Microscopic Calculation of the Black Hole Entropy

C. Bachas, LPT-ENS

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Dans la série de cours du prof. G.Veneziano

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A reminder: Thermodynamics versus Statistical Mechanics

Consider a ferro/para-magnetic material in a magnetic field B, and otherwise isolated so that its total energy E is fixed.

The state of the system is given by (B, E)

Thermodynamics introduces a potential, the entropy function: $S(B,E)$

This equation of state is not universal, it depends on details of the system.

From it, using thermodynamic identities, we can compute other macroscopic quantities

Temperature: $$\frac{\partial S}{\partial E} = \frac{1}{T}$$

Magnetization: $$-\frac{\partial S}{\partial B} = M$$

Given a microscopic description of the system, it is in principle possible to calculate, rather than postulate, the equation of state *S(B, E)*.

Boltzmann-Gibbs statistical entropy:

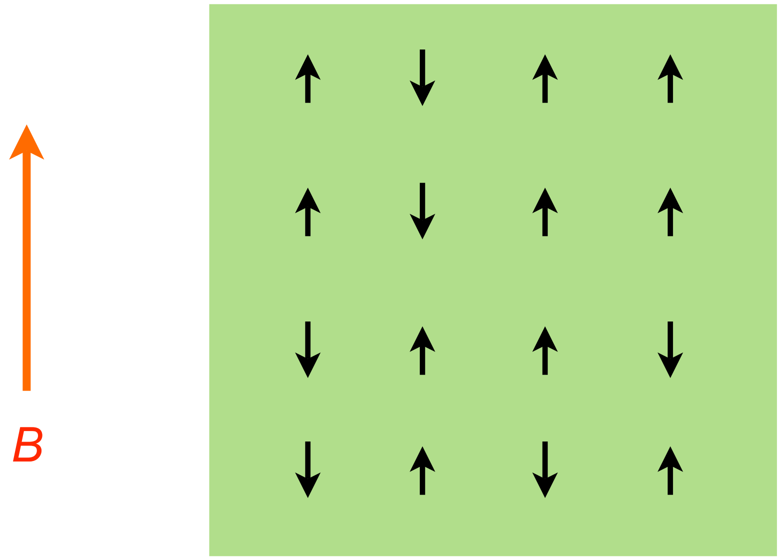
$$S = k_B \log \mathcal{N}(B, E)$$

Number of states of the system, for given energy and magnetic field

In most cases it is not easy to calculate S, though it can be done in principle with a sufficiently large computer.

Explicit calculations are possible for non-interacting degrees of freedom, or when the interactions are weak and can be treated perturbatively.

Suppose for example that our paramagnet is described, at the microscopic level, by N non-interacting spins:



$$E := N\epsilon = -\sum 2\mu_B \vec{\sigma}\cdot\vec{B}$$
$$= \mu_B\, B(N_+ - N_-)$$

Bohr magneton

number of spins UP

number of spins DOWN

Calculating the entropy is a simple counting problem:

$$S = \log\begin{pmatrix} N \\ N_+ \end{pmatrix} \simeq -N(y_+ \log y_+ + y_- \log y_-)\,,\ \text{where}\quad y_\pm := \frac{1 \pm \epsilon/\mu_B B}{2}$$

For interacting spins, e.g. $E_{\mathrm{int}} = -J \sum_{<ij>} \vec{\sigma}_i \cdot \vec{\sigma}_j$ the calculation isn't simple.

At very strong coupling, even the choice of degrees of freedom may be inadequate, and one has to go back to the description in terms of <u>atoms</u> and <u>electrons</u>.

2 A second reminder: Thermodynamics of Black Holes

(voir cours de G.Veneziano)

The geometry of a charged BH is described by the Reissner-Nordström metric:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_2^2 \ , \qquad \text{where}$$

$$f(r) = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r}) \qquad \text{with} \quad r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G_N Q^2}$$

charge in units where Coulomb's constant =1 .

The outer horizon is at $r = r_+$, and $\sqrt{G_N} M \geq Q$ by the cosmic-censorship hypothesis (no naked singularities). The Schwarzschild BH is found for $Q=0$, while the extremal BH is obtained when the inequality is saturated.

NB: Astrophysical black holes have zero charge; but in our discussion we will focus on near-extremal BHs, so the charge is essential .

To an **in-falling observer** nothing special happens as he crosses the horizon!

River as a rowers' black hole:

asymptotic $(r \simeq \infty)$

horizon $(r \simeq r_+)$

singularity $(r \simeq 0)$

velocity field $\rightarrow$ $\longrightarrow$ turbulent !*?#

> top speed of Olympic champ

Passing the horizon seems very innocent while it is happening. It's like being in a rowboat above Niagara Falls. If you accidentally pass the point where the current is moving faster than you can row, you are doomed. But there is no sign—DANGER! POINT OF NO RETURN—to warn you. Maybe on the river there are signs but not at the horizon of a black hole.

(Lenny Susskind, CA Literary Review)

To a **distant observer** the horizon looks thermal with temperature T_H !

This is shown by Hawking's semiclassical calculation of thermal radiation emission. A quicker alternative calculation is to go to imaginary time, and choose its periodicity so as to avoid a conical singularity. Changing radial coordinate near the horizon:

$$r - r_+ = \left(\frac{r_+ - r_-}{4r_+^2}\right)\rho^2 \quad \Longrightarrow \quad ds^2 \simeq \underbrace{d\rho^2 + \rho^2\left(\frac{2\pi T_H}{\hbar}dt_E\right)^2}_{\text{Tip of cigar}} + r_+^2 d\Omega_2^2 \ ,$$

where the Hawking temperature is

$$T_H \ := \ \hbar\,\frac{r_+ - r_-}{4\pi r_+^2} \ = \begin{cases} \dfrac{\hbar}{8\pi G_N M} & \text{Schwarzschild} \\ 0 & \text{extremal} \end{cases}$$

Choosing the periodicity of the time coordinate so that $T = T_H$ results in a non-singular geometry. This allows the definition of a KMS state [defined by functional integral] thereby showing that the BH is at equilibrium with the asymptotic heat bath.

invariant under imaginary time translations

If BHs have a temperature, then from the first law of thermodynamics they must also have an entropy:

$$dM = T_H dS + V dQ \quad \Longrightarrow \quad \boxed{S_{BH} = \frac{4\pi r_+^2}{4 G_N \hbar}}$$

horizon area

Bekenstein-Hawking

- Valid for all kinds of black holes, provided *M*, *Q* ... are large

thermodynamic limit

- Einstein's gravity "knows" the equation of state !

- Can we compute S_{BH} by counting microstates?

Check consistency of quantum gravity & uncover horizon ``degrees of freedom"

3

The simplest String-theory Black Hole

Schwarzschild BHs have negative specific heat $\quad \frac{\partial M}{\partial T} < 0 \quad$ **unstable**

Near-extremal BHs have positive specific heat $\quad \frac{\partial M}{\partial T} > 0 \quad$ **(marginally) stable**

We will need to extrapolate parameters, so want *near-extremal BHs.*

But the Reissner-Nordström BH is NOT a solution of string theory:

Recall the effective 4D action of Kaluza-Klein theory:

$$ds^2 = e^{-\phi} g_{\mu\nu} dx^\mu dx^\nu + (e^\phi dx^5 + A_\mu dx^\mu)^2 \qquad x^5 \equiv x^5 + 2\pi$$

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left(-\mathcal{R} + \frac{3}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{3\phi} F_{\mu\nu} F^{\mu\nu} \right)$$

Suppose we had a spherically-symmetric charged BH with smooth horizon:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_2^2 \ ,$$

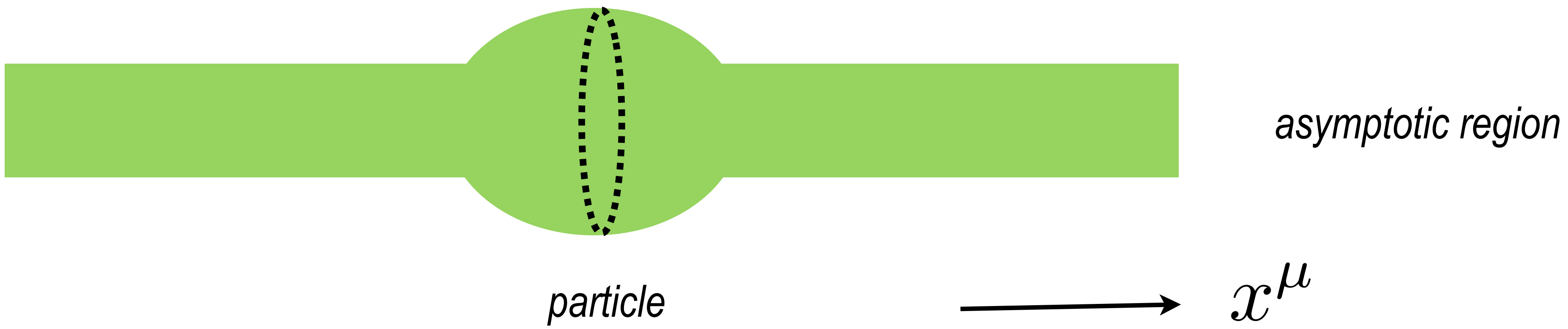
$$\vec{E} = \frac{Q\vec{r}}{r^3}$$ Gauss' law

$$f(r_+) = 0$$ for finite horizon size r_+

The equation for the radius field $$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = e^{-3\phi}\frac{Q^2}{4r^4}$$

has no solution near the horizon where the **radius wants to go to infinity** .

This is because in KK theory charge = momentum in 5th dimension. For a massless particle: $M = E = \dfrac{n}{R}$, so the radius wants to be as large as possible.



To balance the pressure in the 5th dimension, we need a string that carries both momentum and winding.

The winding # appears as charge of a second U(1) gauge field $(B_{\mu 5})$

$$S_{\text{eff}} = \int d^4x\sqrt{-g}\left(-\mathcal{R}+\frac{3}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{3\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\,e^{-\phi}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

a winding string pushes the radius at the horizon to zero:

but in the presence of both momentum and winding the radius field is fixed at the horizon to the potential minimum:

$$\boxed{e^{4\phi} = \frac{3Q^2}{\tilde{Q}^2}}$$

"attractor mechanism"

This 2-charge BH is still NOT a solution of string theory. String theory has a large number of scalar fields called "moduli" : *size and shape of 6d compact space, and a universal dilaton field determining the string coupling constant* g_s

All of these moduli must have equilibrium values at the BH horizon.

The important combination is the mass in units of the (effective) Newton's constant:

$$\frac{1}{\# g_s^2 \alpha'^4} \int d^{10}x \sqrt{-g}\, \mathcal{R} \rightarrow \frac{V^{(10-d)}}{\# g_s^2 \alpha'^4} \int d^d x \sqrt{-g}\, \mathcal{R} \equiv \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\, \mathcal{R}$$

For the fundamental strings, the dimensionless parameter: $\boxed{M_s (G_N)^{\frac{1}{d-2}} \sim g_s^{\frac{2}{d-2}}}$

so the string coupling is pushed to zero at the BH horizon.

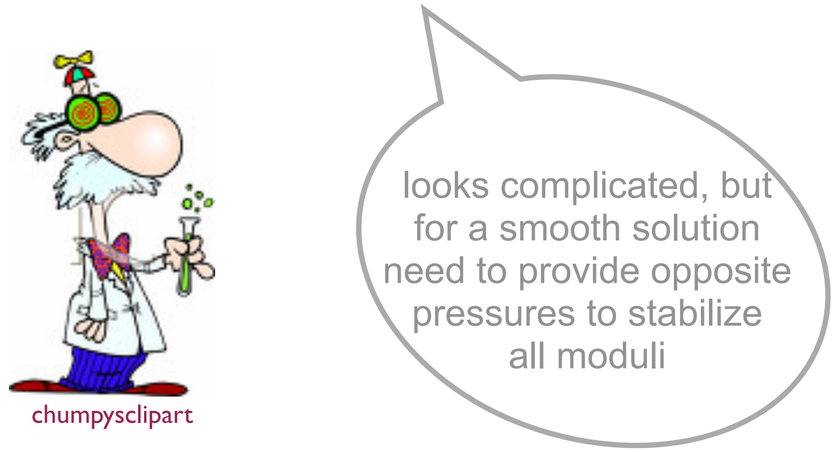
To counterbalance this "pressure" we need to endow the BH with *D-brane charge:*

$$\boxed{M_D (G_N)^{\frac{1}{d-2}} \sim g_s^{-1+\frac{2}{d-2}}}$$

The simplest example is a solution of type-IIB theory compactified on $T^4 \times S^1$.

The **Strominger-Vafa** BH:

N_5	D5-branes wrapping	$T^4 \times S^1$
N_1	D-strings wrapping	S^1
N_p	units of KK momentum along	S^1



Seen from a distance, this will look like a particle in 4+1 non-compact dimensions, carrying three different types of charge.

We need to find the corresponding BH solution of the effective 5d supergravity. This is a straightforward generalization of the 4d Reissner-Nordström BH.

Non-trivial input: the relation between mass and integer charges is completely fixed by String Theory !

The corresponding extremal solution is:

$$ds^2 = -f^{-2/3}dt^2 + f^{1/3}(dr^2 + r^2 d\Omega_3^2) \ , \quad \text{where} \quad f(r) = H_1(r)H_5(r)H_p(r)$$

with $\quad H_i = 1 + \frac{r_i^2}{r^2} \quad$ and

$$\left.\begin{aligned} r_1^2 &= \frac{(2\pi)^4\alpha'^{\,3}g_s}{V_4}\,N_1 \\ r_5^2 &= g_s\alpha' N_5 \\ r_p^2 &= \frac{(2\pi)^6\alpha'^{\,4}g_s^2}{V_4 R}\,N_p \ . \end{aligned}\right\} \text{charge normalization}$$

The horizon is at r = 0 , and its area is $2\pi^2 r_1 r_5 r_p$.

radius of S1 and volume of T4

Using the value of the 5D Newton's constant, $\quad \frac{1}{16\pi G_N} = \frac{RV_4}{(2\pi)^7\alpha'^{\,4}g_s^2}$

leads to the BH entropy:

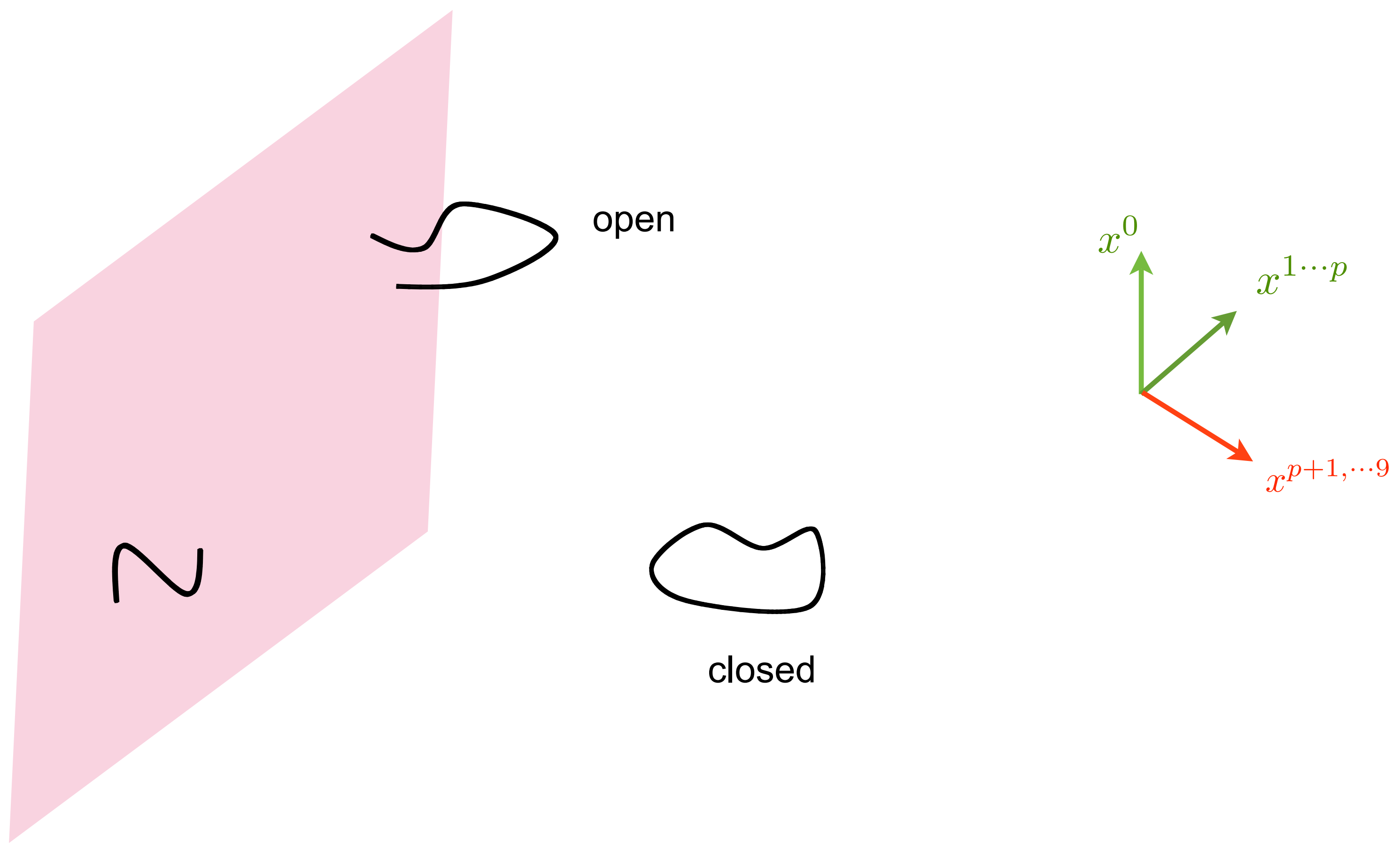
$$S_{BH} = \frac{A}{4G_N} = 2\pi\sqrt{N_1 N_5 N_p}$$

moduli-independent index

4 One last reminder: Dirichlet branes

D*p*-branes are soliton-like excitations of string theory extending in ***p*** spatial dimensions (p=0 particle, p=1 string, p=2 membrane, etc). Their worldvolumes are spacetime hypersurfaces to which open-string endpoints can be attached.

Polchinski '95



D-branes interact with the closed strings [e.g. an open string can emit a closed one].

They have, in particular, *Ramond-Ramond charge density* and *tension*: ρ_p and T_p

$$S_0 = \rho_0 \int d\tau \, \partial_\tau X^\mu A_\mu$$ D-particle

$$S_1 = \rho_1 \int d\tau ds \, \partial_\tau X^\mu \partial_s X^\nu A_{\mu\nu}$$ D-string

$$S_2 = \rho_2 \int d\tau ds_1 ds_2 \, (\partial_\tau X^\mu \partial_{s_1} X^\nu \partial_{s_1} X^\rho) A_{\mu\nu\rho}$$ D-membrane

$$\vdots$$ $\vdots$

The antisymmetric RR forms obey duality relations:

$$F_{\mu_{n+1}\cdots\mu_{10}} = \frac{1}{n!} \, \epsilon_{\mu_1\cdots\mu_{10}} F^{\mu_1\cdots\mu_n}$$

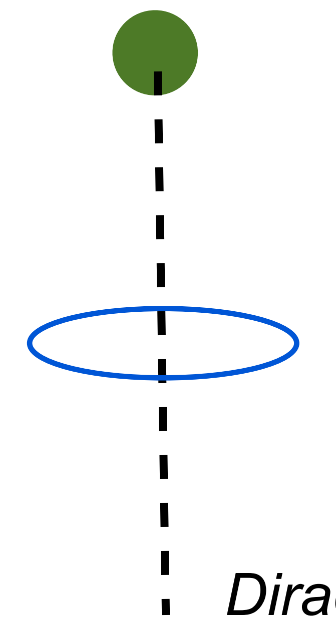
In standard electromagnetism:

$$F_{\mu_1\mu_2} \equiv \partial_{\mu_1} A_{\mu_2} - \partial_{\mu_2} A_{\mu_1}$$ electric charge: $q_e \int A_\mu dX^\mu$

$$\frac{1}{2} \, \epsilon_{\mu_1\cdots\mu_4} F^{\mu_3\mu_4} \equiv \tilde{F}_{\mu_1\mu_2} \equiv \partial_{\mu_1} \tilde{A}_{\mu_2} - \partial_{\mu_2} \tilde{A}_{\mu_1}$$ magnetic charge: $q_m \int \tilde{A}_\mu dX^\mu$

So Dp-branes/D(6-p)-branes behave like electric/magnetic charges .

Dirac quantization condition:



in polar coordinates: $A_\phi = q_m (1 - \cos\theta)$

no Aharonov-Bohm phase implies :

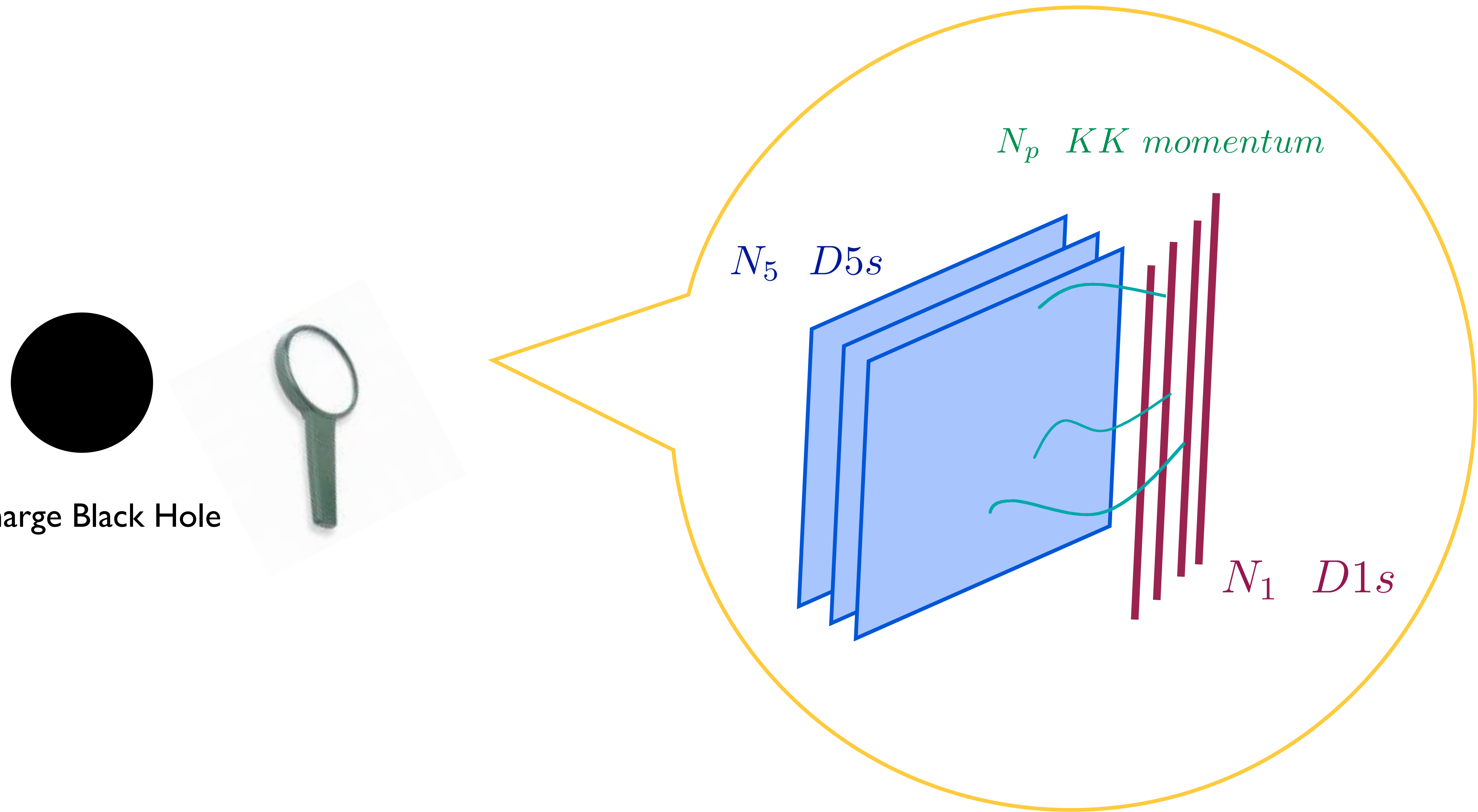
$$2 q_e q_m = N \hbar$$

Nepomechie-Teitelboim condition: $2\kappa^2 \rho_p \rho_{(6-p)} = 2\pi N$

is satisfied with *N=1*. So D-branes are elementary RR charges, they cannot be decomposed into more elementary constituents !

5

Microscopic description of 3-charge BH



Our task is to count the **number of quantum states** with the lowest energy (extremality condition) for the given values of integer charges.

The minimal-energy condition simplifies the problem enormously:

No brane/anti-brane pairs

No excited fundamental strings

All fundamental strings move in same direction

What are the lowest states of the fundamental strings ?

(5,5) strings: gauge bosons of $U(N_5)$ theory & susy partners

(1,1) strings: gauge bosons of $U(N_1)$ theory & susy partners

(1,5) & (5, 1) strings: $N_1 N_5$ hypermultiplets

oriented

1 hypermutliplet = 4 bosons + 4 fermions

The (1,5) strings have 2 coordinates with Neumann-Neumann boundary cns $\mu = 0, 1$

4 coordinates with Dirichlet-Neumann boundary cns $\mu = 2, 3, 4, 5$

4 coordinates with Dirichlet-Dirichlet boundary cns $\mu = 6, 7, 8, 9$

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n} a_n^\mu \, e^{in\tau} \cos n\sigma$$ **NN**

$$X^\mu = i\sqrt{2\alpha'} \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n} a_n^\mu \, e^{in\tau} \sin n\sigma$$ **DD**

$$X^\mu = i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n} a_n^\mu \, e^{in\tau} \sin n\sigma$$ **DN**

Mass-shell condition: $$\alpha' M^2 = \sum_{i\in\{2\cdots 9\}} \sum_{n>0} (\alpha_n^i)^\dagger \alpha_n^i + E_0$$

excitations

zero-point mass

$$\frac{E_0}{\ell} = \sum_{n>0} \frac{n}{2\ell} e^{-n\epsilon/\ell} - \frac{\#\ell}{\epsilon^2} = \begin{cases} -\frac{1}{24} & \textbf{DD} \\ \frac{1}{24} & \textbf{DN} \end{cases}$$

$\sigma \in [0, \pi\ell]$
standard convention sets
$\ell = 1$

& likewise for fermions, so lowest-lying (15) strings are **massless**.

The anticommuting coordinates of the superstring have b.cs. :

	Neveu-Schwarz	Ramond
DD	A	P
DN	P	A

so there are in both sectors *four anticommuting zero modes* whose algebra is realized on **4 mass-degenerate string states**.

The effective low-E theory on the D-branes [neglecting string excitations and the KK modes on T4] is a $\frac{1}{2}N_{max}$ supersymmetric $U(N_1) \times U(N_5)$ gauge theory, with $N_1^2 + N_5^2 + N_1 N_5$ hypermultiplets. Its details are a little complicated to discuss here but the upshot is that only the $N_1 N_5$ states can be filled by the string gas.

for a technical review, see e.g. David, Mandal, Wadia hep-th/0203048

The problem finally boils down to a combinatorial question:

Count # of ways to distribute the total KK momentum N_p in a gas of free fundamental strings, if there are $4N_1N_5$ bosonic and $4N_1N_5$ fermionic single-string states for each integer value of momentum.

Generating function (quantum-statistical partition function):

$$\left(\prod_{m=1}^{\infty} \frac{(1+q^m)}{(1-q^m)} \right)^{4N_1N_5} \equiv: \sum_{N_p=0}^{\infty} q^{N_p} \mathcal{N}(N_1N_5, N_p)$$

Compute by saddle-point method for $N_1N_5, N_p \gg 1$:

$$S = \log \mathcal{N} \simeq 2\pi\sqrt{N_1N_5N_p}$$

in agreement with semi-classical computation !

6

Why did it work ?

The two calculations have a priori very different ranges of validity :

- The **gravity calculation** requires that all volumes and curvatures are much larger than both the string scale and the Planck scale,

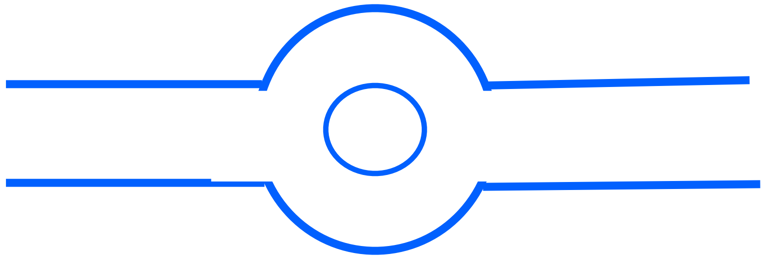
  in particular $r_1, \, r_5, \, (V_4)^{1/4} \gg \sqrt{\alpha'} \;, \; G_N^{1/3}$

  which imply (see solution) $N_1 g_s \;, \; N_5 g_s \gg 1$

- The **string calculation** requires that the strings be free, or at least weakly- coupled.

  This is the case if $N_1 g_s, N_5 g_s \ll 1$

because



$\sim \, g_s^2 N^2$

The day is saved by **supersymmetry**: what we were counting are the supersymmetric ground states in a given charge sector [*1/8*-BPS black holes]

can be sometimes checked

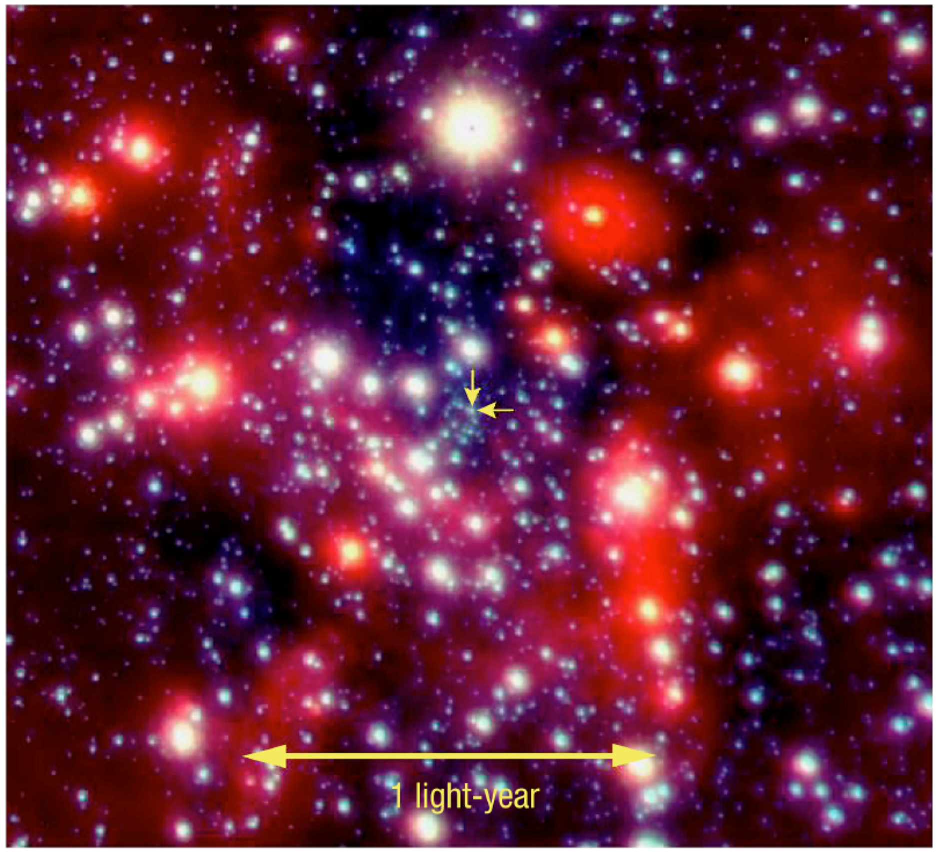
This is (modulo a mild assumption) an **index** , which does not change as theory parameters, such as g_s , vary continuously.

An important step forward was taken with the understanding that a large number of (semi)classical gravity calculations should match those in a **holographically dual large-N quantum field theory** at strong coupling.

get rid of strings on the D-brane side !

This is the **AdS/CFT correspondence**, about which you will hear more later in this course.

The End



The Centre of the Milky Way
(VLT YEPUN + NACO)
ESO PR Photo 23a/02 (9 October 2002)