A PROOF OF HALL'S THEOREM

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Theorem 0.1. *Let $K_1, K_2 \subset \mathbb{R}$ be Cantor sets such that $\tau(K_1)\,\tau(K_2) > 1$. Then at least one of the following properties hold*

1. *K_1 is contained in a connected component of $\mathbb{R} \setminus K_2$;*
2. *K_2 is contained in a connected component of $\mathbb{R} \setminus K_1$;*
3. *$K_1 \cap K_2 \neq \emptyset$.*

Proof. We assume that none of the properties in the theorem is satisfied and obtain a contradiction. We choose, for $i = 1, 2$, trivializations $h_i : \mathcal{A}^{\mathbb{Z}_+} \to K_i$ such that $\tau(K_1, h_1)\,\tau(K_2, h_2) > 1$.

Two non-trivial compact intervals $J_1, J_2 \subset \mathbb{R}$ are said to be *entangled* if J_1, J_2 intersect but one has neither $J_1 \subset J_2$ nor $J_2 \subset J_1$. Writing $J_i = [a_i, b_i]$, it means that either $a_1 < a_2 \leqslant b_1 < b_2$ or $a_2 < a_1 \leqslant b_2 < b_1$.

As none of the first two properties in the theorem is satisfied , the intervals $J(h_1, \emptyset)$ and $J(h_2, \emptyset)$ intersect. We may assume that $|J(h_2, \emptyset)| \leqslant |J(h_1, \emptyset)|$. We first show

Lemma 0.2. *There exists a finite word Θ on the alphabet $\mathcal{A}$ such that the intervals $J(h_2, \emptyset)$, $J(h_1, \Theta)$ are entangled.*

Proof. Indeed, if $J(h_1, \emptyset)$ and $J(h_2, \emptyset)$ are entangled, we take $\Theta = \emptyset$. If not, we have $J(h_2, \emptyset) \subset J(h_1, \emptyset)$. We cannot have $J(h_2, \emptyset) \subset G(h_1, \emptyset)$ because K_2 is not contained in a connected component of $\mathbb{R} \setminus K_1$. Therefore there exists $\theta_1 \in \mathcal{A}$ such that $J(h_2, \emptyset)$ intersects $J(h_1, \theta_1)$. If $J(h_2, \emptyset)$, $J(h_1, \theta_1)$ are not entangled, we must have $J(h_2, \emptyset) \subset J(h_1, \theta_1)$: indeed the reverse inclusion $J(h_2, \emptyset) \supset J(h_1, \theta_1)$ is not possible because it would imply that $J(h_2, \emptyset)$ and $J(h_1, \theta_1)$ have a common endpoint, contradicting $K_1 \cap K_2 = \emptyset$. From $J(h_2, \emptyset) \subset J(h_1, \theta_1)$, we obtain by the same argument that there exists $\theta_2 \in \mathcal{A}$ such that either $J(h_2, \emptyset)$ and $J(h_1, \theta_1\,\theta_2)$ are entangled or $J(h_2, \emptyset) \subset J(h_1, \theta_1\,\theta_2)$. As $|J(h_1, \Theta)|$ converges to 0 as $|\Theta| \to \infty$, the process cannot go home indefinitely and we obtain the conclusion of the lemma $\square$

The lemma gives the starting point of an induction. The induction step is given by the

Lemma 0.3. *Let Θ_1, Θ_2 be finite words such that $J(h_1, \Theta_1)$ and $J(h_2, \Theta_2)$ are entangled. Then there exist words Θ_1', Θ_2' such that*

1. *For $i = 1, 2$, Θ_i is an initial word of Θ_i' with $\Theta_i' = \Theta_i$ or $|\Theta_i'| = |\Theta_i| + 1$. Moreover, there exists $i \in \{1, 2\}$ such that $|\Theta_i'| = |\Theta_i| + 1$.*
2. *$J(h_1, \Theta_1')$ and $J(h_2, \Theta_2')$ are entangled.*

Proof. One may assume that $h_1(\Theta_1\,\bar{0}) < h_2(\Theta_2\,\bar{0}) \leqslant h_1(\Theta_1\,\bar{1}) < h_2(\Theta_2\,\bar{1})$. One must actually have $h_2(\Theta_2\,\bar{0}) < h_1(\Theta_1\,\bar{1})$ as $K_1 \cap K_2 = \emptyset$.

If $h_1(\Theta_1\,1\,\bar{0}) < h_2(\Theta_2\,\bar{0})$, the words $\Theta_1' := \Theta_1\,1$ and $\Theta_2' := \Theta_2$ satisfy the conditions of the lemma. The equality $h_1(\Theta_1\,1\,\bar{0}) = h_2(\Theta_2\,\bar{0})$ is impossible because $K_1 \cap K_2 = \emptyset$.

Similarly, if $h_1(\Theta_1\,\bar{1}) < h_2(\Theta_2\,0\,\bar{1})$, the words $\Theta_1' := \Theta_1$ et $\Theta_2' := \Theta_2\,0$ satisfy the conditions of the lemma. The equality $h_1(\Theta_1\,\bar{1}) = h_2(\Theta_2\,0\,\bar{1})$ is impossible because $K_1 \cap K_2 = \emptyset$.

In the remaining case, one has $h_1(\Theta_1\,1\,\bar{0}) > h_2(\Theta_2\,\bar{0})$ and $h_1(\Theta_1\,\bar{1}) > h_2(\Theta_2\,0\,\bar{1})$. If $h_1(\Theta_1\,1\,\bar{0}) < h_2(\Theta_2\,0\,\bar{1})$, the words $\Theta_1' := \Theta_1\,1$, $\Theta_2' := \Theta_2\,0$ satisfy the conditions of the lemma. The equality $h_1(\Theta_1\,1\,\bar{0}) = h_2(\Theta_2\,0\,\bar{1})$ is impossible because $K_1 \cap K_2 = \emptyset$. Finally, the case $h_1(\Theta_1\,1\,\bar{0}) > h_2(\Theta_2\,0\,\bar{1})$ subdivides as follows

- If $h_2(\Theta_2\,1\,\bar{0}) < h_1(\Theta_1\,\bar{1})$, one takes $\Theta_1' := \Theta_1$, $\Theta_2' := \Theta_2\,1$.
- Si $h_1(\Theta_1\,0\,\bar{1}) > h_2(\Theta_2\,\bar{0})$ one takes $\Theta_1' := \Theta_1\,0$, $\Theta_2' := \Theta_2$
- As $K_1 \cap K_2 = \emptyset$ the equalities $h_2(\Theta_2\,1\,\bar{0}) = h_1(\Theta_1\,\bar{1})$ and $h_1(\Theta_1\,0\,\bar{1}) > h_2(\Theta_2\,\bar{0})$ are impossible.
- If one had $h_2(\Theta_2\,1\,\bar{0}) > h_1(\Theta_1\,\bar{1})$ and $h_1(\Theta_1\,0\,\bar{1}) < h_2(\Theta_2\,\bar{0})$, one would have $J(h_1, \Theta_1\,1) \subset G(h_2, \Theta_2)$ and $J(h_2, \Theta_2\,0) \subset G(h_1, \Theta_1)$. But this is not compatible with $\tau(K_1, h_1, \Theta_1)\,\tau(K_2, h_2, \Theta_2) > 1$.

□

From the two lemmas, one get two sequence of words $(\Theta_1(n))_{n\geqslant 0}$, $(\Theta_2(n))_{n\geqslant 0}$ with the following properties:

- for all $i = 1, 2$ and $0 \leqslant m \leqslant n$, $\Theta_i(m)$ is an initial word of $\Theta_i(n)$;
- for all $n \geqslant 0$, one has $|\Theta_1(n)| + |\Theta_2(n)| \geqslant n$;
- for all $n \geqslant 0$, $J(h_1, \Theta_1(n))$ and $J(h_2, \Theta_2(n))$ are entangled.

The second property implies that $\lim_{n\to\infty} \inf(|J(h_1, \Theta_1(n))|, |J(h_2, \Theta_2(n))|) = 0$. The third property implies that $J(h_1, \Theta_1(n)) \cap J(h_2, \Theta_2(n)) \neq \emptyset$. This contradicts the assumption that $K_1 \cap K_2 = \emptyset$.

□