On a problem of A. Katok[1]

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A. Katok asked the following question:

Does there exist a volume-preserving diffeomorphism of a compact manifold with zero topological entropy such that the differential of the iterates grows faster than any polynomial at every point?

Answer: Yes, even a C^ω-diffeomorphism.

We will construct minimal translations R_α, R'_α on $\mathbb{T}$, real analytic complex-valued functions φ, φ' on $\mathbb{T}$ of mean value zero, and an increasing sequence A_k going to $+\infty$ such that the Birkhoff sums

$$S_n\varphi(\theta) := \sum_{j=0}^{n-1} \varphi(\theta + j\alpha) \text{ and } S_n\varphi'(\theta) := \sum_{j=0}^{n-1} \varphi'(\theta + j\alpha')$$

satisfy

$$\begin{aligned} |S_n\varphi(\theta)| &\geq \sqrt{n} \quad \text{for all } k \geq 1,\ A_{2k-1} \leq n \leq A_{2k},\ \theta \in \mathbb{T}\,, \\ |S_n\varphi'(\theta')| &\geq \sqrt{n} \quad \text{for all } k \geq 1,\ A_{2k} \leq n \leq A_{2k+1},\ \theta' \in \mathbb{T}\,. \end{aligned}$$

Write $\varphi = \varphi_1 + i\varphi_2$, $\varphi' = \varphi_3 + i\varphi_4$. Let Γ be a cocompact lattice of $SL(2,\mathbb{R})$ and F be the diffeomorphism of $\mathbb{T}^2 \times \left(SL(2,\mathbb{R})_{/\Gamma}\right)^4$ defined by:

$$(\theta, \theta', y_1, y_2, y_3, y_4) \mapsto (\theta + \alpha, \theta' + \alpha', A_1(\theta)y_1, A_2(\theta)y_2, A_3(\theta')y_3, A_4(\theta')y_4)$$

where, for $i = 1, 2, 3, 4$,

$$A_i(\theta) = \begin{pmatrix} e^{\varphi_i(\theta)} & 0 \\ 0 & e^{-\varphi_i(\theta)} \end{pmatrix}.$$

The diffeomorphism F is analytic and volume preserving. By unique ergodicity of R_α and $R_{\alpha'}$, the C^1-norm of the iterates of F grows subexponentially, hence the topological entropy of $F$is zero. On the other hand, for $n \geq 0$, up to a multiplicative constant one has at every point:

$$\|DF^n(\theta, \theta', y_1, y_2, y_3, y_4)\| \geq exp\left(\sup\left(|S_n\varphi_1(\theta)|, |S_n\varphi_2(\theta)|, |S_n\varphi_3(\theta')|, |S_n\varphi_4(\theta')|\right)\right) \geq e^{\sqrt{\frac{n}{2}}}.$$

To construct $\alpha, \alpha', \varphi, \varphi'$ we follow partially Yoccoz [1], Appendice 1, p. 215-224.

Let $(p_k/q_k)_{k\geq 0}$ be the convergents of α (to be constructed). Define $\varphi(\theta) = \sum_{k\geq 1} 2^{-q_k} exp(2\pi i q_k \theta)$.

This is a complex-valued analytic function on $\mathbb{T}$ with mean value zero. For $n \geq 0$, we have

$$S_n\varphi(\theta) = \sum_{k\geq 1} 2^{-q_k} \frac{\sin \pi n q_k \alpha}{\sin \pi q_k \alpha} exp(2\pi i q_k \theta).$$

[1]Ce document extrait des archives de Michel Herman a été préparé par F. Laudenbach et J.-C. Yoccoz, qui a apporté une simplification par rapport au manuscrit original.

From the classical estimates of the convergents

$$\frac{1}{2q_{k+1}} < |q_k\alpha - p_k| < \frac{1}{q_{k+1}}\,,$$

we obtain

$$\frac{\sin \pi n q_k \alpha}{\sin \pi q_k \alpha} \leq \min\left(\frac{q_{k+1}}{2}, n\right), \quad \text{for all } n,$$

and

$$\frac{\sin \pi n q_k \alpha}{\sin \pi q_k \alpha} \geq \frac{2n}{\pi}, \quad \text{for } n \leq \frac{q_{k+1}}{2}.$$

When $\pi^2 2^{2q_k} < n \leq \frac{q_{k+1}}{2}$, we have for every $\theta \in \mathbb{T}$:

$$\left| 2^{-q_k} \frac{\sin \pi n q_k \alpha}{\sin \pi q_k \alpha} exp(2\pi i q_k \theta) \right| \geq 2^{-q_k} \frac{2n}{\pi} \geq 2\sqrt{n},$$

$$\left| \sum_{\ell<k} 2^{-q_\ell} \frac{\sin \pi n q_\ell \alpha}{\sin \pi q_\ell \alpha} exp(2\pi i q_\ell \theta) \right| \leq \sum_{\ell<k} 2^{-q_\ell} \frac{q_{\ell+1}}{2} < q_k,$$

$$\left| \sum_{\ell>k} 2^{-q_\ell} \frac{\sin \pi n q_\ell \alpha}{\sin \pi q_\ell \alpha} exp(2\pi i q_\ell \theta) \right| \leq n \sum_{\ell>k} 2^{-q_\ell} \leq q_{k+1} 2^{-q_{k+1}} < 1.$$

It follows that $|S_n\varphi(\theta)| \geq \sqrt{n}$ when $\pi^2 2^{2q_k} < n \leq \frac{q_{k+1}}{2}$.

We define similarly φ' from the sequence of convergents $(p'_k/q'_k)_{k\geq 0}$ of α'. To obtain the required property (with $A_{2k} = \frac{q_{k+1}}{2}$, $A_{2k+1} = \frac{q'_{k+1}}{2}$) it is therefore sufficient to pick α, α' such that their convergents satisfy, for all k,

$$\frac{q'_k}{2} \leq \pi^2 2^{2q'_k} \leq \frac{q_{k+1}}{2} \leq \pi^2 2^{2q_{k+1}} \leq \frac{q'_{k+1}}{2}.$$

References

[1] J.-C. Yoccoz, *Petits diviseurs en dimension 1*, Astérisque 231 (1995), Soc. Math. France.