# A PROOF OF THE SIEGEL-BRJUNO THEOREM

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**Theorem 0.1.** *Let $\alpha$ be a Brjuno number, $\lambda = \exp(2\pi i\alpha)$, and $F(z) = \lambda z + O(z^2)$ a germ of holomorphic diffeomorphism. Then $F$ is analytically linearizable.*

*Proof.* Write

$$F(z) = \lambda z \sum_{n \geqslant 0} F_n z^n$$

with $F_0 = 1$, $\lambda = \exp 2\pi i\alpha$. We look for a conjugacy

$$h(z) = z \sum_{n \geqslant 0} h_n z^n,$$

with $h_0 = 1$, satisfying $F \circ h(z) = h(\lambda z)$. The coefficients $h_n$ are inductively given for $n > 0$ by

$$h_n = (\lambda^n - 1)^{-1} \sum_{k \geqslant 1, n_j \geqslant 0, n_0 + \ldots + n_k = n-k} F_k h_{n_0} \ldots h_{n_k}.$$

After rescaling if necessary, we may assume that $|F_n| \leqslant 1$ for $n \geqslant 1$. Also, for every real number $x$, we have

$$|\exp 2\pi i x - 1| \geqslant 4||x||$$

where $||x||$ is the distance to the nearest integer. Thus, if $\overline{h}_n$ is the sequence inductively define by $\overline{h}_0 = 1$ and

$$\overline{h}_n = (4||n\alpha||)^{-1} \sum_{k \geqslant 1, n_j \geqslant 0, n_0 + \ldots + n_k = n-k} \overline{h}_{n_0} \ldots \overline{h}_{n_k},$$

we have $|h_n| \leqslant \overline{h}_n$ for all $n \geqslant 0$. It is therefore sufficient to show that the sequence $\overline{h}_n$ grows at most exponentially fast.

Next, we deal with the number of terms: define inductively $\widetilde{h}_n$ by $\widetilde{h}_0 = 1$ and

$$\widetilde{h}_n = (4||n\alpha||)^{-1} \max_{k \geqslant 1, n_j \geqslant 0, n_0 + \ldots + n_k = n-k} \widetilde{h}_{n_0} \ldots \widetilde{h}_{n_k}$$

for $n > 0$. Write $\overline{h}_n = \widetilde{h}_n H_n$. We then have $H_0 = 1$ and

$$H_n \leqslant \sum_{k \geqslant 1, n_j \geqslant 0, n_0 + \ldots + n_k = n-k} H_{n_0} \ldots H_{n_k}$$

for $n > 0$. The sequence $H_n$ grows at most exponentially fast, because it is termwise dominated by the coefficients of the conjugacy for the attractive germ $F_0(z) := \frac{1}{2} z(1 - \sum_{n>0} z^n)$. It is therefore sufficient to see that the sequence $\widetilde{h}_n$ grows at most exponentially fast.

Denote by $\{x\}$ the fractional part of a real number $x$. Let $j$ be a nonnegative integer. define, for $n > 0$

$$\sigma_j^+(n) = \begin{cases} 1 & \text{if } 0 \leqslant \{n\alpha\} \leqslant \frac{1}{2}2^{-2^j} \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$\sigma_j^-(n) = \begin{cases} 1 & \text{if } 0 \leqslant 1 - \{n\alpha\} \leqslant \frac{1}{2}2^{-2^j} \\ 0 & \text{otherwise} \end{cases}$$

Observe that we have, for any $n > 0$

$$(4||n\alpha||)^{-1} \leqslant \prod_{j\geqslant 0} 2^{2^j\sigma_j^+(n)} \prod_{j\geqslant 0} 2^{2^j\sigma_j^-(n)}.$$

Indeed, if $\{n\alpha\} \in (\frac{1}{4}, \frac{3}{4})$, we have $\sigma_j^+(n) = \sigma_j^-(n) = 0$ for all $j$ and $4||n\alpha|| \geqslant 1$. If $\{n\alpha\} \in (\frac{1}{2}2^{-2^m}, \frac{1}{2}2^{-2^{m-1}}]$ for some $m > 0$, we have $\sigma_j^-(n) = 0$ for all $j$, $\sigma_j^+(n) = 0$ for $j \geqslant m$ and $\sigma_j^+(n) = 1$ for $j < m$, hence

$$\prod_{j\geqslant 0} 2^{2^j\sigma_j^+(n)} = \frac{1}{2}2^{2^m} \geqslant (4||n\alpha||)^{-1}.$$

The case where $\{n\alpha\} > \frac{3}{4}$ is symmetric.

For $j \geqslant 0$, we define inductively $\Sigma_j^+(n)$ by $\Sigma_j^+(0) = 0$ and, for $n > 0$

$$\Sigma_j^+(n) = \sigma_j^+(n) + \max_{k\geqslant 1, n_j\geqslant 0, n_0+\ldots+n_k=n-k} (\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k)).$$

Define similarly $\Sigma_j^-(n)$. From the inequality above, we get

$$\widetilde{h}_n \leqslant \prod_{j\geqslant 0} 2^{2^j\Sigma_j^+(n)} \prod_{j\geqslant 0} 2^{2^j\Sigma_j^-(n)}.$$

It remains to estimate $\Sigma_j^+(n)$ and $\Sigma_j^-(n)$.

**Lemma 0.2.** *Let $q(j)$ be the smallest positive integer $q$ such that $||q\alpha|| < \frac{1}{2}2^{-2^j}$. Then we have $\Sigma_j^\pm(n) = 0$ for $n < q(j)$ and, for all $n \geqslant q(j)$*

$$\Sigma_j^\pm(n) \leqslant 2\lfloor \frac{n}{q(j)} \rfloor - 1,$$

*where $\lfloor x \rfloor$ denotes the integer part of $x$.*

*Proof.* We have $\sigma_j^\pm(n) = 0$ for $n < q(j)$, hence $\Sigma_j^\pm(n) = 0$ for $n < q(j)$. For $n \geqslant q(j)$, we will prove the inequality of the lemma by induction on $\lfloor \frac{n}{q(j)} \rfloor$. We deal for instance with $\Sigma_j^+(n)$. Observe that if $n, n'$ are two distinct integers such that $\sigma_j^+(n) = \sigma_j^+(n') = 1$, one must have $|n - n'| \geqslant q(j)$, hence there is at most one integer $n^*$ with a given value of $\lfloor \frac{n^*}{q(j)} \rfloor$ such that $\sigma_j^+(n^*) = 1$.

First consider the initial step $\lfloor \frac{n}{q(j)} \rfloor = 1$. Define $n^*$ to be the integer in $[q(j), 2q(j))$ such that $\sigma_j^+(n^*) = 1$ if such an integer exists; otherwise, set $n^* = 2q(j)$. It is clear that $\Sigma_j^+(n) = 0$ for $n < n^*$ and $\Sigma_j^+(n) = 1$ for $n^* \leqslant n < 2q(j)$, hence the inequality of the lemma holds when $\lfloor \frac{n}{q(j)} \rfloor = 1$.

Consider now the case where $\lfloor \frac{n}{q(j)} \rfloor = \ell > 1$. Define $n^*$ to be the integer in $[\ell q(j), (\ell + 1)q(j))$ such that $\sigma_j^+(n^*) = 1$ if such an integer exists; otherwise, set $n^* = (\ell + 1)q(j)$.

We first see (by induction) that $\Sigma_j^+(n) \leqslant 2\ell - 2$ for $n \in [\ell q(j), n^*)$. Let $n \in [\ell q(j), n^*)$, $k \geqslant 1$ and let $n_0, \ldots, n_k$ be nonnegative integers with $n_0 + \ldots + n_k = n - k$. One has then

$$\lfloor \frac{n_0}{q(j)} \rfloor + \ldots + \lfloor \frac{n_k}{q(j)} \rfloor \leqslant \ell$$

and even

$$\lfloor \frac{n_0}{q(j)} \rfloor + \ldots + \lfloor \frac{n_k}{q(j)} \rfloor \leqslant \ell - 1$$

when $n = \ell q(j)$. By the induction hypothesis, this gives

$$\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k) \leqslant 2\ell - 2$$

if $n = \ell q(j)$ and otherwise

$$\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k) \leqslant 2\ell - k',$$

where $k'$ is the number of $n_s$ which are $\geqslant q(j)$.

- If $k' > 1$, we have
$$\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k) \leqslant 2\ell - 2;$$
- If $k' = 1$, say $n_0 \geqslant q(j)$, we have also (by induction)
$$\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k) = \Sigma_j^+(n_0) \leqslant 2\ell - 2;$$
- If $k' = 0$, we have
$$\Sigma_j^+(n_0) + \ldots + \Sigma_j^+(n_k) = 0 \leqslant 2\ell - 2;$$

In all cases, we are able to conclude that $\Sigma_j^+(n) \leqslant 2\ell - 2$ for $n \in [\ell q(j), n^*)$. In the same way, one shows that $\Sigma_j^+(n) \leqslant 2\ell - 1$ for $n \in [n^*, (\ell + 1)q(j))$. □

Thus, we see that if the series

$$\sum_{j \geqslant 0} \frac{2^j}{q(j)}$$

are convergent, then $\widetilde{h}_n$ grows at most exponentially fast and the formal conjugacy defines an holomorphic map in the neighborhood of the origin.

But the convergence of this series follows from (actually is equivalent to) the Brjuno condition: denoting by $(\frac{p_n}{q_n})$ the convergents of $\alpha$, one has $q(j) = q_{n(j)}$, with

$$(2q_{n(j)+1})^{-1} < ||q(j)\alpha|| \leqslant \frac{1}{2} 2^{-2^j},$$

hence

$$\frac{2^j}{q(j)} \leqslant \frac{\log q_{n(j)+1}}{q_{n(j)} \log 2};$$

(several $q(j)$ can be equal; in this case, we only keep the term with higher $j$). This concludes the proof!

□