A QUESTION OF BARRY SIMON

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[1]

Introduction. We propose to show that the Lyapunoff exponent of $R_\alpha \times \begin{pmatrix} \lambda\cos(2\pi\theta) & -1 \\ 1 & 0 \end{pmatrix} = R_\alpha \times A_{\lambda,0}$ acting on $\mathbb{T}^1 \times \mathbb{R}^2$ by $(\theta, y) \mapsto (\theta+\alpha, A_{\lambda,0}(\theta)y)$ is equal to 0 when α satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \, Log\, |e^{2\pi i n\alpha} - 1|^{-1} = 0 \tag{0.1}$$

and $|\lambda| \leq 2$. This also implies that the Lyapunoff exponent of $R_\alpha \times A_{\lambda,0}$ is equal to zero when α belongs to a dense G_δ of $\mathbb{T}^1$ (i.e. the generic Liouville numbers). This is a counter example to a conjecture of Barry Simon as reported by Tom Spencer in the fall 1988 at Princeton University.

For $\lambda \in \mathbb{C}$, $E \in \mathbb{R}$, $\alpha \in \mathbb{R}$ we denote

$$A_{\lambda,E}(\theta) = \begin{pmatrix} \lambda\cos(2\pi\theta) + E & -1 \\ 1 & 0 \end{pmatrix}$$

$R_\alpha : \theta \mapsto \theta + \alpha$, for $\theta \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

We suppose that α satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \, Log\, |e^{2\pi i n\alpha} - 1|^{-1} = 0 \tag{0.2}$$

(e.g. α is a diophantine number).

Proposition. *If α satisfies (??) and $|\lambda|/2 \geq 1$ then* Lyapunoff$(R_\alpha, A_{\lambda,0}) =$ $Log\,|\lambda/2|$ *(we suppose $(R_\alpha, A_{\lambda,0)})$ acts on $\mathbb{T}^1 \times \mathbb{R}^2$ in its natural way).*

Proof. We denote

$$B_\lambda(z) = \begin{pmatrix} \frac{\lambda}{2}(1+z^2) & -z \\ z & 0 \end{pmatrix}, \qquad z \in \mathbb{C},$$

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$r_\alpha(z) = e^{2\pi i\alpha}z$ and $\mathbb{S}_r = \{z \in \mathbb{C}, |z| = r\}$. (r_α, B_λ) acts on $\mathbb{S}_r \times \mathbb{C}^2$ by $G : (z, y) \mapsto (r_\alpha(z), B_\lambda(z)y)$. We fix $\alpha \in \mathbb{R}$; then the following function is subharmonic

$$\begin{aligned}\lambda_+(r) &= \text{Lyapunoff}((r_\alpha, B_\lambda) \text{ acting on } \mathbb{S}_r \times \mathbb{C}^2\} \\ &= \inf_{n\geq 1} \int_0^1 \frac{1}{n} Log\, |||B^{(n)}(re^{2\pi i\theta})|||d\theta\end{aligned}$$

where $B^n = B_\lambda \circ r_{(n-1)\alpha} B_\lambda \circ r_{(n-2)\alpha} \cdots B_\lambda$; $\|\cdot\|$ is a norm on $\mathbb{C}^2$ and $|||\cdot|||$ denotes the induced operator norm.

Hence the function $u \mapsto \lambda_+(e^u)$ is convex for $u \in \mathbb{R}$ and therefore continuous.

We have $\lambda_+(1) = \text{Lyapunoff}(R_\alpha, A_{\lambda,0})$. It is enough to prove, when $r < 1$,

$$\lambda_+(r) = Log\, |\lambda/2|.$$

We look for $z \in \mathbb{D} = \{|z| < 1\} \to v(z) = \begin{pmatrix}\eta(z)\\ \eta_1(z)\end{pmatrix} \in \mathbb{C}^2$ an analytic function such that

$$B_\lambda(z)v(z) = \frac{\lambda}{2} v \circ r_\alpha(z), \quad \|v(z)\| \neq 0,\ \forall z \in \mathbb{D}$$

and $v(0) = \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{C}^2$.

As $\det B_\lambda(z) = z^2$ and on $\mathbb{S}_r$ v is continuous, for $0 < r < 1$, (r_α, B_λ) has a uniform hyperbolic structure (acting on $\mathbb{S}_r \times \mathbb{C}^2$) and

$$\lambda_+(r) = Log\, |\frac{\lambda}{2}|.$$

We have to show the existence of the function v; $v(z) = \begin{pmatrix}\eta(z)\\ \eta_1(z)\end{pmatrix}$ satisfies

$$\begin{aligned}\frac{\lambda}{2}(1+z^2)\eta - z\eta_1 &= \frac{\lambda}{2}\eta \circ r_\alpha \\ z\eta &= \frac{\lambda}{2}\eta_1 \circ r_\alpha\end{aligned}$$

hence

$$\frac{\lambda}{2}(1+z^2)\eta - \frac{2e^{-2\pi i\alpha}}{\lambda} z^2 \eta \circ r_{-\alpha} = \frac{\lambda}{2}\eta \circ r_\alpha$$

$$\Updownarrow$$

$$\eta \circ r_\alpha - \eta = z^2\eta - 4\frac{e^{-2\pi i\alpha}}{\lambda^2} z^2 \eta \circ r_{-\alpha}.$$

We write

$$\eta(z) = \sum_{n\geq 0} a_n z^n, \qquad a_0 = 1$$

hence

$$a_{2p+1} = 0, \quad p \geq 1 \quad p \in \mathbb{N}$$

and

$$a_{n+2} = \frac{1}{\lambda_\alpha^{n+2} - 1}(1 - \frac{4}{\lambda^2}\bar{\lambda}_\alpha \lambda_\alpha^{-n})a_n, \qquad \lambda_\alpha = e^{2\pi i\alpha}$$

therefore

$$a_{2n} = \prod_{p=1}^{n} \frac{1}{\lambda_\alpha^{2p} - 1}\varphi((2p-2)\alpha)$$

with

$$\varphi(\theta) = (1 - \frac{4}{\lambda^2}\bar{\lambda}_\alpha e^{-2\pi i\theta}).$$

For $|\lambda/2| \geq 1$, we have

$$\int_0^1 Log\,|\varphi(\theta)|d\theta = 0$$

$$= \int_0^1 Log\,|\varphi(-\theta)|d\theta = \Re\frac{1}{2\pi i}\int_0^1 Log(1 - \frac{4}{\lambda^2}z)\frac{dz}{z} = 0$$

When α satisfies (**??**), 2α satisfies (**??**) and by a result of Hardy and Littlewood

$$\lim_{n\to\infty}\left(\prod_{p=1}^{n}\frac{1}{|\lambda_\alpha^{2p} - 1|}\right)^{1/n} = 1.$$

If $|\lambda/2| > 1$, $\lim_{n\to\infty} \| \prod_{p=1}^{n} \varphi((2p-2)\alpha + \theta)\|_{C^0}^{1/n} = 1$, but when $|\lambda/2| = 1$ it is not difficult to see that

$$\limsup_{n\to\infty} \| \prod_{p=1}^{n} \varphi((2p-2)\alpha + \theta)\|_{C^0}^{1/n} \leq 1.$$

In conclusion, when $|\lambda/2| \geq 1$, we find a solution η analytic on $\mathbb{D}$ (since $\limsup_{n\to\infty} |a_{2n}|^{1/2n} \leq 1$) with $\eta(0) = 1$ and such that

$$v(z) = \begin{pmatrix} \eta(z) \\ \frac{2\bar{\lambda}_\alpha}{\lambda} z\eta \circ r_{-\alpha}(z) \end{pmatrix}$$

satisfies

$$B_\lambda(z)v(z) = \frac{\lambda}{2} v \circ r_\alpha(z).$$

If for some $z_0 \in \mathbb{D}$, $\|v(z_0)\| = 0$, as $\alpha \in \mathbb{R} - \mathbb{Q}$, we have $\|v(z_0 e^{2\pi i\theta})\| = 0$, $\forall\theta \in \mathbb{R}$; this implies $\eta(z) = 0$ on $\mathbb{D}$ what contradicts $\eta(0) = 1$! □

Consequences

1) By Aubry's duality and Thouless formula Lyapunov$(R_\alpha, A_{\lambda,0}) = 0$ when $|\lambda/2| \leq 1$.

2) Given λ, $|\lambda/2| \leq 1$, then there exists a dense G_δ, $G \subset \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$ such that if $\alpha \in G$,

$$\text{Lyapunoff}(R_\alpha, A_{\lambda,0}) = 0.$$

This follows from the fact that $l : \alpha \in \mathbb{T}^1 \to \text{Lyapunov}(R_\alpha, A_{\lambda,0}) \in \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ is upper semi-continuous. $l^{-1}(0)$ is therefore a G_δ; it is dense since it contains the numbers that satisfy (**??**). $l^{-1}(0) \cap (\mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z}))$ is also a dense G_δ.

Point 2) contradicts a conjecture (?) of B. Simon ?? [2]

[2]Question marks are by M.R Herman