Particules Élémentaires, Gravitation et Cosmologie
Année 2009-'10

Théorie des Cordes: une Introduction
Cours III&IV: 5 février 2010

Dualité de DHS et la fonction Beta

- Dualité de DHS et un "bootstrap" bon marché
- Une solution "exacte" (et simple)
- Quelques propriétés remarquables de la solution

Superconvergence (S. Fubini ~ 1966)

Regge-Chew-Mandelstam theory can be combined with analyticity to get some interesting "sum rules"

The first case, superconvergence, applies to the case in which the t-channel quantum numbers (either internal or related to helicity flips) are such that *A*(s,t) decreases at infinity faster than 1/s. Writing a fixed-t (unsubtracted) dispersion relation for *A*,

$$A(s,t) = \frac{1}{\pi} \int ds' \frac{ImA(s',t)}{s' - s - i\epsilon}$$

and imposing that s*A*-->0 at large s we must have:

$$\int ds \; ImA(s,t) = 0$$

Inserting low-energy "data" met with very reasonable success

Finite-energy sum rules (FESR)

In this case we use our theoretical (Regge) model at high energy and write a superconvergence relation for a subtracted amplitude: $A^{(sub)} = A(s,t) - A^{(R)}(s,t)$ so that $s\, A^{(sub)}$ goes to zero at large s. Limiting the integral to a finite value s_0 we get:

$$\int_0^{s_0} ds\ ImA(s,t) \sim \int_0^{s_0} ds\ ImA^{(R)}(s,t)$$

s_0 has to be taken judiciously. Using two such reasonable s_0

$$\int_{s_1}^{s_2} ds\ ImA(s,t) \sim \int_{s_1}^{s_2} ds\ ImA^{(R)}(s,t) = \sum \beta_i(t) \frac{s_2^{\alpha_i(t)+1} - s_1^{\alpha_i(t)+1}}{\alpha_i(t)+1}$$

Unitarity relates Im A to s-channel intermediate states hence we get a relation between s and t-channel quantities

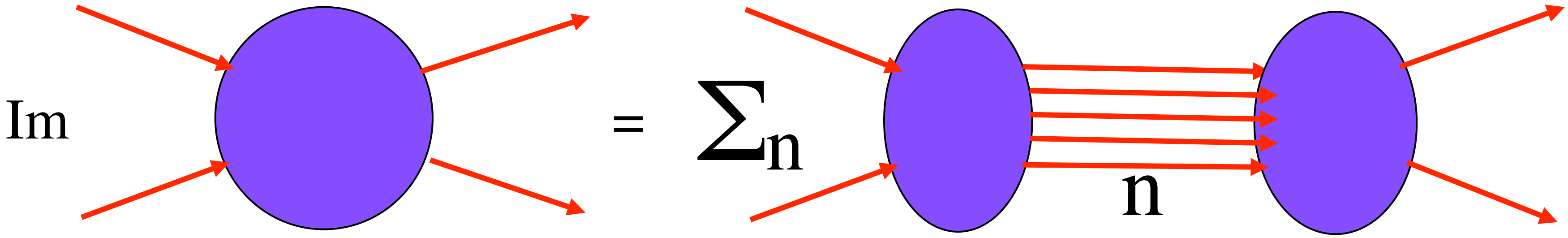
Thanks to Regge-Chew-Mandelstam we think we know what to put on the t-channel (r.h.) side of the FESR.

The question is: what should we put on the s-channel (l.h.) side of the FESR?

Giving the correct answer to this question turned out to be one of the crucial steps towards the ultimate discovery of string theory...

In very special cases one can use actual data. But in most cases one is forced to use some theoretical model.

Unitarity relates Im $A(s,t)$ to a sum over all the physical intermediate states that can appear in the s-channel with total energy $s^{1/2}$.

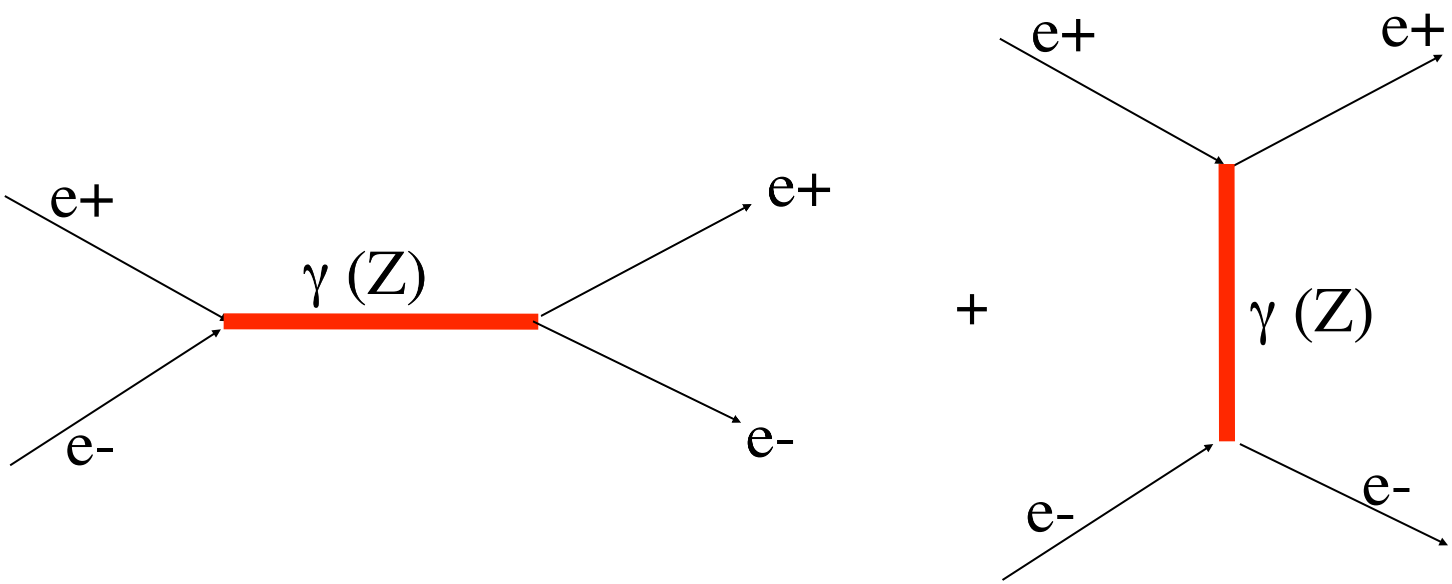


One obvious contribution was the one due to the resonances that could be produced in the s-channel (supposedly lying on that channel's Regge trajectories)

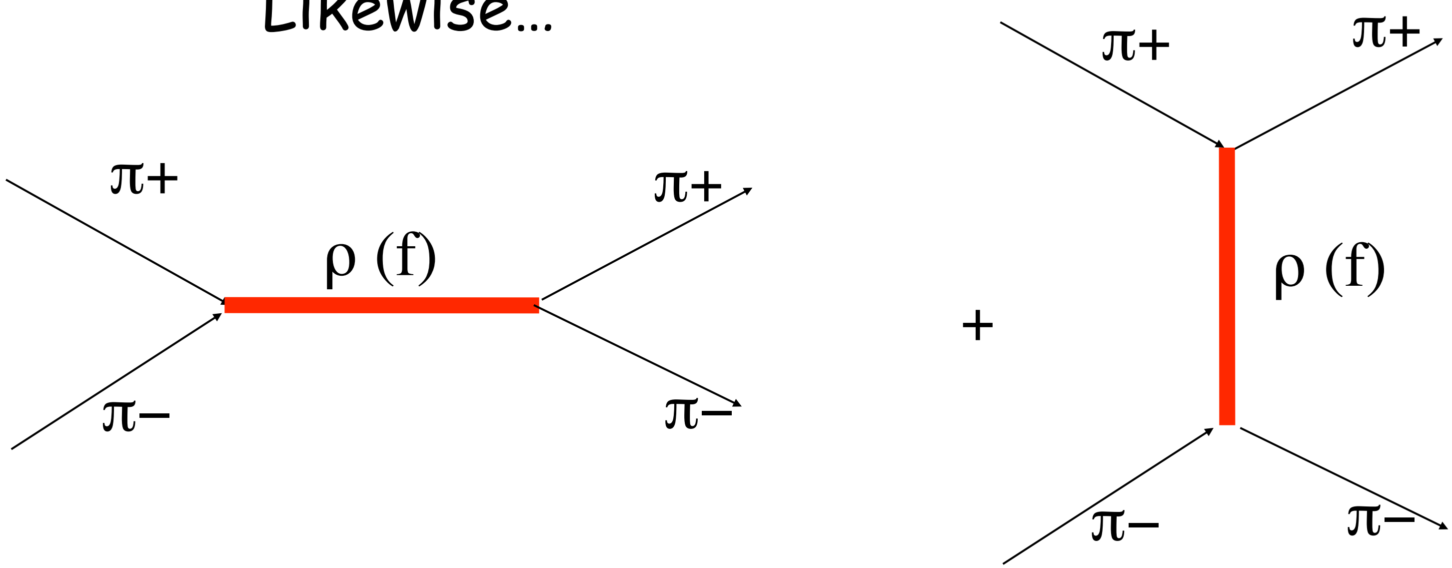
But what about other contributions that were making up the imaginary part of the t-channel Regge pole itself?

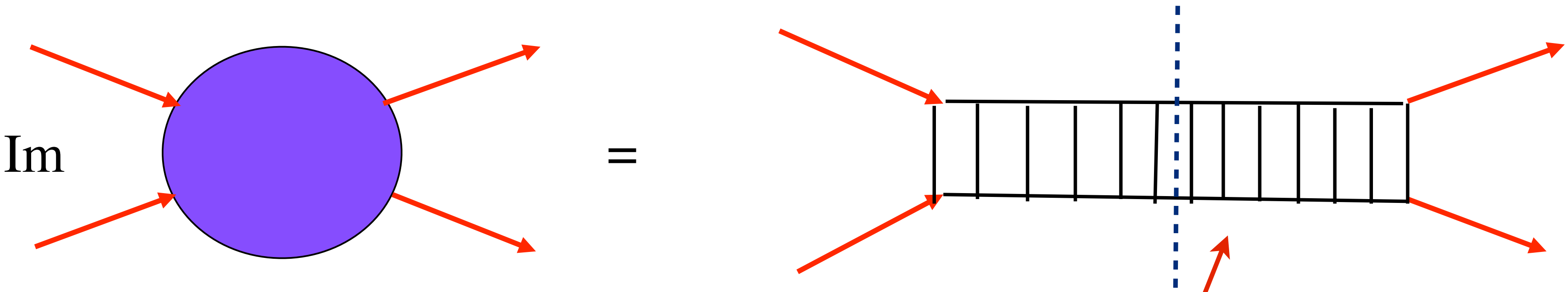
The prevailing belief at the time was that those two contributions had nothing to do with each other and that, therefore, should be added.

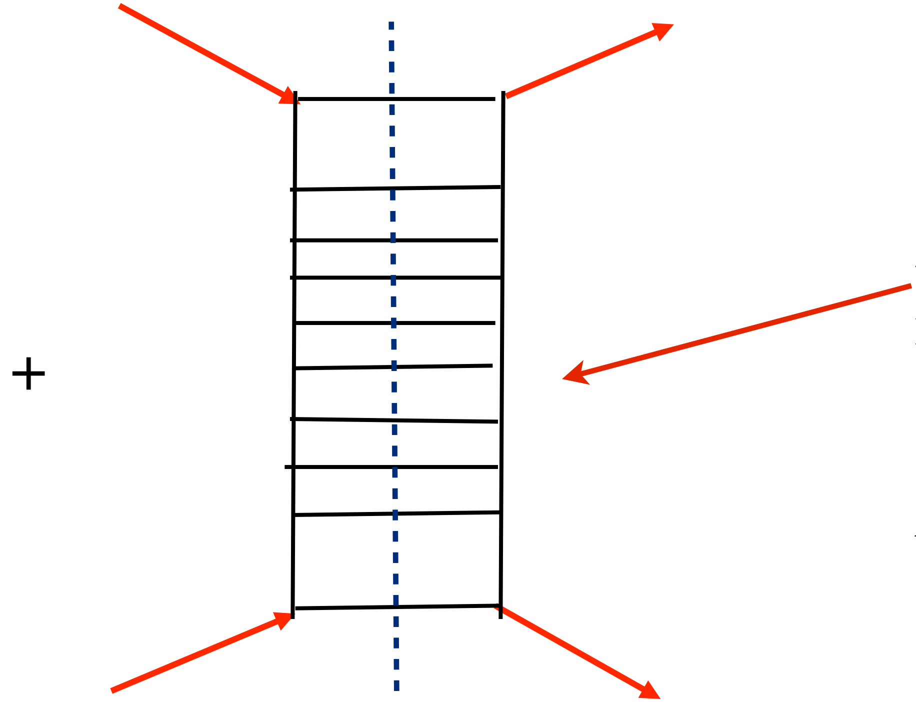
This was supported by QED calculations and also by QFT models for Regge poles



Likewise…







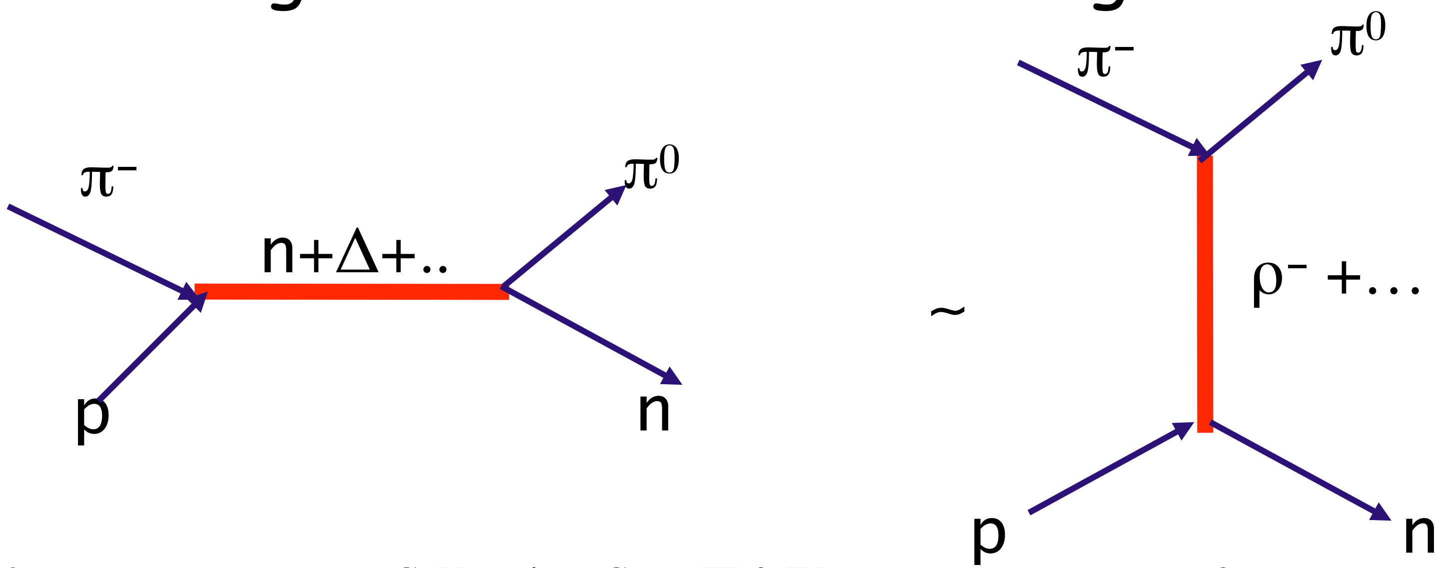
AFS multiperipheral model for Regge poles (also from sixties) definitely suggests adding both

DHS duality

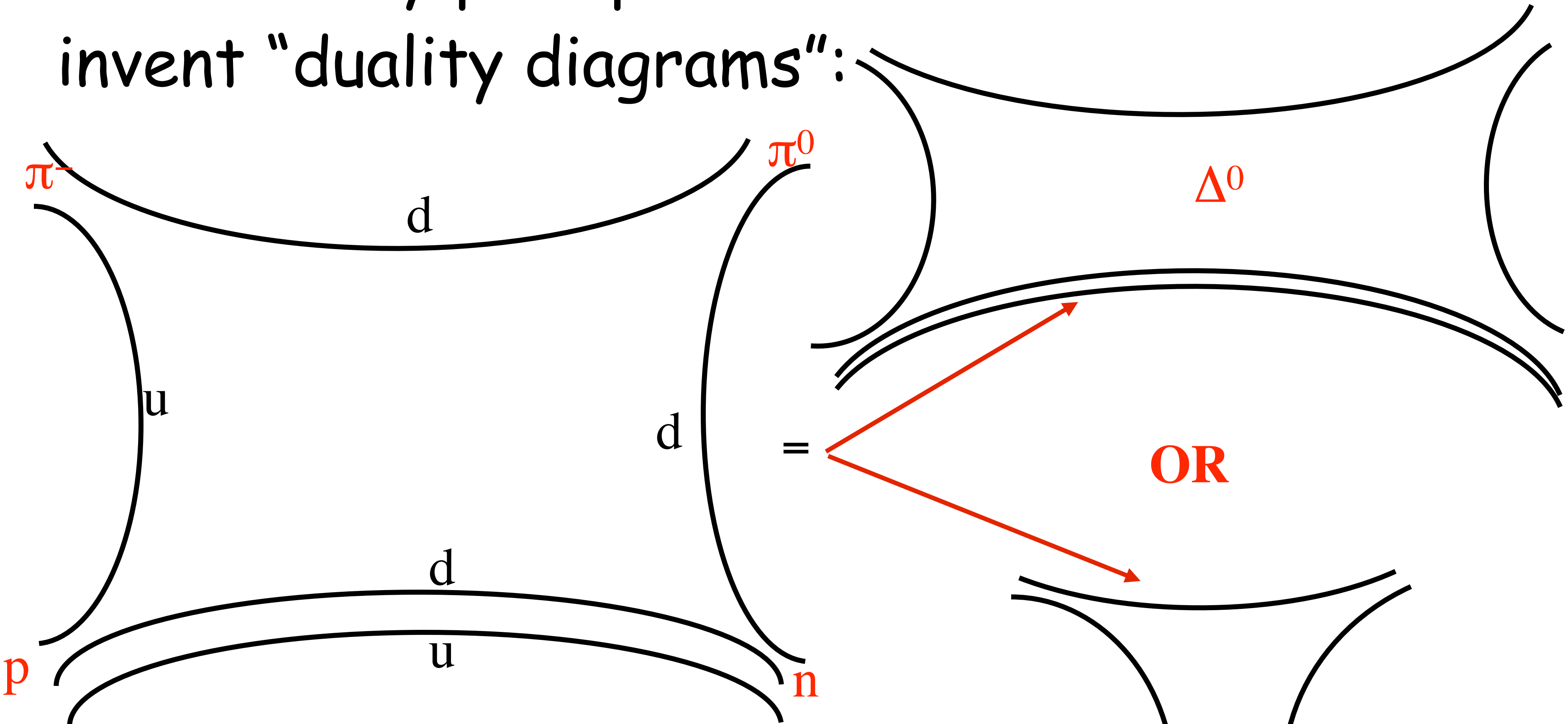
Erice, 1967: Gell Mann bringing news from Caltech:

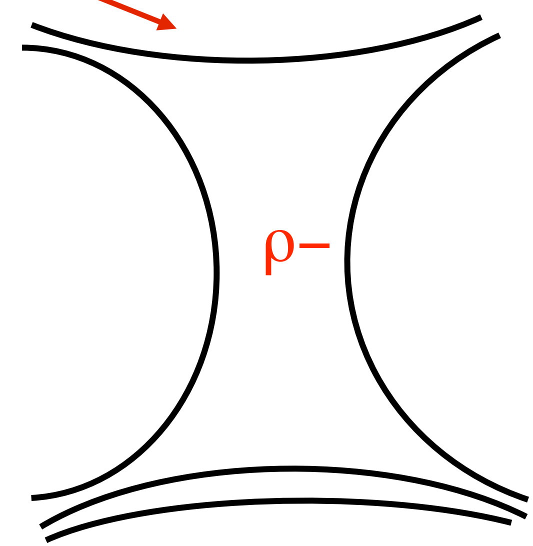
Dolen-Horn-Schmit duality: s-and t-channel descriptions are roughly equivalent, complementary, DUAL (Cf. QM's particle/wave duality)

Adding them = double counting!



DHS duality prompted Harari and Rosner to invent "duality diagrams":





NB: Quarks were just a mnemonics for QN's in those days

N.B. The Pomeron was dual to BKGND

π N scattering looked too complicated

We* decided to consider a simpler case:

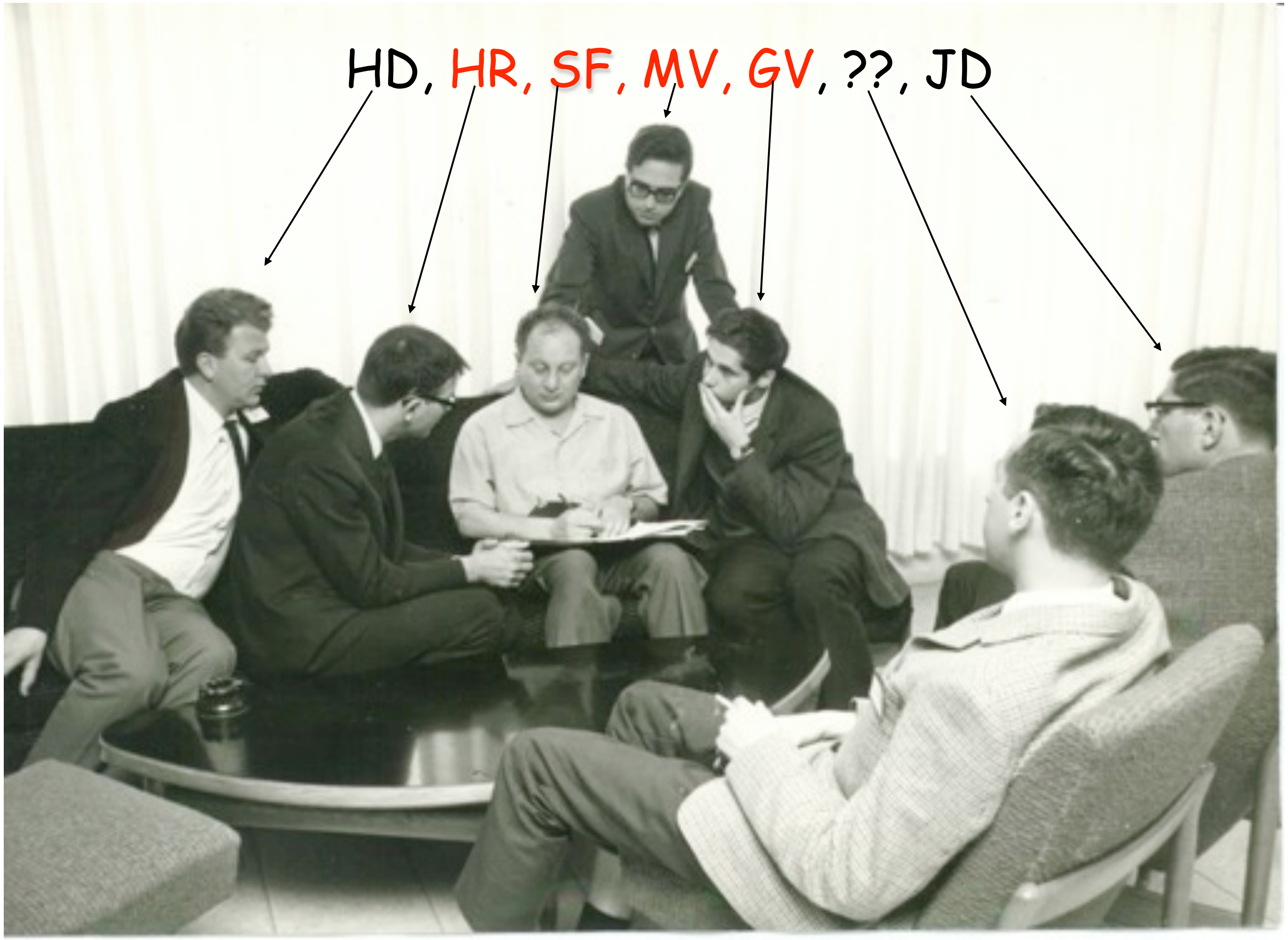
$\pi\,\pi \to \pi\,\omega$ (Very symmetric & very selective in QN's)

(ρ, ρ^*..)

Between the fall of 1967 and the summer of 1968 we made much progress in finding solutions to this "Easy Bootstrap".

*) Ademollo, Rubinstein, Virasoro, GV (+Bishari & Schwimmer) with much advice and encouragement by Sergio Fubini

Weizmann Institute, 1967



An exact (and simple) solution

The ARVV ansatz that worked amazingly well for the DHS bootstrap in $\pi\,\pi \to \pi\,\omega$ was simply*:

$$Im\,A(s,t) = \frac{\beta(t)}{\Gamma(\alpha(t))}(\alpha' s)^{\alpha(t)-1}(1+O(1/s))$$

with:

$$\beta(t) \sim const.,\ \alpha(t) = \alpha_0 + \alpha' t$$

i.e. a linear leading Regge trajectory accompanied by parallel "daughter" trajectories. The latter, if suitably tuned, were improving the agreement in an increasingly large range of t

Which was the road that led from the above ansatz to an "exact solution"? Three main ingredients were used:

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*The extra 1/s is due to the non-zero helicity of ω

1. Look at A rather than at Im A (A = analytic function)
2. Impose exact crossing symmetry : A(s,t) = A(t,s)
3. Emphasize resonances over Regge (A ~ meromorphic)

1. Easy to show that

$$Im\, A(s,t) = \frac{\beta(t)}{\Gamma(\alpha(t))} (\alpha' s)^{\alpha(t)-1} (1 + O(1/s))$$

corresponds to (up to a trivial factor π):

$$A(s,t) = \beta(t)\Gamma(1-\alpha(t))(-\alpha' s)^{\alpha(t)-1}(1+O(1/s))$$

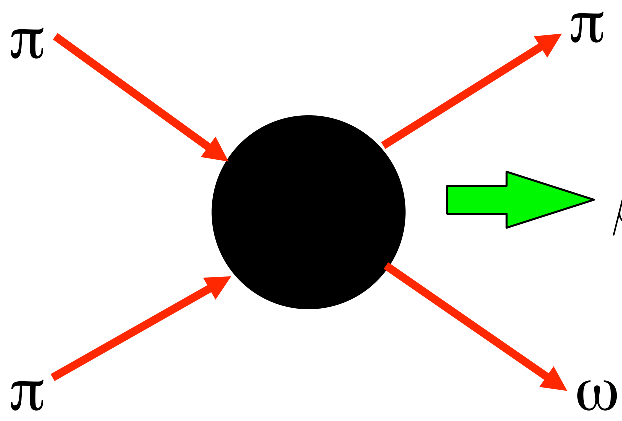
after using a well known formula: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin\pi x}$

3. A(s,t) already exhibits resonances (poles) in the t-channel but still only a smooth Regge behaviour in s: However, using

$$\frac{\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(s)-\alpha(t))} \to (-\alpha' s)^{\alpha(t)-1}(1+O(1/s))$$

we can satisfy both 2. and 3. by simply writing (GV's formula):

$$A(s,t) = \beta \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} = \beta\, B(1-\alpha(s), 1-\alpha(t))$$



Full x-ing symmetry then implies:

$$\begin{aligned} \beta^{-1} A(s,t) &= B\,(1-\alpha(s), 1-\alpha(t)) \\ &+ B\,(1-\alpha(u), 1-\alpha(t)) \\ &+ B\,(1-\alpha(s), 1-\alpha(u)) \end{aligned}$$

Exact DHS duality is implied by:
1. Analyticity (dispersion relations);
2. All singularities are poles corresponding to resonances
3. Good (Regge) asymptotics!
=> duality between two infinite sets of resonances in "dual" channels!

Properties best analyzed by using a well-known integral representation of the Beta-function:

$$B\left(1-\alpha(s), 1-\alpha(t)\right) = \int_0^1 dx\ x^{-\alpha(s)}(1-x)^{-\alpha(t)}$$

Analytic cont. needed: only converges for suff. negative s,t.

$$\beta(t) \sim const., \alpha(t) = \alpha_0 + \alpha' t$$

$$B\left(1-\alpha(s),1-\alpha(t)\right)=\int_0^1 dx\; x^{-\alpha(s)}(1-x)^{-\alpha(t)}$$

1. Crossing symmetry: x --> (1-x)
2. All singularities are poles: e.g. expanding integrand in powers of x (or of (1-x)) gives s-t duality in a nicer form:

$$\sum_{n=0}^{\infty}\frac{C_n(t)}{s-m_n^2}=\sum_{n=0}^{\infty}\frac{C_n(s)}{t-m_n^2}$$

3. Good (Regge) asymptotics: as s becomes very large (with t fixed) integral is dominated by x ~1 region. Implies duality!

$$\int_0^1 dx\;(1-x)^{-\alpha_s}x^{-\alpha_t}\sim\int_0^{\cdots} dx\; e^{x\alpha_s}x^{-\alpha_t}$$

$$\sim(-\alpha_s)^{\alpha_t-1}\int_0^{\infty} dz e^{-z}z^{-\alpha_t}=(-\alpha_s)^{\alpha_t-1}\Gamma(1-\alpha_t)$$

$$B\left(1-\alpha(s),1-\alpha(t)\right)=\int_0^1 dx\ x^{-\alpha(s)}(1-x)^{-\alpha(t)}$$

4. High energy, fixed angle (t/s = - (1-cos θ)/2 = z)

$$\int_0^1 dx\ x^{-\alpha_s}(1-x)^{-\alpha_t}=\int_0^1 dx\ e^{-\alpha' s[logx+zlog(1-x)]}\sim e^{-\alpha' s[logx^*+zlog(1-x^*)]}$$

where x* denotes the saddle point of the integrand:

$$\frac{1}{x^*}-\frac{z}{(1-x^*)}=0\Rightarrow x^*=\frac{1}{1+z}$$

giving (for t, s < 0)

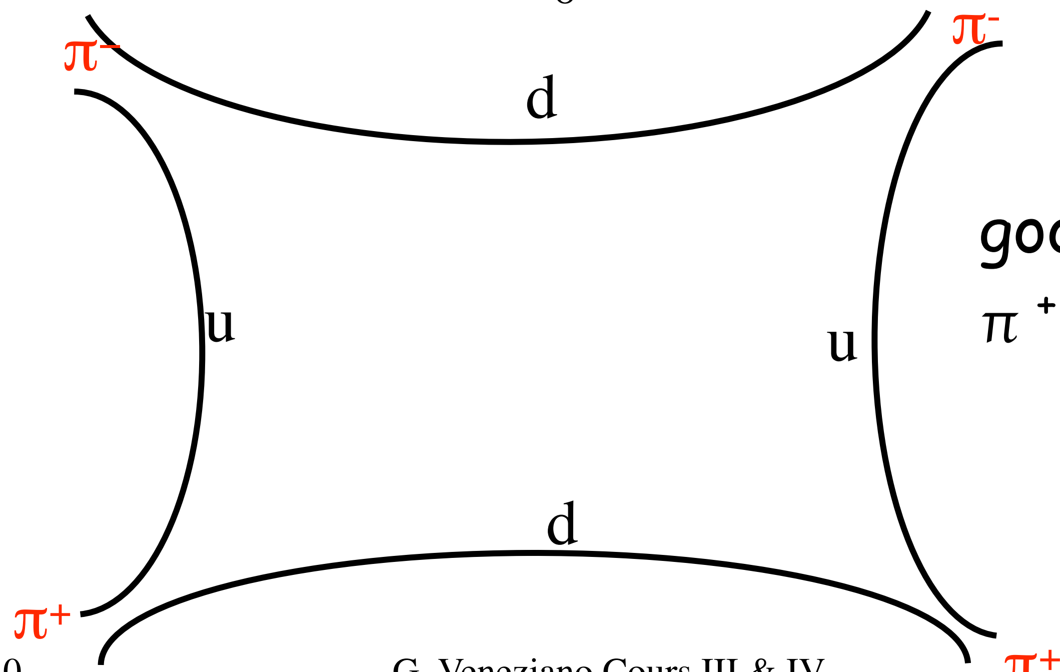
$$A(s,t)\rightarrow e^{\alpha' s\ log(1+\frac{t}{s})+\alpha' t\ log(1+\frac{s}{t})}$$
$$=e^{-\alpha' s\ log(s)-\alpha' t\ log(t)-\alpha' u\ log(u)}$$

i.e. mysteriously symmetric in s,t,u.

In the following our starting point will be the 2->2 scattering amplitude for spinless particles (e.g. $\pi\pi$ scattering). This is obtained by the simple replacement:

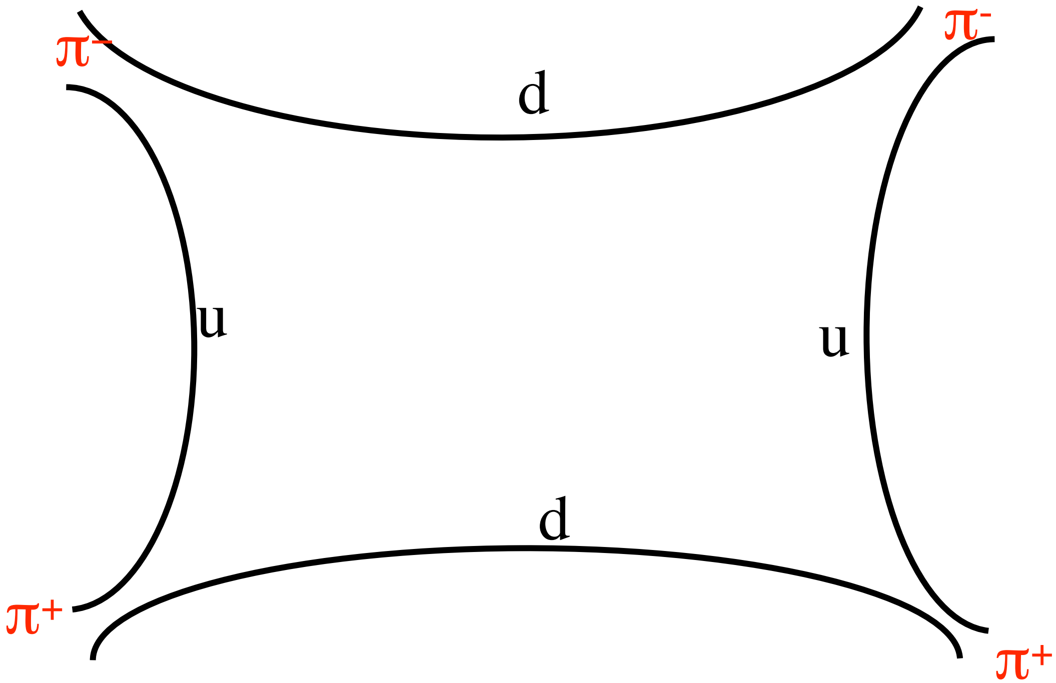
$$\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} \to \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$$= \quad B(-\alpha(s),-\alpha(t)) = \int_0^1 dx\; x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}$$



good to describe $\pi^+\pi^-$ scattering?

A more succesful (but still phenomenological) model was proposed by Lovelace giving rise to great hopes...



$$A(\pi^+\pi^- \to \pi^+\pi^-) = g^2 \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}$$

$$\alpha(t) = \alpha_\rho(t) \sim 0.5 + 0.9t \ \mathrm{GeV}^{-2}$$

It took quite a while before it was realized that the DRM was a theory of strings. Till about 1972 it looked like a very strange kind of theory, mysteriously different from anything that had been seen before, like QFT or GR.

As such it polarized the community with the opponents (particularly within the establishment) outnumbering the (mostly young) enthusiasts.

Even modern string theory remembers its very unconventional origins and uses concepts and methods that are very different from those people are accustomed to

"A piece of physics from the 21st century that fell, by accident, in the 20th"
(S. Fubini, ~1969)