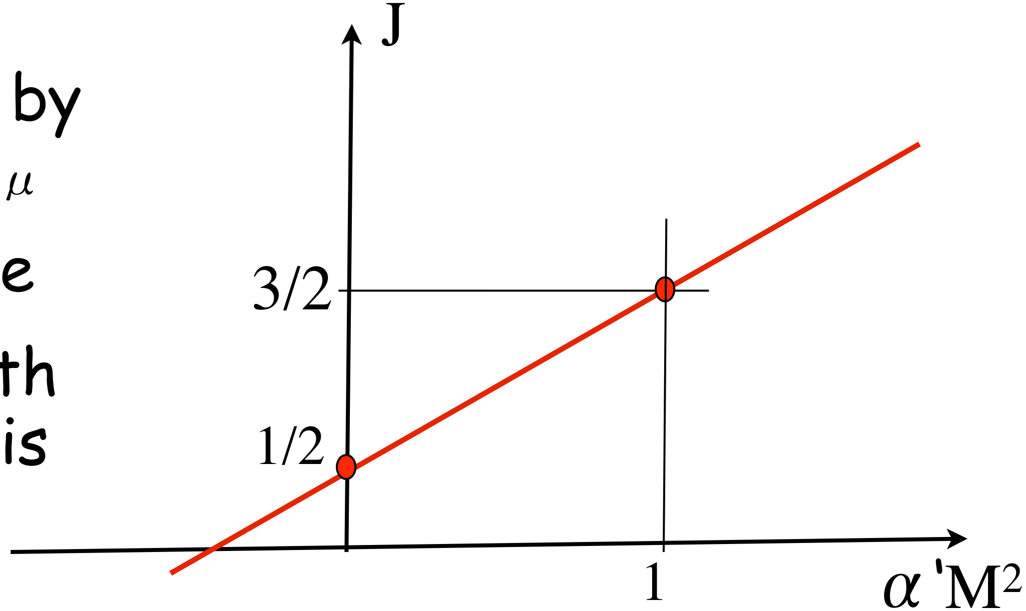


The open string spectrum (R sector)

In this (fermionic) sector the mass formula is:

$$\alpha' M^2 = \sum_{n=1}^{\infty} n a_n^{\dagger\mu} a_{n\mu} + \sum_{n=1}^{\infty} n d_{-n\mu} d_n^{\mu}$$

The lowest state |0> is physical and massless. It provides a representation of the Dirac algebra in D=10. We can excite it by applying to it either $a^{+}_{1\mu}$ or $d_{-1\mu}$ In both cases we get a spin 3/2 state with $\alpha' M^2 = +1$. This sector is tachyon free!



Light-cone quantization

The residual gauge freedom to perform superconformal transformations allows us to fix one bosonic and one fermionic coordinate. The light-cone gauge corresponds to choosing:

$$X^+(\sigma,\tau) = 2\alpha' p^+ \tau \quad ; \quad X^\pm(\sigma,\tau) \equiv \frac{X^0 \pm X^{D-1}}{\sqrt{2}}$$

$$\psi^+(\sigma,\tau) = 0 \quad ; \quad \psi^\pm(\sigma,\tau) \equiv \frac{\psi^0 \pm \psi^{D-1}}{\sqrt{2}} = 0$$

The constraints (do not confuse ± for WS and target space!)

$$\begin{aligned} \partial_\pm X \cdot \partial_\pm X \quad &+ \quad \frac{i}{2}\psi \cdot \partial_\pm \psi = 0 \\ \psi \cdot \partial_\pm X \quad &= \quad 0 \end{aligned}$$

can be solved for X^- and ψ^- in terms of the transverse operators (they become bilinear in the latter)

We can then construct the generators of the Lorentz group and check their algebra. This can be done for arbitrary D and n.o. constant α_0. The tricky commutators is again $[M_{+i}, M_{+j}] = 0$. For general D, a, one finds (for NS case):

$$[M_{+i}, M_{+j}] \propto \sum_{n=1}^{\infty} \left[n^2 \left(\frac{D-2}{8} - 1 \right) - \left(\frac{D-2}{8} - 2\alpha_0 \right) \right] \left(a_n^{\dagger i} a_n^j - a_n^{\dagger j} a_n^i \right)$$

Thus the Lorentz algebra has an anomaly unless D=10 and $\alpha_0 = 1/2$. A similar calculation for the R sector gives again D= 10 but $\alpha_0 = 0$. These results are again confirmed by the covariant (BRST) quantization procedure (now with BRST ghosts and superghosts).

Conclusions on NSR open string

With respect to the bosonic string the NSR model is much richer: it has also fermions (with no tachyon) and a bosonic trajectory with intercept 1/2 (with a tachyon on it). It also has an amusing (though only partial) degeneracy between the bosonic and fermionic spectra. This last point will be a central theme of the seminar...