NON TOPOLOGICAL CONJUGACY OF SKEW PRODUCTS IN $SU(2)$

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VERY PRELIMINARY VERSION

[1]

0. Introduction

We fix $\alpha \in \mathbb{R} - \mathbb{Q}$. Given $A \in C^\infty(\mathbb{S}^1, SU(2))$ then $G_{\alpha,A} = r_\alpha \times A$ acts naturally on $\mathbb{S}^1 \times \mathbb{S}^3$ ($\mathbb{S}^3 \cong SU(2)$) where $r_\alpha : z \mapsto e^{2\pi i\alpha} z$.

Let

$$V_\alpha = \{A \in C^\infty(\mathbb{S}^1, SU(2)), (G^n_{\alpha,A})_{n\in\mathbb{Z}} \text{ acting on } \mathbb{S}^1 \times \mathbb{S}^3 \text{ is not equicontinuous}\}.$$

We propose to show that the closure of V_α for the C^∞ topology contains $SU(2)$ ($B \in SU(2) \subset C^\infty(\mathbb{S}^1, SU(2))$) is identified to the constant map $x \in \mathbb{S}^1 \to B \in SU(2)$).

Let us note for that every $A \in V_\alpha$, $G_{\alpha,A}$ is not C^0 conjugated on $\mathbb{S}^1 \times SU(2)$ to action of $R_\alpha \times B$ on $\mathbb{S}^1 \times SU(2)$ for any $B \in SU(2)$.

1. Notations

1.1. $\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$, $SU(2) = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$.

If $A \in SU(2)$, $A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix} = {}^t\bar{A}$.

$$C^\infty(\mathbb{S}^1, SU(2)) = \{A : \mathbb{S}^1 \to SU(2) \mid \text{is a map of class } C^\infty\}.$$

If $\alpha, \beta \in \mathbb{R}$, $\lambda_\alpha = e^{2\pi i\alpha}$, $\lambda_\beta = e^{2\pi i\beta}$, $r_\alpha : z \to e^{2\pi i\alpha} z$.

1.2. On $G = \mathbb{S}^1 \times C^\infty(\mathbb{S}^1, SU(2))$ we put the group law

$$(\lambda_\alpha, A) \cdot (\lambda_\beta, B) = (\lambda_{\alpha+\beta}, A \circ r_\alpha B).$$

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G acts on $\mathbb{S}^1 \times \mathbb{S}^3 \subset \mathbb{S}^1 \times \mathbb{C}^2$, $\mathbb{S}^3 = \{(z_1, z_2), |z_1|^2 + |z_2|^2 = 1\}$, $g = (\lambda_\alpha, A) \in G$ by

$$(1.3) \quad (z, y) \to (r_\alpha(z), A(z)y) = g \cdot (z, y)$$

and if $g_1, g_2 \in G$, $(g_2 g_1)(z, y) = (g_2(g_1(z, y))$ (i.e. $\mathbb{S}^1 \times \mathbb{S}^3$ is a G-space).

Let $G_0 = \{(1, A), A \in C^\infty(\mathbb{S}^1, SU(2))\}$. G_0 is a normal subgroup of G.

1.3 We put on G the C^∞ topology. We denote by d_∞ a metric on G such that d_∞ defines the C^∞ topology on G and G is complete for d_∞. G therefore is a *Baire space* as well as all its closed subsets. For the C^∞ topology G is a topological group, and G_0 is closed in G. G is a Polish topological group.

2. Let $\alpha \in \mathbb{R} - \mathbb{Q}$. If $x \in \mathbb{R}$ we denote $\|x\|_a = \inf_{p \in \mathbb{Z}} |x + p|$ (group metric on $\mathbb{R}/\mathbb{Z}$).

Let $\psi(n) = e^{-e^n}$, $n \in \mathbb{N}$.

We fix $\alpha \in \mathbb{R} - \mathbb{Q}$.

Let $l(\beta) = \inf_{\substack{n \geq 1 \\ n \in \mathbb{N}}} \|n\alpha + 2\beta\|_a (\psi(n))^{-1}$.

2.1.Lemma. *$l^{-1}(0)$ a dense G_δ of $\mathbb{R}$.*

Proof. $\beta \to l(\beta) \in \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ is upper semicontinuous, hence $l^{-1}(0)$ is a G_δ. It is dense since it contains $\{2n\alpha + p, p \in \mathbb{Z}, -n \in \mathbb{N}^*\}$. □

2.2 The set $G_\alpha = l^{-1}(0) - \{\frac{n\alpha}{2} + \frac{p}{q}, n \in \mathbb{Z}, p \in \mathbb{Z}, q \in \mathbb{N}^*\}$ is also a dense G_δ of $\mathbb{R}$. If $\beta \in G_\alpha$, $2\beta + n\alpha \notin \mathbb{Q}$ for every $n \in \mathbb{Z}$.

3. Let $\alpha \in \mathbb{R} - \mathbb{Q}$ fixed. We define

$$O_\alpha^\infty(\mathbb{S}^1) = \{g^{-1}(\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar{\lambda}_\beta \end{pmatrix})g \mid \beta \in \mathbb{R}, g \in G_0\}$$

We denote by $\overline{O}_\alpha^\infty$ the closure for the C^∞ topology of O_α^∞ in G. $\overline{O}_\alpha^\infty$ with the induced C^∞ topology is a Baire space (cf. 1.3).

4. Let $\delta = (\delta_1, \delta_2) \in \mathbb{S}^3$, (i.e. $|\delta_1|^2 + |\delta_2|^2 = 1$)

$$A_{\delta,n}(z) = \begin{pmatrix} \delta_1 & -\bar{\delta}_2 \bar{z}^n \\ \delta_2 z^n & \bar{\delta}_1 \end{pmatrix} \in SU(2).$$

We have

$$A_{\delta,n}^{-1} \circ r_\alpha(z) = \begin{pmatrix} \bar\delta_1 & \bar\delta_2 \bar z^n \bar\lambda_\alpha^n \\ -\delta_2 z^n \lambda_\alpha^n & \delta_1 \end{pmatrix}$$

$$(4.1) \quad \left(A_{\delta,n}^{-1} \circ r_\alpha \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar\lambda_\beta \end{pmatrix} A_{\delta,n} \right)(z) \equiv C_{\delta,n,\beta}(z) =$$

$$\begin{pmatrix} \bar\delta_1 & \bar\delta_2 \bar z^n \bar\lambda_\alpha^n \\ -\delta_2 z^n \lambda_\alpha^n & \delta_1 \end{pmatrix} \begin{pmatrix} \lambda_\beta \delta_1 & -\lambda_\beta \bar\delta_2 \bar z^n \\ \bar\lambda_\beta^n \delta_2 z^n & \bar\lambda_\beta \bar\delta_1 \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_\beta \delta_1 \bar\delta_1 + \delta_2 \bar\delta_2 \bar\lambda_\alpha^n \bar\lambda_\beta & \bar\delta_1 \bar\delta_2 \bar z^n [\bar\lambda_a^n \bar\lambda_\beta - \lambda_\beta] \\ \delta_1 \delta_2 z^n [-\lambda_\beta \lambda_a^n + \bar\lambda_\beta] & \bar\lambda_\beta \bar\delta_1 \delta_1 + \delta_2 \bar\delta_2 \lambda_\alpha^n \lambda_\beta \end{pmatrix}$$

since $\delta_2 \bar\delta_2 = (1 - \delta_1 \bar\delta_1)$

$$(4.2) \quad C_{\delta,n,\beta}(z) = \begin{pmatrix} [-\bar\lambda_\alpha^n \bar\lambda_\beta + \lambda_\beta] \delta_1 \bar\delta_1 + \bar\lambda_\alpha^n \bar\lambda_\beta & \bar\delta_1 \bar\delta_2 \bar z^n [\bar\lambda_\alpha^n \bar\lambda_\beta - \lambda_\beta] \\ \delta_1 \delta_2 z^n [-\lambda_\beta \lambda_\alpha^n + \bar\lambda_\beta] & \lambda_\alpha^n \lambda_\beta + \delta_1 \bar\delta_1 [\bar\lambda_\beta - \lambda_\alpha^n \lambda_\beta] \end{pmatrix}$$

from (4.2) we conclude

$$A_{\delta,n}^{-1} \circ r_{p\alpha} \begin{pmatrix} \lambda_\beta^n & 0 \\ 0 & \bar\lambda_\beta^n \end{pmatrix} A_{\delta,n} =$$

$$\begin{pmatrix} \bar\delta_1 \delta_1 [\lambda_{p\beta} - \lambda_{-pn\alpha - p\beta}] + \bar\lambda_\alpha^{n\beta} \bar\lambda_\beta^p & \bar\delta_1 \bar\delta_2 z^n [\bar\lambda_\alpha^{np} \bar\lambda_\beta^p - \lambda_\beta^p] \\ \delta_1 \delta_2 z^n [-\lambda_{p\beta + np\alpha} + \lambda_{-p\beta}] & \lambda_\alpha^{np} \lambda_\beta^p + \delta_1 \bar\delta_1 [\bar\lambda_\beta^p - \lambda_\alpha^{pn} \lambda_\beta^p] \end{pmatrix}$$

5. We choose $\delta = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. We note $h_n = (1, A_{\delta,n}) \in G_0$. We consider

$$c_{n,\alpha,\beta} = h_n^{-1}(\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar\lambda_\beta \end{pmatrix}) h_n \in O_\alpha^\infty$$

We write $c_{n,\alpha,\beta} = (\lambda_\alpha, C_{n,\beta})$. We suppose that $\beta \in G_\alpha$. Using 2.1 and 2.2 we can find a sequence $n_j \to \infty$ such that

$$|\bar\lambda_\alpha^{n_j} \bar\lambda_\beta^2 - 1| \le C \|n_j \alpha + 2\beta\|_a \le C e^{-e^{n_j}}$$

(C is a universal constant)

We conclude from (4.2)

5.1 Proposition 1 $\forall \beta \in G_\alpha$, $\exists n_j \to \infty$ *such that*

$$c_{n_j,\alpha,\beta} \to (\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar\lambda_\beta \end{pmatrix})$$

in the C^∞ *topology.*

We write for $p \in \mathbb{N}$

$$c_{n_j,\alpha,\beta}^p = (\lambda_{p\alpha}, C_{n_j,\beta}^{(p)}) = h_n^{-1}(\lambda_{p\alpha}, \begin{pmatrix} \lambda_\beta^p & 0 \\ 0 & \bar\lambda_\beta^p \end{pmatrix}) h_n.$$

We suppose that $\beta \in G_\alpha$. We have (cf. (4.3)) [2]

$$(5.2)\qquad \sup_{p\geq 1}(\|C^{(p)}_{n_j,\beta}(1)\begin{pmatrix}1\\0\end{pmatrix} - C^{(p)}_{n_j,\beta}(e^{i\pi/n_j})\begin{pmatrix}1\\0\end{pmatrix}\| = 2$$

(This follows from the fact that

$$\|C^{(p)}_{n_j,\beta}(1)\begin{pmatrix}1\\0\end{pmatrix} - C^{(p)}_{n_j,\beta}(z)\begin{pmatrix}1\\0\end{pmatrix}\| = |\lambda_{pn_j\alpha+2p\beta} - 1||1-z^n|/2$$

and $\sup_{p\geq 1}|\lambda_{p(n_j\alpha+2\beta)} - 1| = 2$ since $n_j\alpha + 2\beta \notin \mathbb{Q}$, when $\beta \in G_\alpha$ (see (2.2)).

6. Let $h = (1,\beta) \in G_0$, $\beta \in G_\beta$; then $F_j = h^{-1}C_{n_j,\alpha,\beta}h \to h^{-1}(\lambda_\alpha, \begin{pmatrix}\lambda_\beta & 0\\ 0 & \bar\lambda_\beta\end{pmatrix})h = \hat F$ in the C^∞ topology, where n_j is the sequence given by proposition 1.

6.1. Proposition 2. *Given $h \in G_0$, $\beta \in G_\alpha$ and n_j the sequence given by proposition 1, let $F^p_j = (\lambda_{p\alpha}, Q^{(p)}_j)$, $p \in \mathbb{N}^*$. We can find $y_j, y'_j \in \mathbb{S}^3$ such that*

$$(6.2)\qquad \sup_{p\geq 1}\|Q^{(p)}_j(1)y_j - Q^{(p)}_j(e^{i\pi/n_j})y'_j\| > 1$$

and

$$(6.3)\qquad \lim_{j\to\infty}\|y_j - y'_j\| = 0.$$

Proof. Let $y_j = B^{-1}(1)\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{S}^3$, $y'_j = B^{-1}(e^{i\pi/n_j})\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{S}^3$. Since $\theta \mapsto B^{-1}(\theta)$ is continuous $\|y_j - y'_j\| \to 0$ when $j \to \infty$. We have

$$\|Q^{(p)}_j(1)y_j - Q^{(p)}_j(e^{i\pi/n_j})y'_j\| \leq$$
$$\|B^{-1}(\lambda_{p\alpha})C^{(p)}_{n_j,\beta}(1)\begin{pmatrix}1\\0\end{pmatrix} - B^{-1}(\lambda_{p\alpha})C^{(p)}_{n_j,\beta}(e^{i\pi/n_j})\begin{pmatrix}1\\0\end{pmatrix}\| +$$
$$2|||B^{-1}(\lambda_{p\alpha}) - B^{-1}(\lambda_{p\alpha}e^{i\pi/n_j})|||$$
$$= \|C^{(p)}_{n_j,\beta}(1)\begin{pmatrix}1\\0\end{pmatrix} - C^{(p)}_{n_j,\beta}(e^{i\pi/n_j})\| + 2|||B^{-1}(\lambda_{p\alpha}) - B^{-1}(\lambda_{p\alpha}e^{i\pi/n_j})|||.$$

Since $\theta \to B^{-1}(\theta)$ is continuous, the second term $\to 0$ when $j \to \infty$ and the proposition follows from 5.2.

□

[2](*) On $\mathbb{C}^2$ we put the norm $\|(z_1,z_2)\|^2 = |z_1|^2 + |z_2|^2$; $|||\cdot|||$ denotes the induced operator norm.

6.4 Let us formulate the proposition in another way. We define the metric d on $\mathbb{S}^1 \times \mathbb{S}^3$ by

$$d((z,y),(z',y')) = \sup(|z-z'|, \|y-y'\|)$$

where $(z,y),(z',y') \in \mathbb{S}^1 \times \mathbb{S}^3$.

6.5 *F_j acts on $\mathbb{S}^1 \times \mathbb{S}^3$ by 1.3 and we can find $v_j = (1, y_j)$ $v'_j = (e^{i\pi/n_j}, y'_j)$ such that $d(v_j, v'_j) \to 0$ as $j \to \infty$ and*

$$\sup_{p\geq 1} d(F_j^p(v_j), F_j^p(v'_j)) > 1$$

7. Given $\varepsilon > 0$ we define the set

$$U_\varepsilon = \{F = (\lambda_\alpha, B) \in \overline{O}_\alpha^\infty \mid \exists v, v' \in \mathbb{S}^1 \times \mathbb{S}^3 \text{ such that } d(v,v') \leq \varepsilon \text{ and } \sup_{p\geq 1} d(F^p(v), F^p(v')) > 1\}$$

7.1. Lemma. *The set U_ε is open in $\overline{O}_\alpha^\infty$ for the C^∞ topology.*

Proof.

$$U_\varepsilon = \bigcup_{\substack{v,v' \\ d(v,v')\leq\varepsilon}} \{F, \sup_{p\geq 1} d(F^p(v), F^p(v')) > 1\}$$

i.e. U_ε is the union of the sets $\{F, \sup_{p\geq 1} d(F^p(v), F^p(v')) > 1, v, v' \in \mathbb{S}^1 \times \mathbb{S}^3, d(v,v') \leq \varepsilon\}$; each set $\{F, \sup_{p\geq 1} d(F^p(v), F^p(v')) > 1\}$ is open since G is a topological group and for fixed v, v' and $p \in \mathbb{N}$, $F \in G \to d(F^p(v), F^p(v'))$ is continuous; hence $F \mapsto \sup_{p\geq 1} d(F^p(v), F^p(v'))$ is lower semi continuous.

□

7.2. Proposition. *For every $\varepsilon > 0$, U_ε is dense in $\overline{O}_\alpha^\infty$ for the C^∞ topology.*

Proof. It is enough to show $\bar{U}_\varepsilon$ (the closure of U_ε in $\overline{O}_\alpha^\infty$) contains the following dense set of O_α^∞, for the C^∞ topology on $\overline{O}_\alpha^\infty$:

$$V = \{h^{-1}(\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar{\lambda}_\beta \end{pmatrix})h, \beta \in G_\alpha, h \in G_0\} \subset O_\alpha^\infty(\mathbb{S}^1)$$

(G_α is dense in $\mathbb{R}$ and $O_\alpha^\infty(\mathbb{S}^1)$ is dense in $\overline{O}_\alpha^\infty(\mathbb{S}^1)$ by definition of $\overline{O}_\alpha^\infty$). Given $\hat{F} = h^{-1}(\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar{\lambda}_\beta \end{pmatrix} h$, by 6.5 we can find a sequence

$(F_j)_{j\in\mathbb{N}} \subset O^\infty_\alpha(\mathbb{S}^1)$, $F_j \to \hat{F}$ in the C^∞ topology, when $j \to \infty$ such that, when j is large enough, $F_j \in U_\varepsilon$. □

8. Let $\varepsilon_j > 0$, $\varepsilon_j \to 0$; by 7.1 and 7.2 $K_\alpha = \bigcap_j U_{\varepsilon_j}$ is a dense G_δ of $\overline{O}^\infty_\alpha$ ($\overline{O}^\infty_\alpha$ is a Baire space for the C^∞ topology) (everything is always for the C^∞ topology !)

Theorem. *Given $\alpha \in \mathbb{R} - \mathbb{Q}$, and $\beta \in \mathbb{R}$, we can find $H_j = (\lambda_\alpha, B_j) \in \overline{O}^\infty_\alpha$, $j \in \mathbb{N}$, $H_j \to (\lambda_\alpha, \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \bar{\lambda}_\beta \end{pmatrix})$ in the C^∞ topology when $j \to \infty$ and for every j there does not exist a homeomorphism k of $\mathbb{S}^1 \times \mathbb{S}^3$ that conjugates H_j acting on $\mathbb{S}^1 \times \mathbb{S}^3$ to any linear map $(\lambda_{\alpha'}, \begin{pmatrix} \lambda_{\beta'} & 0 \\ 0 & \bar{\lambda}_{\beta'} \end{pmatrix})$ acting on $\mathbb{S}^1 \times \mathbb{S}^3$, $\alpha', \beta' \in \mathbb{R}$.*

Proof. If $H = (\lambda_\alpha, B) \in K_\alpha \subset \overline{O}^\infty_\alpha$ then $\forall \varepsilon_j > 0$ $\exists v_j, v'_j \in \mathbb{S}^1 \times \mathbb{S}^3$, $d(v_j, v'_j) \leq \varepsilon_j$ such that

$$\sup_{p\geq 1} d(H^p(v_j), H^p(v'_j)) > 1;$$

hence the sequence of diffeomorphisms $(H^p)_{p\in\mathbb{Z}}$ acting on $\mathbb{S}^1 \times \mathbb{S}^3$ is *not uniformly equicontinuous.* □