Particules Élémentaires, Gravitation et Cosmologie

Année 2009-'10

Théorie des Cordes: une Introduction

Cours VI: 19 février 2010

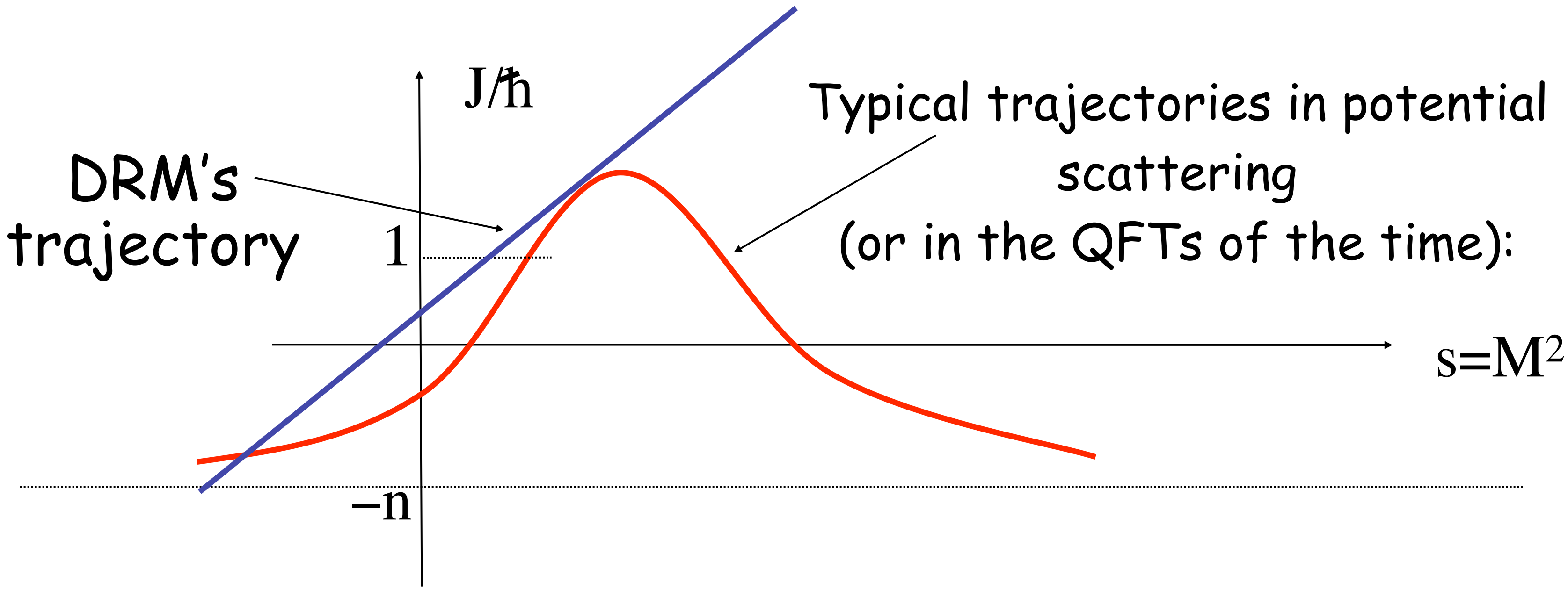
Naissance de la Théorie des Cordes

- Missed hints
- The Nambu-Goto action
- Simplest classical string motions
- GGRT quantization

Missed hints of an underlying string ?

1. From linear Regge trajectories
2. From duality diagrams
3. From the harmonic oscillators
4. From Q(z) and its correlators
5. From the DDF «transverse» states

From linear Regge trajectories

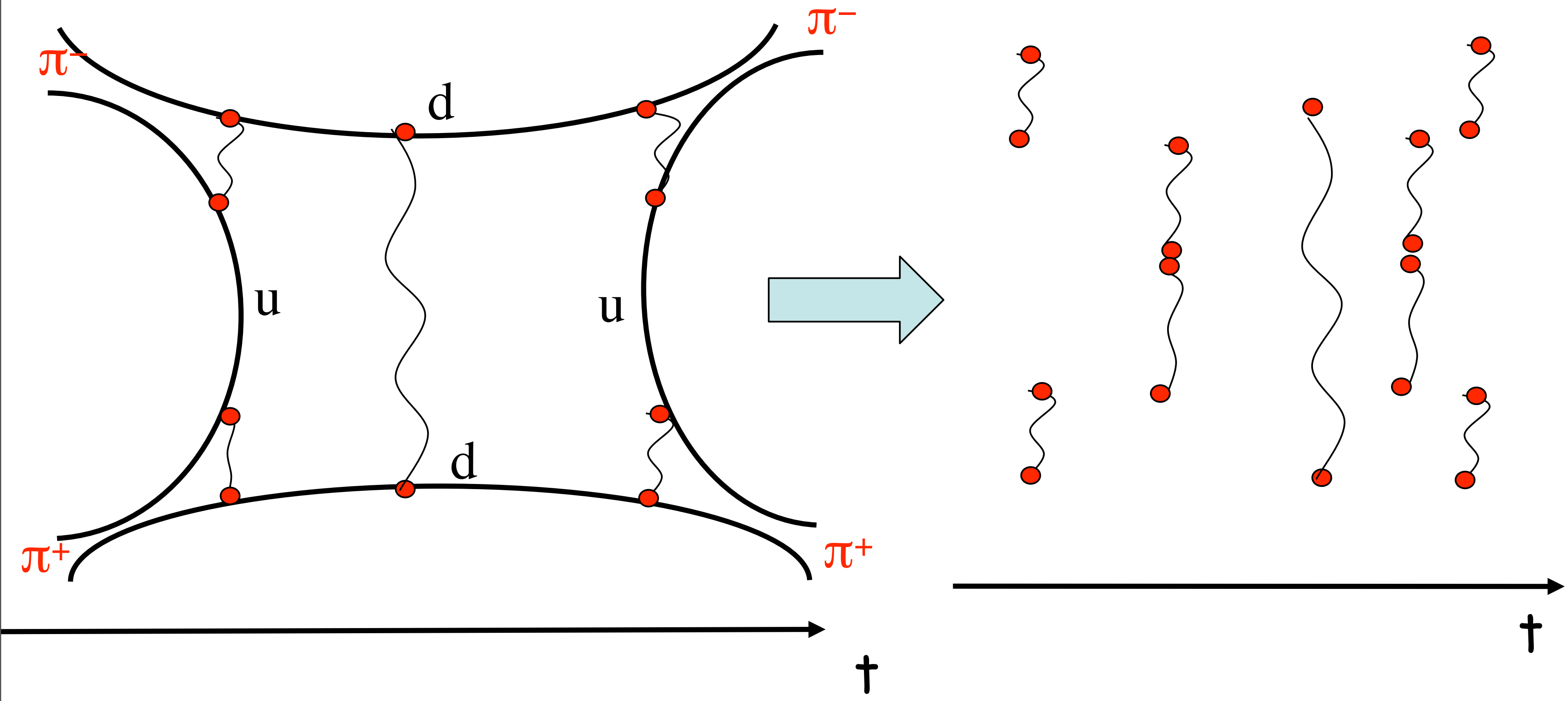


$\alpha' = dJ/dM^2 \sim 10^{-13}$ cm/GeV ~ constant.

Its inverse, 10^{13} GeV/cm, has dimensions of a string tension

(NB, c=1 but no ħ needed)!

From duality diagrams



From the harmonic oscillators

This was certainly the clearest hint since a string is a collection of harmonic oscillators whose frequencies are multiples of a fundamental frequency.

From DDF «transverse» states

The (small) vibrations of a string are orthogonal to the string itself: the number of physical dof should therefore be proportional to (D-2), like for the DDF states.

From Q(z) and its correlators

The commutator of two Q's looks like that of scalar fields of a QFT in D=2.

The Nambu-Goto action

After some handwaving attempts to formulate a string theory that would reproduce the DRM (Nielsen, Susskind, Nambu), a decisive step forward was made in 1970-'71 by Nambu and, independently, by Goto.

They wrote a geometric action for the relativistic string in strict analogy with the well-known action of the relativistic particle (see last year's course).

The latter can be written in a general (but given, fixed) spacetime metric $g_{\mu\nu}(x)$ and for any D as:

$$S_p = -mc \int ds = -mc \int d\tau \sqrt{-\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau} g_{\mu\nu}(x(\tau))}$$

The action of a point-particle is thus proportional (with mc as the proportionality constant) to the proper length of the "world-line" described by the particle's motion and parametrized by $x^{\mu}(\tau)$. The classical motion is the one minimizing that length (a geodesic in the given metric).

We may regard the quantity:

$$\gamma(\tau) = \frac{dx^{\mu}(\tau)}{d\tau}\frac{dx^{\nu}(\tau)}{d\tau} g_{\mu\nu}(x(\tau))$$

as the induced metric on the world line, since:

$$ds^2 = \gamma(\tau) d\tau^2$$

In complete analogy, for a relativistic string NG wrote a geometric action proportional to the area of the surface ("world-sheet" in analogy with "world-line") swept by the string. T, the string tension, is the proportionality constant. The string's motion is parametrized by $X^\mu(\sigma, \tau)$ where:

$\mu = 0, 1, \ldots D-1;\ 0 < \sigma < \pi$ (by convention), τ unconstrained.

NG did this in Minkowski spacetime ($g_{\mu\nu}(x) = \eta_{\mu\nu}(x)$) but,

like for point particles, the construction can be easily generalized to an arbitrary metric $G_{\mu\nu}(x)$ and for any D.

$$S_{NG} = -T \int d(Area) = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}} \equiv -T \int d\xi^0 \int_0^\pi d\xi^1 \sqrt{-det\gamma_{\alpha\beta}}$$

where

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

is the induced metric on the world sheet.

NB. $G_{\mu\nu}(x)$ is not Einstein's tensor!

Again by analogy with the point-particle, the classical motion of the string is obtained by varying the action and corresponds to minimizing the area of the surface swept.

But, unlike in the case of point-particles, the problem is already non trivial even in Minkowski spacetime (particularly at the quantum level as we shall see).

Another major difference is that for point particles we can add in the action different particles of different mass thus introducing many free parameters.

Also, interactions have to be added by hand (and it is not so simple!) and are quite arbitrary.

This will not be the case for the string: there will be only one T and interactions will be automatically included in a "geometric" way!

The classical constraints

S_p is invariant under reparametrization of the world-line, $\tau \to \tau'(\tau)$. This leads to 1 constraint (easy to check):

$$p_\mu(\tau) \equiv \frac{\delta S_p}{\delta \dot{x}^\mu(\tau)} \Rightarrow p_\mu(\tau) p_\nu(\tau) g^{\mu\nu}(x(\tau)) = -m^2$$

Similarly, S_{NG} is invariant under reparametrization of the world-sheet by an arbitrary redefinition $\xi^\alpha \to \xi'^\alpha(\xi^\alpha)$

This leads now to 2 constraints (easy again to check):

$$P_\mu(\xi) \equiv \frac{\delta S_{NG}}{\delta \dot{X}^\mu(\xi)} \Rightarrow P_\mu(\xi) X'^\mu(\xi) = 0$$

$$P_\mu(\xi) P_\nu(\xi) G^{\mu\nu}(X(\xi)) + T^2 X'^\mu(\xi) X'^\nu(\xi) G_{\mu\nu}(X(\xi)) = 0$$

$$\dot{X}^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^0}, \quad X'^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^1}$$

Strings in Minkowski spacetime: action and equations of motion

Since classical string motion (and even more quantization) is already non-trivial in Minkowski spacetime let us consider that case (also needed for connection with DRM) but let's keep the dimensionality of spacetime D arbitrary.

$$S_{NG} = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}}$$

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

becomes: $$S_{NG} = -T \int d^2\xi \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}$$

$$\dot{X}^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \tau} \ , \quad X'^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \sigma}$$

Lorentz product understood

The equations of motion for the point-particle are trivial while for the string they look quite frightening:

$$\delta S_{NG} \propto \int d\tau \int_0^\pi d\sigma \left[\left(\frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X'^\mu} \right) \delta X^\mu - \frac{\partial}{\partial \sigma} \left(\frac{\partial L}{\partial X'^\mu} \delta X^\mu \right) \right]$$

$$\begin{aligned} \frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^\mu} \quad &+ \quad \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X'^\mu} = 0 \\ \frac{\partial L}{\partial \dot{X}^\mu} \quad &= \quad T \frac{\dot{X}_\mu X'^2 - X'_\mu (\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} \\ \frac{\partial L}{\partial X'^\mu} \quad &= \quad T \frac{X'_\mu \dot{X}^2 - \dot{X}_\mu (\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} \end{aligned}$$

Boundary conditions

Boundary conditions are very important and they differ in a crucial way for open and closed strings. We need, at all τ,

$$\left(\frac{\partial L}{\partial X'^{\mu}}\delta X^{\mu}\right)(\sigma = 0) = \left(\frac{\partial L}{\partial X'^{\mu}}\delta X^{\mu}\right)(\sigma = \pi) \quad ; \quad (\text{no sum over } \mu)$$

For closed strings the points $\sigma = 0$ and $\sigma = \pi$ are physically

the same point. If spacetime is topologically trivial this implies $X^{\mu}(0, \tau) = X^{\mu}(\pi, \tau)$ and the b.c. is satisfied.

For open strings we have two options:

Neumann b.c. $$\frac{\partial L}{\partial X'^{\mu}} = 0 \quad , \quad \sigma = 0, \pi$$

Dirichlet b.c. $$\delta X^{\mu} = 0 \quad , \quad \sigma = 0, \pi$$

For the moment we will consider N. b.c. for open strings

A convenient choice of coordinates

We can use WS reparametrization invariance to impose two useful conditions (defining the orthonormal gauges):

$$\begin{aligned} \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \\ \dot{X} \cdot X' &= 0 \quad , \quad \text{i.e.} \ (\dot{\mathrm{X}} \pm \mathrm{X}')^2 = 0 \end{aligned}$$

Also: $$\frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu = P_\mu \ ; \ \frac{\partial L}{\partial X'^\mu} = -TX'_\mu$$

and the simple equation of motion: $\ddot{X}_\mu = X''_\mu$

w/ solution: $X_\mu(\sigma,\tau) = F_\mu(\tau - \sigma) + G_\mu(\tau + \sigma)$

Note that the transformations $\tau \pm \sigma \to f_\pm(\tau \pm \sigma)$

keep us in the ON gauge (residual gauge transformations).

General solution of $\ddot{X}_\mu = X''_\mu$ and of b.c.

Open (Neumann) strings ($X'_\mu(\sigma=0, \pi)=0$). Def. $2\pi\alpha' = 1/T$

$$X_\mu(\sigma,\tau) = q_\mu + 2\alpha' p_\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-in\tau} - \frac{a^*_{n,\mu}}{\sqrt{n}}e^{in\tau}\right] cos(n\sigma)$$

Closed strings $X_\mu(\sigma=0) = X_\mu(\sigma=\pi)$

$$\begin{aligned} X_\mu(\sigma,\tau) &= q_\mu + 2\alpha' p_\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-2in(\tau-\sigma)} - \frac{a^*_{n,\mu}}{\sqrt{n}}e^{2in(\tau-\sigma)}\right] \\ &+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\tilde{a}_{n,\mu}}{\sqrt{n}}e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^*_{n,\mu}}{\sqrt{n}}e^{2in(\tau+\sigma)}\right] \end{aligned}$$

to be added

$$\begin{aligned} \dot{X}\cdot X' &= 0 \\ \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \end{aligned}$$

Simplest classical solution: open string, the rotating rod

The equations to be solved are $\ddot{X}_\mu = X''_\mu$

subject to the constraints

$$\begin{aligned} \dot{X} \cdot X' &= 0 \\ \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \end{aligned}$$

and to the b.c. $X'_\mu(\sigma = 0, \pi) = 0 \Rightarrow \sum_i \left(\frac{dX_i}{dX_0}\right)^2 (\sigma = 0, \pi) = 1$

A simple solution is:

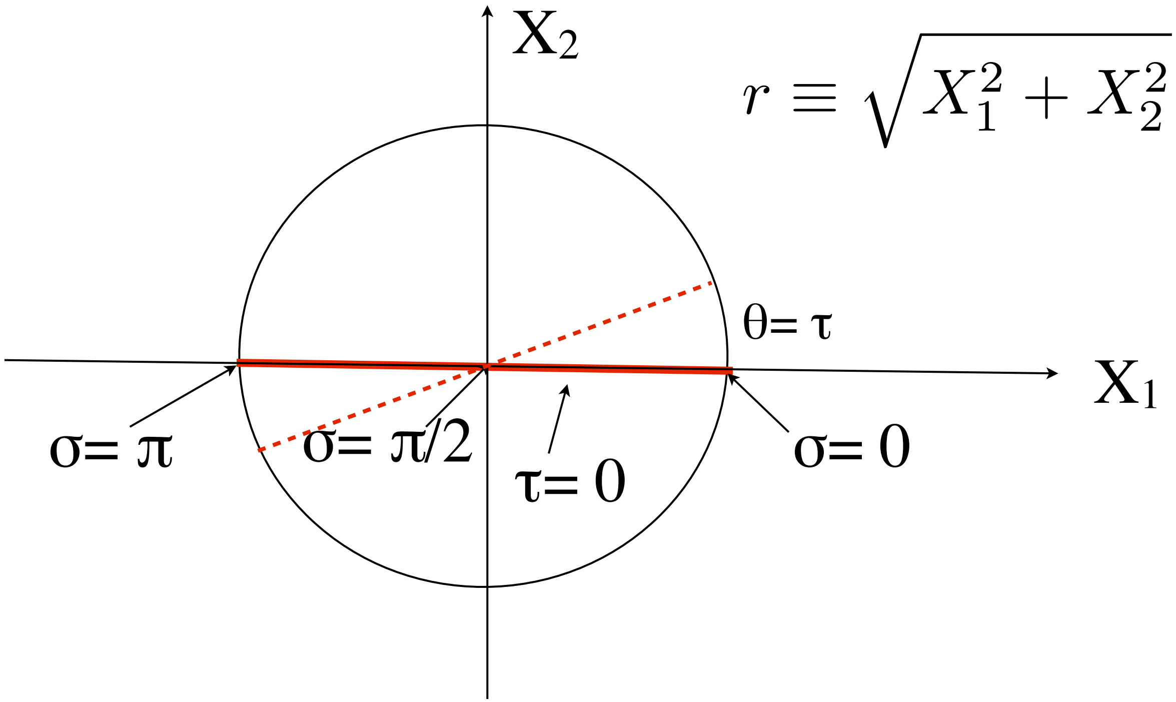
$$X_0 = A\tau \quad ; \quad X_1 = A\, cos\tau\, cos\sigma \;,\; X_2 = A\, sin\tau\, cos\sigma$$

$$X_i = 0 \;,\; (i = 3, 4, \ldots D - 1)$$

e.o.m., constraints and b.c. easily checked!

$$X_0 = A\tau \quad ; \quad X_1 = A\ cos\tau\ cos\sigma\ ,\ X_2 = A\ sin\tau\ cos\sigma$$

$$X_i \quad = \quad 0 \quad , \quad (i = 3, 4, \ldots D - 1)$$



$$r \equiv \sqrt{X_1^2 + X_2^2} = A|cos\sigma|$$

A rigid, rotating rod whose ends move with the speed of light since $dl/dX_0 = r\ d\,\theta\ /Ad\,\tau\ = r/A = |\cos\sigma|$.

Let us now compute the energy (mass) and angular momentum of this classical string.

$$X_0 = A\tau \quad ; \quad X_1 = A\ cos\tau\ cos\sigma\ ,\ X_2 = A\ sin\tau\ cos\sigma$$

$$X_i \quad = \quad 0 \quad , \quad (i = 3, 4, \ldots D-1)$$

$$p_i = \int_0^\pi d\sigma P_i(\sigma) = T \int_0^\pi d\sigma \dot{X}_i = 0$$

$$E = M = \int_0^\pi d\sigma P_0(\sigma) = T \int_0^\pi d\sigma \dot{X}_0 = \pi T A$$

$$\begin{aligned} J_{12} \quad &= \quad \int_0^\pi d\sigma (X_1 P_2 - X_2 P_1)(\sigma) \\ &= \quad TA^2 \int_0^\pi d\sigma (sin^2\tau cos^2\sigma + cos^2\tau cos^2\sigma) = \frac{\pi}{2} TA^2 \end{aligned}$$

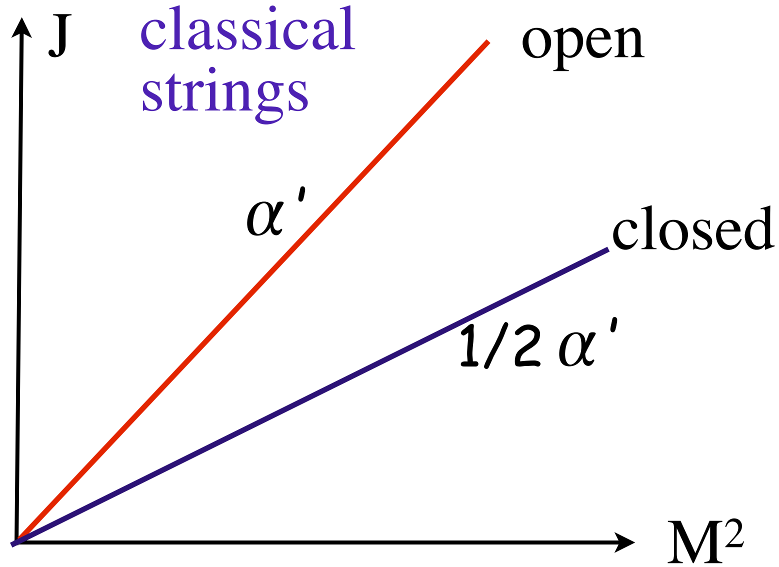
thus $$J = \frac{M^2}{2\pi T} = \alpha' M^2\ ,\ \alpha' \equiv \frac{1}{2\pi T}$$

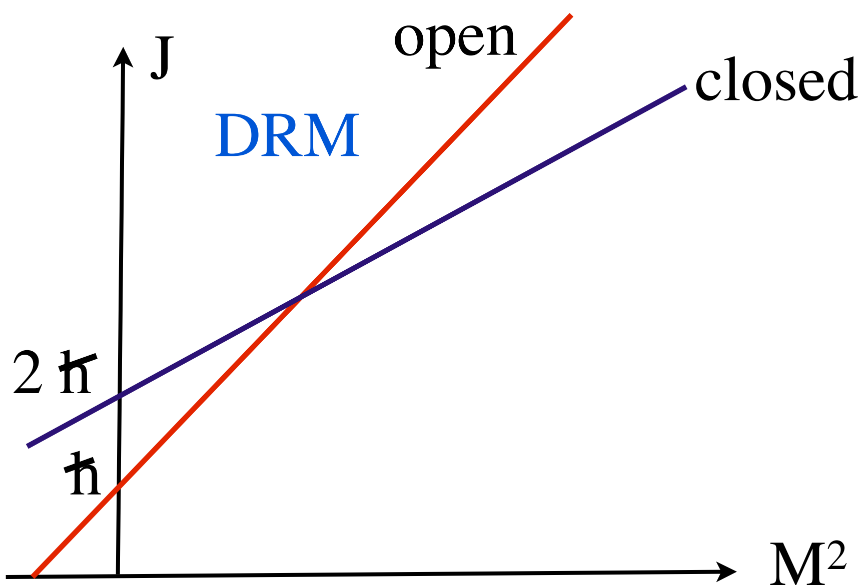
It is quite obvious that this solution maximizes the ratio J/M^2. The relation is very similar to the one given by the linear Regge trajectory we have been discussing in DRM.

For closed strings one finds the same relation between J and M^2 except for T --> 2 T (simple interpretation: for the same total length the closed string is half as big since it has to "come back on itself"). Hence α' -> 1/2 α'

Yet classical strings and DRM differ in crucial way.

In the classical theory, J and M^2 can take any real value with $J < \alpha' M^2$ => classical strings cannot have J without having mass! But in the DRM there were such states:





Classical vs. quantum strings

The discrepancy between strings and the DRM disappears completely once we move from classical to quantum strings

This is where points and strings start to differ in a fundamental, qualitative way.

Point particles can be quantized in an arbitrary background metric. This turns out not to be true for strings!

As we shall see, even Minkowski spacetime is in general forbidden... unless D takes a (so-called) critical value!

String quantization is not trivial and can be done in many different ways. Fortunately the end results are always the same.

Here we shall discuss the first succesful attempt to quantize strings based on the light-cone gauge.

In the seminar we will see more modern ways...

Quantization (open string)

In order to quantize the string we proceed as in any D=2 QFT and promote X and P to non-commuting operators:

$$X_\mu(\sigma,\tau) = q_\mu + 2\alpha' p_\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-in\tau} - \frac{a^\dagger_{n,\mu}}{\sqrt{n}}e^{in\tau}\right] cos(n\sigma)$$

$$\begin{aligned} X_\mu(\sigma,\tau) &= q_\mu + 2\alpha' p_\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-2in(\tau-\sigma)} - \frac{a^\dagger_{n,\mu}}{\sqrt{n}}e^{2in(\tau-\sigma)}\right] \\ &+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\tilde{a}_{n,\mu}}{\sqrt{n}}e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^\dagger_{n,\mu}}{\sqrt{n}}e^{2in(\tau+\sigma)}\right] \end{aligned}$$

Using the standard h. osc. c.r. we get the desired result:

$$[X_\mu(\sigma,\tau), P_\nu(\sigma',\tau)] = i\eta_{\mu\nu}\delta(\sigma-\sigma') \quad , \quad (\hbar = 1)$$

The only tricky things to take care of are the constraints!

Light-cone quantization (Goddard, Goldstone, Rebbi & Thorn, 1973)

The residual freedom to perform conformal transformations:

$$\tau \pm \sigma \to f_\pm(\tau \pm \sigma)$$

allows us to (almost completely) fix one of the coordinates. For the rigid rod we took $X_0 = A\,\tau$, but for the general case it is more useful to perform an infinite boost and fix instead:

$$X^+(\sigma,\tau) = 2\alpha' p^+ \tau \quad ; \quad X^\pm(\sigma,\tau) \equiv \frac{X^0 \pm X^{D-1}}{\sqrt{2}}$$

$$P^+(\sigma,\tau) \; = \; T\dot{X}^+(\sigma,\tau) = \frac{1}{\pi} p^+$$

The constraints can be solved for X^- since one must have:

$$2\dot{X}^-\dot{X}^+ = \sum_{i=1}^{(D-2)} \left(\dot{X}_i^2 + {X'_i}^2\right) \; ; \; X'^-\dot{X}^+ = \sum_{i=1}^{(D-2)} \dot{X}_i X'_i$$

$$\dot{X}^- = \frac{1}{4\alpha' p^+} \sum_{i=1}^{(D-2)} \left(\dot{X}_i^2 + X_i'^2 \right) \; ; \; X'^- = \frac{1}{2\alpha' p^+} \sum_{i=1}^{(D-2)} \dot{X}_i X_i'$$

These equations allow to express the a^-_n oscillators in terms of the transverse ones (while the a^+_n are zero). Note that the a^-_n oscillators become bilinear in the transverse ones.

Therefore in this gauge we were able to solve the constraints and to reduce the physical spectrum to the one generated by (D-2) spacelike oscillators (Cf. DDF).

At this point it looks as if we managed to eliminate all the ghosts without getting any constraint on α_0 or on D.

Does it mean that the string is smarter than the DRM? Unfortunately the answer is negative.

The problem is that the l.c. gauge breaks explicit Lorentz invariance: we have to check that L.I. is still there, even if hidden ...

A Lorentz anomaly?

We have to check the O(D-1,1) Lorentz algebra:

$$
\begin{aligned}
i[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} \\
M_{\mu\nu} &\equiv \int d\sigma \; (P_\mu X_\nu - P_\nu X_\mu)
\end{aligned}
$$

The check is easy for the compact O(D-2) subgroup (L_{ij} are bilinear in the oscillators) but becomes highly non-trivial for the components of the Lorentz generators involving the ± directions (these may involve three oscillators). In particular $[M_{+i}, M_{+j}]$ should vanish (recall that $\eta_{++} = 0$) while a long but straightforward calculation (GGRT) gives:

$$
[M_{+i}, M_{+j}] \propto \sum_{n=1}^{\infty} \left[n^2 \left(\frac{D-2}{24} - 1 \right) - \left(\frac{D-2}{24} - \alpha_0 \right) \right] \left(a_n^{\dagger i} a_n^j - a_n^{\dagger j} a_n^i \right)
$$

We thus find that the Lorentz algebra is broken unless $\alpha_0 = 1$ and D = 26, i.e. the same conditions we found in the DRM!

Relation to DRM formalism

What is the relation between the operator Q(z) of the DRM and the string position operator X(σ , τ)?

It turns out that Q(z) describes the motion of one end of the open string:

$$Q^\mu(z) = X^\mu(\tau, \sigma = 0) \text{ with } z = e^{i\tau}$$

Our real variable z corresponds to having made all calculation using Euclidean world-sheet time: $\tau_E = -i\,\tau$.

The relation is more direct for closed strings since in the DRM their treatment needs a complex z-variable and this can be traded for both τ (giving |z|) and σ (providing the phase of z):

$$Q^\mu(z, \bar{z}) = X^\mu(e^{-\tau_E}e^{-i\sigma}, e^{-\tau_E}e^{+i\sigma})$$

Counting physical states

- In the l.c. quantization the mass-shell condition becomes:

$$L_0 = 1 \Rightarrow \alpha' M^2 + 1 = \sum_{n,\mu} n \; a^\dagger_{n,\mu} \; a^\mu_n \Rightarrow \sum_{n,i=1}^{D-2} n \; a^\dagger_{n,i} \; a^i_n$$

and the number of physical states w/ $\alpha' M^2 = (N-1)$ is given by the number of solutions of the equation (in the integers $N_{n,i}$):

$$\alpha' M^2 + 1 = N = \sum_{n,i=1}^{D-2} n N_{n,i}$$

- This is the famous "Partitio Numerorum" problem solved long ago by the Hardy-Ramanujan formula (for D-2 =1):

$$d(N) = N^{-p} e^{2\pi\sqrt{\frac{N(D-2)}{6}}} = N^{-p} e^{2\pi\sqrt{\frac{\alpha'(D-2)}{6}}M}$$

- Although unexpected, this was just the behaviour postulated by R. Hagedorn a few years earlier (~1965) on a more phenomenological basis (e.g. a Boltzmann factor in the "transverse energy" of particles produced in high energy hadronic collisions).

- Taken at face value, such a density of states leads to a limiting (maximal, Hagedorn) temperature (FV, 1969) since:

$$Z(\beta) \equiv Tr[e^{-\beta H}] \sim \int dM\ d(M) e^{-\beta M} \sim \int dM\ e^{cM - \beta M}$$

diverges for $\beta = 1/(k_B T) < c$.

- Here the limiting temperature T_H is given by:

$$T_H = \frac{1}{2\pi\sqrt{\alpha'}}\sqrt{\frac{6}{D-2}} \rightarrow \frac{1}{4\pi\sqrt{\alpha'}} \quad \text{if} \quad D = 26$$

i.e. a maximal temperature of order a few hundred MeV (if we set D=4 and take for α' the experimental value)!

- T_H has an interesting reinterpretation in QCD as the deconfining temperature above which hadronic matter (protons, Nuclei) "melts" and makes a phase transition to the quark-gluon plasma phase (something that must have happened in the very early Universe in the opposite direction).
- In string theory such an interpretation is absent.