An example of non convergence of Birkhoff sums[1]

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We consider the 2-torus $\mathbb{T}^2$and a diffeomorphism $f : (\theta, r) \mapsto (\theta + \ell(r), g(r))$, where ℓ is a C^∞ function and g is the projective transformation $g(r) = \frac{r}{r+1}$. We suppose $\ell(r) = r$ for r small, hence, in a neighbourhood of $\mathbb{T}^1 \times \{0\}$ we have

$$f(\theta, r) = \left(\theta + r, \frac{r}{r+1}\right) .$$

Let φ be a non-negative C^∞ function on $\mathbb{T}^2$, such that:

$$\varphi(x) = 1, \text{ if } x \in B_{1/20},$$

$$\varphi(x) = 0, \text{ if } x \notin B_{1/10},$$

where B_R is the Euclidian ball of radius R. Let $S_n(\theta, r) = \frac{1}{n} \sum_0^{n-1} \varphi \circ f^k(\theta, r)$ denote the nth Birkhoff sum.

Proposition. *For $r \neq 0$, the sequence $S_n(\theta, r)$ does not have a limit when n goes to $+\infty$.*

Proof. As $r \neq 0$, we have $0 < g^n(r) < 1/20$ for large n. Therefore, it is enough to prove the proposition when $0 < r < 1/20$. Let $f^n(\theta, r) = (\theta_n, r_n)$. We have

$$(1) \qquad r_n = \frac{r}{1 + nr} = \frac{1}{n} - \frac{1}{rn^2} + O\left(\frac{1}{n^3}\right),$$

and $\theta_n = \theta_0 + r_0 + r_1 + \cdots + r_{n-1}$. As $\sum_0^{n-1} r_j \to \infty$ and $r_n \to 0$ when n goes to ∞, the sequence (θ_n, r_n) passes through $B_{1/20}$ and $\mathbb{T}^2 \setminus B_{1/10}$ infinitely often.
Given $N >> 1$, let $n_1 > N$ be the first integer such that $(\theta_{n_1}, r_{n_1}) \in B_{1/20}$ and $(\theta_{n_1-1}, r_{n_1-1}) \notin B_{1/20}$; let $n_1' > n_1$ be the first integer such that $(\theta_{n_1'}, r_{n_1'}) \in B_{1/20}$ and $(\theta_{n_1'+1}, r_{n_1'+1}) \notin B_{1/20}$. We define similarly integers $n_2' > n_2 \geq N$ for the set $\mathbb{T}^2 \setminus B_{1/10}$. We have

$$\sum_{n_1}^{n_1'} r_j = \frac{1}{10} + O\left(\frac{1}{n_1}\right),$$

$$\sum_{n_2}^{n_2'} r_j = \frac{4}{5} + O\left(\frac{1}{n_2}\right),$$

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and hence, using (1),

$$\frac{n_1'}{n_1} = e^{1/10} + O\left(\frac{1}{n_1}\right), \tag{2}$$

$$\frac{n_2'}{n_2} = e^{4/5} + O\left(\frac{1}{n_2}\right). \tag{3}$$

Therefore we have

$$S_{n_1'}(\theta_0, r_0) = e^{-1/10} S_{n_1}(\theta_0, r_0) + (1 - e^{-1/10}) + O\left(\frac{1}{n_1}\right), \tag{4}$$

$$S_{n_2'}(\theta_0, r_0) = e^{-4/5} S_{n_2}(\theta_0, r_0) + O\left(\frac{1}{n_2}\right). \tag{5}$$

Assume that the sequence $S_n(\theta, r)$ converges to a limit a. From (4) and (5), we have

$$a = e^{-1/10}a + 1 - e^{-1/10}, \ a = e^{-4/5}a.$$

These equations have no common solution. The proposition follows. $\square$