ON INVARIANT SETS UNDER THE DOUBLING MAP

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Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, $f(x) = 2x \mod 1$. For $x \in \mathbb{R}$ we note $\|x\| = \inf_{p\in\mathbb{Z}} |x+p|$; $\|\cdot\|$ defines a group metric on $\mathbb{T}^1$, i.e. satisfies $\|-x\| = \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$, for x and y in $\mathbb{T}^1$.

Proposition (Veech(?), Douady and al. (?))**.** *Let $K \subset \mathbb{T}^1$ be compact invariant by f, (i.e. $f(K) = K$) such that $f_{|K} = g$ is a homeomorphism; then K is finite.*

Proof. g^{-1} is uniformly continous: there exists $\delta > 0$ such that $\|x - y\| \leq \delta$, $x, y \in K$ implies $\|g^{-1}(x) - g^{-1}(y)\| \leq 1/10$. We can and will suppose that $\delta \leq 1/10$. If $\|x-y\| < \delta$ then $\|g^{-1}(x) - g^{-1}(y)\| \leq \delta/2$. Indeed, $g^{-1}(x) - g^{-1}(y) = \frac{1}{2}(x-y) + \varepsilon/2 \mod 1$ where $\varepsilon = 0, 1$; but as $\|g^{-1}(x) - g^{-1}(y)\| \leq 1/10$ and $\|\frac{1}{2}(x-y)\| = \frac{1}{2}\|x-y\|$, we have $\varepsilon = 0$. Hence, when $\|x-y\| \leq \delta$,

$$\|g^{-j}(x) - g^{-j}(y)\| \leq \delta/2^j,\ j = 1, \ldots \tag{0.1}$$

The sequence $(g^{-j})_{j\geq 1}$ is equicontinuous by Ascoli's theorem: we can find a sequence $0 < j_0 < j_1 < \cdots < j_p < j_{p+1} < \cdots$ such that $(g^{-j_p})_{p\geq 0}$ tends uniformly to a continuous map $\hat{g} : K \to K$ when $p \to \infty$. We need the following lemma:

Lemma. *$\hat{g}$ is surjective.*

Proof. Given any $x \in K$, let $x_{j_p} = g^{j_p}(x)$. We have $g^{-j_p}(x_{j_p}) = x$. We can find a convergent subsequence of $(x_{j_p})_{p\geq 0}$ converging to $z \in K$. We have $\hat{g}(z) = x$ which thus shows that $\hat{g}$ is surjective. □

Equation (**??**) implies that $\hat{g}$ is locally constant and therefore by compactness of K, $\hat{g}$ takes a finite number of values. As $\hat{g}$ is surjective, K is finite. □

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