Particules Élémentaires, Gravitation et Cosmologie
Année 2010-'11

Théorie des cordes: quelques applications

Cours XIV: 1 avril 2011

Comparing perturbations in inflationary and string cosmology

Comparing cosmological perturbations

One of the most important virtues of inflation is that it provides a mechanism for generating an interesting spectrum of cosmological perturbations.

This is also regarded as the best way to test the inflationary paradigm, to select among its many different realizations, and to compare it against alternative cosmologies.

We shall first review how this amazing phenomenon comes about and which are the characteristic properties of the perturbations produced by standard slow-roll inflation.

We will then compare these predictions with those obtained in the context of string cosmology.

Theory of cosmological perturbations

(see also J-Ph. Uzan, cours 2009)

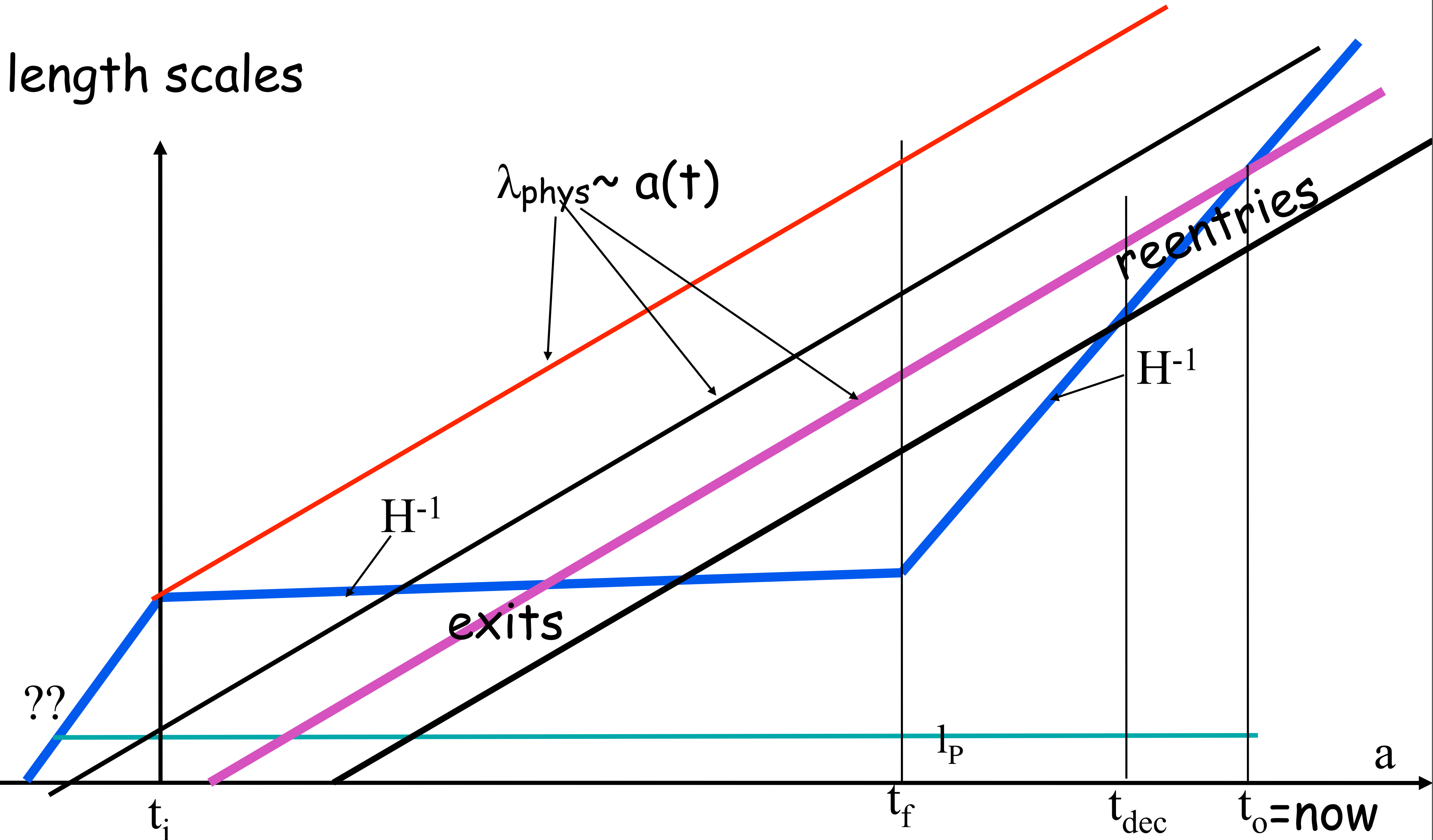
A distinctive property of any inflationary epoch is that, all along its duration, physical scales are continuously pushed outside the horizon. This a direct consequence of the growth of the ratio

$$\frac{\lambda_{phys}}{H^{-1}} \sim aH = \dot{a}$$

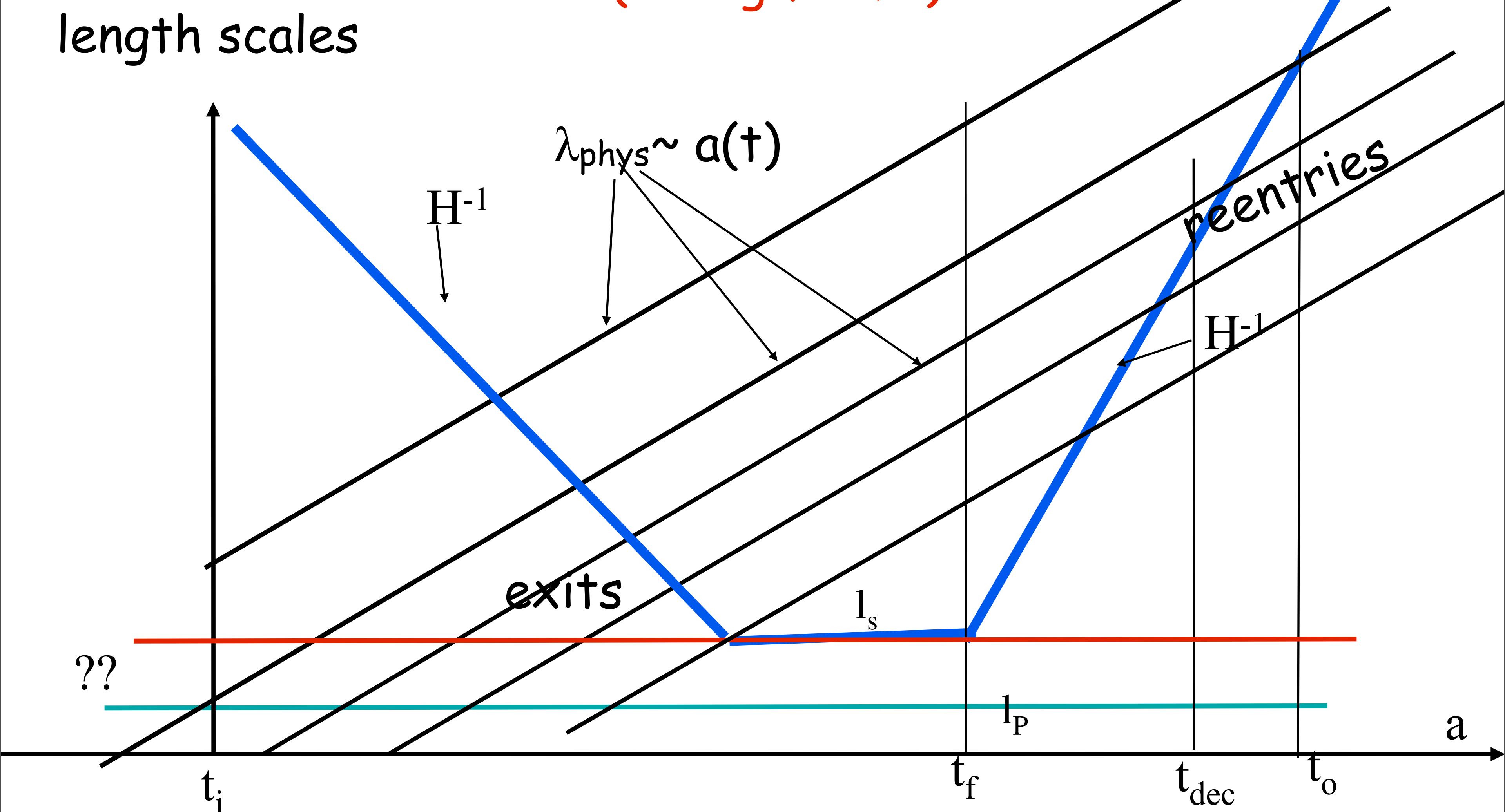
during inflation (the same happens for an accelerated contraction: cf. PBB in the Einstein frame)

During a decelerated expansion, physical scales "re-enter the horizon". The amplification of fluctuations has a lot to do with this basic kinematical fact: scales initially inside the horizon go out during inflation and reenter after inflation.

Kinematics of slow-roll inflation



Kinematics of pre-big bang cosmology (string frame)



Classical considerations

Let's assume that we have a homogeneous solution of the classical cosmological field equations. Let us look for the a general solution describing non-homogeneous small perturbations by expanding every field (the metric as well as the matter fields) around their homogeneous values.

The action describing the dynamics of these perturbations will be quadratic in them (since the action is stationary on the unperturbed solution) but in general is not diagonal. One can diagonalize the kinetic terms of the perturbations and make them canonically normalized.

At lowest order in the derivatives a generic perturbation ψ will enter the action in the form:

$$S_{eff} = -\frac{1}{2}\int d^4x\sqrt{-g}\; Q_\psi(x)\left[\partial_\mu\psi\partial^\mu\psi + m^2(x)\psi^2\right]$$

where Q_ψ is a ψ-dependent scalar field. If the background metric is conformally flat (it soon becomes in ordinary inflation) we can go over to conformal time and the action takes an even simpler form:

$$S_{eff} = \frac{1}{2}\int d^3x d\eta\; P_\psi(\eta)\left[(\psi')^2 - (\partial_i\psi)^2 - m^2a^2\psi^2\right]\;\; ;\;\; \psi' \equiv \partial_\eta\psi = a\partial_t\psi$$

$P_\psi(\eta) = a^2\, Q_\psi$ is called the "pump field" for ψ. Introducing Fourier modes ψ_k wrt the space coordinates different modes decouple and each mode obeys a very simple linear dynamics. We will be mainly concerned with massless perturbations for which:

$$S_{eff} = \frac{1}{2}\int d^3x d\eta\; P_\psi(\eta)\left[(\psi')^2 - (\partial_i\psi)^2\right] = \frac{1}{2}\sum_{\vec{k}} P(\eta)\left[|\psi'_k|^2 - |k\psi_k|^2\right]$$

Comment: the comoving wave vector k is related to the physical wave vector p and wavelength λ_{phys} by:

$$p = \frac{k}{a} \; ; \; \lambda_{phys} = \frac{1}{p} = \frac{a}{k}$$

k is constant in time and has to be compared to aH (the comoving Hubble parameter). A perturbation is inside the horizon if k > (>>) aH = a'/a and is outside if k < (<<) aH. During inflation aH grows, while it decreases afterwards => this is how exits & reentries are seen in comoving variables.

The time evolution of each mode depends crucially on its relation (in or out) wrt the horizon.

It is convenient (also in view of discussing the quantum case) to go over to a Hamiltonian formalism:

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{k}} \left[P^{-1}(\eta)|\Pi_k|^2 + P(\eta)|k\psi_k|^2\right] \; ; \; \Pi_k = P\psi_k'$$

The one-mode Hamiltonian

$$\mathcal{H}_k = \frac{1}{2}\left[P^{-1}(\eta)|\Pi_k|^2 + P(\eta)|k\psi_k|^2\right] \;;\; \Pi_k = P\psi_k'$$

Hamilton's equations:

$$\psi_k' = P^{-1}\Pi_k \;;\; \Pi_k' = -Pk^2\psi_k$$

can be rewritten in terms of some rescaled "canonical variables" as Schroedinger-like equations:

$$\hat{\psi}_k = P^{1/2}\psi_k \;;\; \hat{\Pi}_k = P^{-1/2}\Pi_k$$

$$\hat{\psi}_k'' + \left(k^2 - \frac{(\sqrt{P})''}{\sqrt{P}}\right)\hat{\psi}_k = 0 \;;\; \hat{\Pi}_k'' + \left(k^2 - \frac{(\sqrt{1/P})''}{\sqrt{1/P}}\right)\hat{\Pi}_k = 0$$

and we can distinguish two opposite regimes:

1. When the perturbation is deeply inside the horizon its evolution corresponds to an adiabatically damped oscillator and to a conserved Hamiltonian:

$$\hat{\psi}_k \sim \mathrm{const}\ ;\ \hat{\Pi}_k \sim \mathrm{const}$$

$$\psi_k \sim P^{-1/2}\ ;\ \Pi_k = P^{1/2}\ ;\ \mathcal{H} \sim \mathrm{const.}$$

2. When the perturbation is deeply outside the horizon its evolution corresponds to an overdamped oscillator and the amplitude freezes. This corresponds to an increase in H which can be due either to the freezing of the perturbation or of its conjugate momentum. In both cases H grows.

$$\psi_k \sim \mathrm{const.}\ ;\ \Pi_k =\sim \mathrm{const.}\ ;\ \mathcal{H} \sim \mathrm{Max}(P, P^{-1})\ ;\ \dot{\mathcal{H}} > 0$$

Turning on Quantum Mechanics

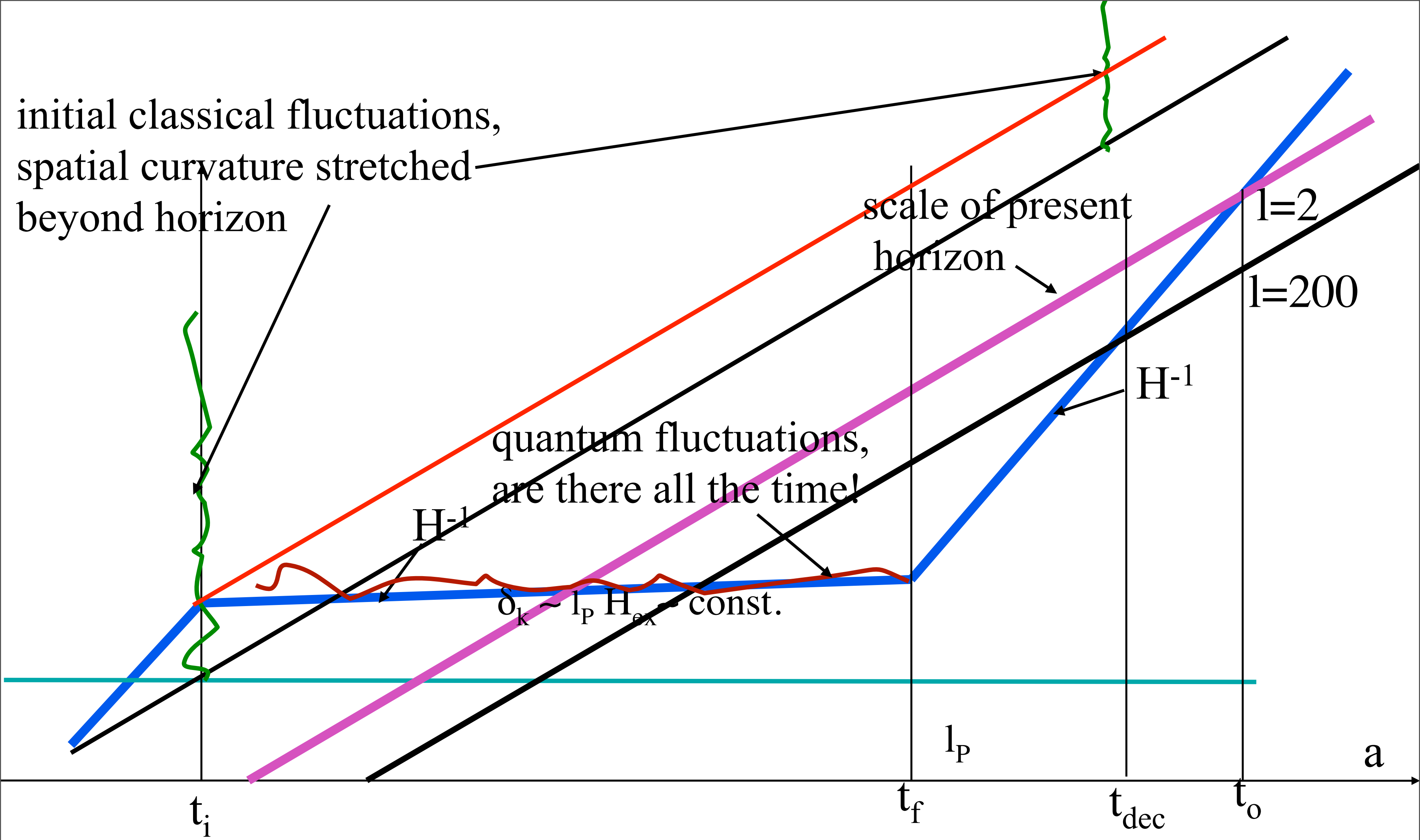
The evolution of perturbations is basically the same in the classical and quantum theory.

What really makes the difference is that classical perturbations are given "initially" and then evolve deterministically. In order to be called classical they involve physical lengths initially larger than l_P.

If inflation lasts long enough such initial perturbations have been stretched way beyond our present horizon H_0^{-1}.

Instead, quantum fluctuations are produced all the time (we cannot turn off h!).

Since they can appear much later than the classical ones they can be still inside our horizon today.


initial classical fluctuations,
spatial curvature stretched
beyond horizon
scale of present
horizon
l=2
l=200
H^{-1}
quantum fluctuations,
are there all the time!
H^{-1}
$\delta_k \sim l_P H_{ex} \sim$ const.
l_P
a
t_i
t_f
t_{dec}
t_o

Perturbations in Conventional Inflation

- Tensor perturbations (GW) generated with $n_T \sim 0$ (approximately scale-invariant);
- Scalar (Density/curvature) perturbations w/ $n_S \sim 1$ (also approx. scale-invariant); WMAP: $n_S \sim 0.95$
- $T/S = O(n_T)$, smallish but perhaps observable in CMB polarization (PLANCK?), too small for direct GW searches;
- Non Gaussian, isocurvature components: small, at least in single-field models;
- EM perturbations: absent since an inflationary metric couples trivially to Maxwell's term and α is constant.

Tensor perturbations in slow-roll inflation

This is one of the most robust predictions of inflation.

Consider a tensor perturbation of the FLRW metric:

$$g_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}(\eta, \vec{x})) \ ; \ \partial^\nu h_{\mu\nu} = h^\mu_\mu = 0$$

The associated "pump field" turns out to be $a^2(\eta)$ so that the Fourier modes of ah satisfy:

$$\hat{h}_k'' + \left(k^2 - \frac{a''}{a}\right)\hat{h}_k = 0 \ ; \ \hat{h}_k = a(\eta) h_k$$

At early enough times the scale 1/k is inside the horizon. (ah) oscillates like exp(i k η) with constant amplitude. In the ground state of this harmonic oscillator QM gives:

$$\hat{h}_k = l_P k^{-1/2} \Rightarrow \delta h(\lambda) = k^{3/2} a^{-1} \hat{h}_k = \frac{l_P}{\lambda} \ ; \ \lambda = a k^{-1} \ ; \ l_P = \sqrt{\frac{8\pi G\hbar}{c^2}}$$

$$\hat{h}_k = l_P k^{-1/2} \Rightarrow \delta h(\lambda) = k^{3/2} a^{-1} \hat{h}_k = \frac{l_P}{\lambda} \; ; \; \lambda = a k^{-1} \; ; \; l_P = \sqrt{\frac{8\pi G \hbar}{c^2}}$$

At a later time the scale 1/k goes out of the horizon and h itself freezes. By matching the solution at exit we find:

$$\boxed{\delta h(\lambda) = \frac{l_P}{\lambda(\eta_{ex})} \sim \frac{H(t_{ex}(\lambda))}{M_P} \; ; \; \eta > \eta_{ex}}$$

Smaller wavelengths have a larger initial amplitude but they exit later and therefore are not amplified as much as longer wavelengths. These two competing effects produce a spectrum that depends on how H changes in time. For slow-roll inflation H(t) is a slowly decreasing function of t and therefore the resulting spectrum is expected to be slightly red-tilted. The amplitude is fixed in terms of H/M_P.

An almost scale-invariant (Harrison-Zel'dovich) spectrum of tensor perturbations from slow-roll inflation!

In slow-roll inflation this calculation can be repeated for "scalar" perturbations. These are coupled perturbations of the inflaton and of the metric (curvature perturbations).

Because of this "mixing" the calculation is more complicated and one has to get rid of possible gauge artifacts.

The end result is that also scalar perturbations have a flattish (and typically red-tilted) spectrum ($n_s \sim 1$).

Their amplitude is not fixed by H/M_P since it is enhanced by 1/(slow-roll parameters).

=> The S contribution to CMB anisotropies dominates over T, but its properties are model-dependent.

We can only put some upper bound on the tensor perturbations in order not to exceed the observed $\Delta T/T$.

Not clear whether PLANCK will see it in B-polarization...

Tensor Perturbations in string cosmology

Unlike in slow-roll inflation H is now rapidly varying in time:

=> we expect all but a flat spectrum of tensor perturbations

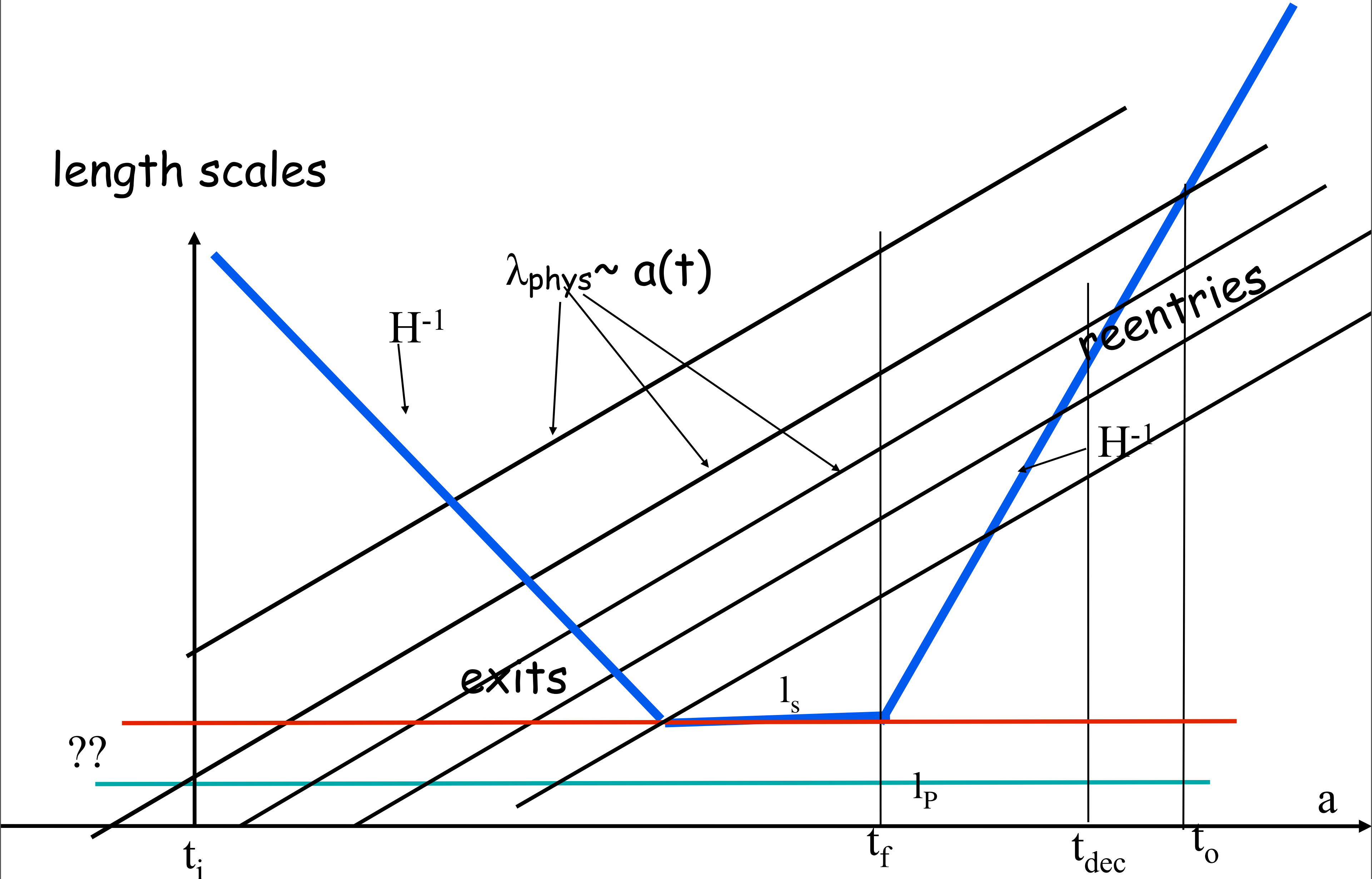
Since H grows in the pre-bang phase the spectrum is expected to be blue-tilted (more power at short scales).

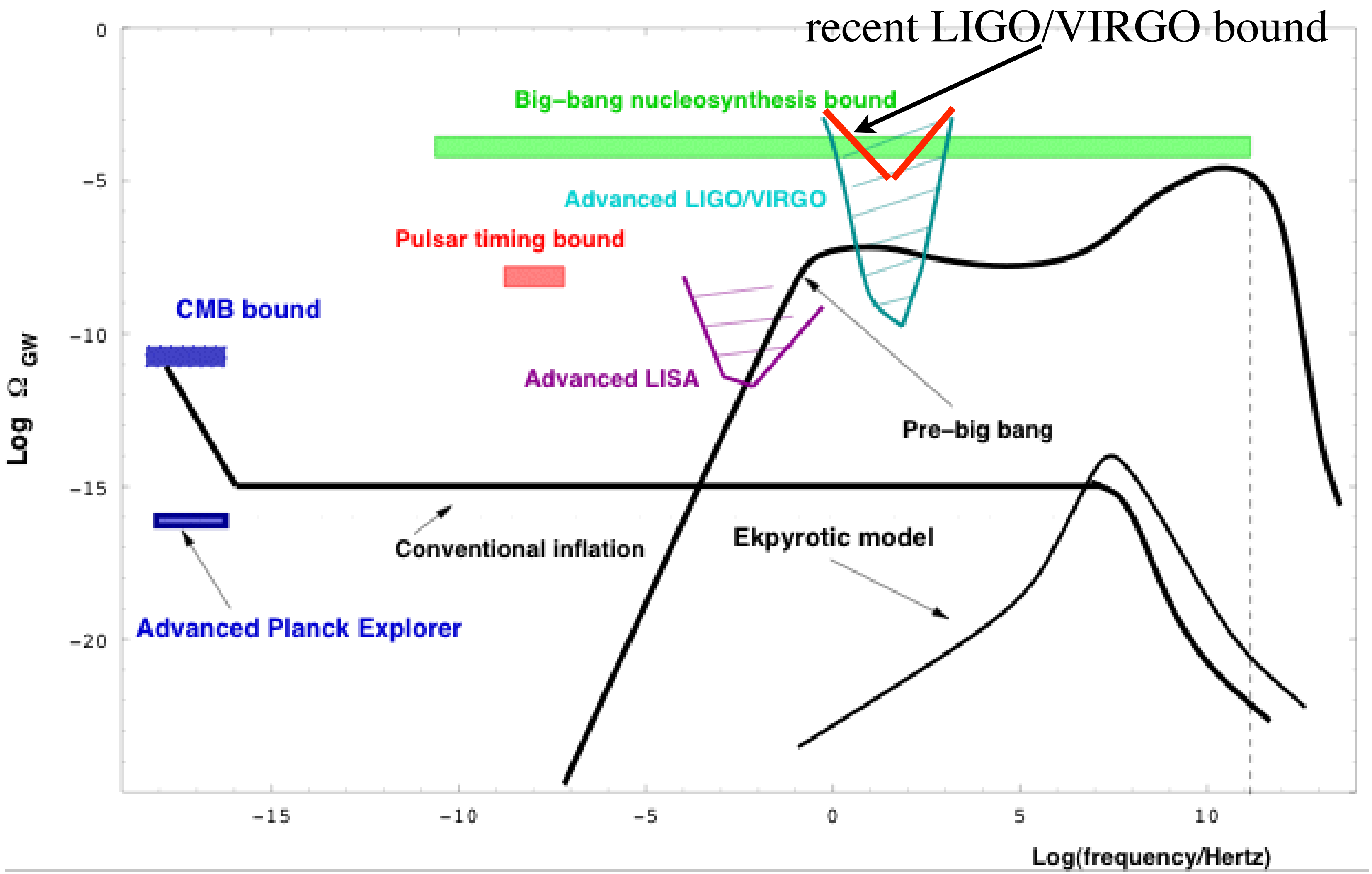
We have to find the "pump field" for tensor perturbations in string cosmology. Because of the way the dilaton enters the effective action one finds that the pump field is

$$P = \exp(-\phi/2)\, a = a_E \sim \eta^{1/2}$$

This implies $n_T = 3$ (as opposed to $n_T = 0$ in SRI)

=> possibly good for detection, irrelevant for CMB, LSS.


length scales
$\lambda_{phys} \sim a(t)$
H^{-1}
reentries
H^{-1}
exits
l_s
??
l_P
a
t_i
t_f
t_{dec}
t_o


recent LIGO/VIRGO bound
Big–bang nucleosynthesis bound
Advanced LIGO/VIRGO
Pulsar timing bound
CMB bound
Advanced LISA
Pre–big bang
Conventional inflation
Ekpyrotic model
Advanced Planck Explorer
Log Ω GW
Log(frequency/Hertz)
0
-5
-10
-15
-20
-15
-10
-5
0
5
10

The reason why the spectrum is not flat in the stringy window is that we do not know the exact dynamics of the string phase. If H is constant but ϕ is not (e.g. has constant positive time derivative) the spectrum is blue-tilted.

A stochastic background of GW should be around us and can be detected, in principle, by looking for cross-correlations in two interferometers (in order to disentangle it from real noise).

Its discovery/measurement would give us a picture of the Universe when gravitons decoupled (like the CMB does for photons) i.e. of the Planck/string epoch, if it ever existed.

LIGO/VIRGO have recently lowered the upper bound on this stochastic background below the so-called NS bound (too many GW would have upset the successes of primordial nucleosynthesis, just like a 4th light neutrino).

Scalar perturbations in string cosmology

Like in ordinary inflation we can compute the spectrum of adiabatic curvature perturbations in string cosmology (coupled dilaton-metric perturbations).

Not surprisingly they also come out blue-tilted ($n_s = 4$) and of the same order as tensor perturbations (no slow-roll enhancement of S/T in SC, in any case they are both tiny!)

Like with tensor perturbations this result is quite insensitive to what the extra dimensions do.

These perturbations are thus irrelevant for CMB, could be detectable in a narrow window of parameter space.

A problem with shear?

There is one potential problem with our scenario. It turns out that spatial isotropy is not necessarily an outcome of PBB evolution. The ratio:

$$\sigma/\theta = \text{(anisotropic expansion)}/\text{(total expansion)}$$

is not diluted during dilaton-driven inflation (stays constant)

Isotropy appears to emerge instead from the stringy phase (similar to de Sitter) or could be a consequence of thermalization after the bounce.

In this connection: we have seen that Kasner-like behaviour was generic but isotropic Kasner was not.

Perturbations in string cosmology

- Gravitational waves: $n_T = 3$

  => irrelevant for CMB, LSS, possibly good for detection

- Adiabatic dilaton/curvature perturbations: $n_S = 4$?

  => Irrelevant for CMB, LSS

- EM perturbations: amplified due to time-dependent ϕ, sensitive to evolution of internal space => Seeds for the dynamo of Cosmic Magnetic fields?

- Axionic perturbations are also amplified and their spectrum depends on the evolution of the extra dimensions in the pre-bounce phase. Can they be used for CMB, LSS?

The big question:

Where does large scale structure come from?