A PROOF OF PYARTLI'S THEOREM

JEAN-CHRISTOPHE YOCCOZ

Theorem 0.1. *Let $\alpha : I \to \mathbb{R}^n$ a map which is (a, b) non-planar at each point of I. Let $\tau > n(n-2)$. There exists a constant $C = C(\tau)$ such that, for all $\gamma > 0$, one has*

$$\mathrm{Leb}\{t \in I,\, \alpha(t) \notin HDC(\gamma, \tau)\} < C(1 + \frac{b}{a}|I|)\left(\frac{\gamma}{a}\right)^{1/(n-1)}.$$

We will first prove the following

Proposition 0.2. *Let $n \geqslant 1$ and let $\alpha : I \to \mathbb{R}$ a C^n map on a compact interval I. Assume that there exist constants $b > a > 0$ such that, for any $t \in I$, one has*

$$b \geq \max_{0 \leq m \leq n} |D^m \alpha(t)| \geq a.$$

Then one has, for any $\gamma > 0$

$$\mathrm{Leb}\{t \in I,\, |\alpha(t)| < \gamma\} < C(1 + \frac{b}{a}|I|)\left(\frac{\gamma}{a}\right)^{1/n},$$

where the constant C depends only on n.

Proof. Replacing α, γ, b by α/a, γ/a, b/a, we may assume that $a = 1$. We may also assume that $Cb\gamma^{1/n} < 1$: otherwise, the right-hand side in the inequality of the proposition is $> |I|$ and there is nothing to prove. Denote by $I(\gamma)$ the open set $\{t \in I,\, |\alpha(t)| < \gamma\}$.

Lemma 0.3. *The open set $I(\gamma)$ has at most finitely many components.*

Proof. Otherwise, there would exist $\varepsilon \in \{-1, +1\}$ and a sequence (x_j) of points of I, converging to a limit x_*, such that $\alpha(x_j) = \varepsilon\gamma$ for all $j \geq 0$. Then we have also $\alpha(x_*) = \varepsilon\gamma$. By Taylor's formula (or Rolle's theorem), we must have $D^m\alpha(x_*) = 0$ for all $1 \leq m \leq n$. This contradicts the hypothesis of the proposition. □

Lemma 0.4. *Each connected component of $I(\gamma)$ has length $\leq C_0(n)\gamma^{1/n}$.*

Proof. Assume that $x_0 < x_1 < \ldots < x_{2^n-1}$ are points of $I(\gamma)$ with $x_{j+1} - x_j = 2\gamma^{1/n}$ for all $0 \leqslant j < 2^n - 1$. By the mean value theorem, there exists, for every $0 \leqslant j \leqslant 2^{n-1} - 1$, a point $x_j^{(1)} \in (x_{2j}, x_{2j+1})$ such that

$$|D\alpha(x_j^{(1)})| = \frac{|\alpha(x_{2j+1}) - \alpha(x_{2j})|}{x_{j+1} - x_j} < \gamma^{\frac{n-1}{n}}.$$

We also have $x_{j+1}^{(1)} - x_j^{(1)} \geqslant 2\gamma^{1/n}$ for $0 \leqslant j < 2^{n-1} - 1$. Proceeding in the same way, we construct, for each $1 \leq m \leq n$, a sequence $x_j^{(m)}$, $0 \leqslant j \leqslant 2^{n-m} - 1$ such that

- $x_0 < x_j^{(m)} < x_{2^n-1}$;
- $|D^m\alpha(x_j^{(m)})| < \gamma^{\frac{n-m}{n}}$;
- $x_{j+1}^{(m)} - x_j^{(m)} \geqslant 2\gamma^{1/n}$ for $0 \leqslant j < 2^{n-m} - 1$.

Let $x_* := x_0^{(n)}$. We have $|D^n\alpha(x_*)| < 1$, hence, from the hypothesis of the proposition, there exists $0 \leqslant m < n$ such that $|D^m\alpha(x_*)| > 1$. On the other hand, we have $|D^m\alpha(x_0^{(m)})| < \gamma^{\frac{n-m}{n}}$. This is not compatible with $|x_* - x_0^{(m)}| < 2^n\gamma^{1/n}$, $|D^{m+1}\alpha| \leqslant b$, $Cb\gamma^{1/n} < 1$ when C is large enough. □

Lemma 0.5. *Let $J_0 = (x_0, y_0), \ldots, J_n = (x_n, y_n)$ be $n+1$ consecutive connected components of $I(\gamma)$. We have $x_n - y_0 \geq \frac{1}{2}b^{-1}$.*

Proof. One has $\alpha(y_i) = \alpha(x_{i+1})$ for $0 \leq i < n$. By Rolle's theorem, there exists $z_i \in (y_i, x_{i+1})$ such that $D\alpha(z_i) = 0$. In the same way, we find, for each $1 \leqslant m \leqslant n$, $(n+1-m)$ distinct zeroes of $D^m\alpha$ in (y_0, x_n). In particular, let x_* be a zero of $D^n\alpha$ in (y_0, x_n). By the hypothesis of the proposition, there exists $0 \leq m < n$ such that $|D^m\alpha(x_*)| \geq 1$. On the other hand, there exists $y_* \in [y_0, x_n]$ such that $|D^m\alpha(y_*)| \leq \gamma$. As $|D^{m+1}\alpha| \leq b$ (and we may assume $\gamma < 1/2$), we must have $x_n - y_0 \geq |x_* - y_*| \geq \frac{1}{2}b^{-1}$. □

We can now prove the proposition. By Lemma 0.3, the open set $I(\gamma)$ has finitely many connected components. Let $J_i = (x_i, y_i)$, $0 \leqslant i \leqslant N$, be those components, written in ascending order. From Lemma 0.4, we have $|J_i| \leq C_0(n)\gamma^{1/n}$ for every $i \in [0, N]$. On the other hand, from Lemma 0.5, we have $x_{i+n} - x_i \geq \frac{1}{2}b^{-1}$ for $0 \leqslant i < i+n \leqslant N$. If $N < n$, we have $|I(\gamma)| \leq n\ C_0(n)\gamma^{1/n}$. If $N \geq n$, we have $|I| \geq \lfloor\frac{N}{n}\rfloor\frac{1}{2}b^{-1}$, hence

$$|I(\gamma)| \leq N\ C_0(n)\gamma^{1/n} \leq 2n\lfloor\frac{N}{n}\rfloor C_0(n)\gamma^{1/n} \leq 4nC_0(n)b|I|\gamma^{1/n}.$$

The proof of the proposition is complete. □

We will now prove Pyartli's theorem. We use the Euclidean operator norm in the definition of non-planarity. Let $\tau > n(n-2)$ and let $\gamma > 0$.

Let $k \in \mathbb{Z}^n$, $k \neq 0$. Define $\alpha_k(t) :=< \frac{k}{\|k\|}, \alpha(t) >$, and[1]

$$I_k := \{t \in I,\ |\alpha_k(t)| \leqslant \gamma\|k\|^{-n-\tau}\}.$$

Let $A(t)$ be the $n\times n$ matrix whose columns are $\alpha(t), D\alpha(t), \ldots D^{n-1}\alpha(t)$. From $\|A(t)\| \leq b$, $\|A(t)^{-1}\| \leq a^{-1}$, we get, for every $t \in I$

$$\frac{a}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}(\sum_0^{n-1} D^m\alpha_k(t)^2)^{1/2} \leq \max_{0\leq m\leq n-1}|D^m\alpha_k(t)| \leq (\sum_0^{n-1} D^m\alpha_k(t)^2)^{1/2} \leq b$$

From Proposition 0.2 we may therefore estimate the measure of I_k:

$$|I_k| \leqslant C(1 + \sqrt{n}\frac{b}{a}|I|)\left(\sqrt{n}\frac{\gamma}{a}\right)^{1/(n-1)}\|k\|^{-\frac{n+\tau}{n-1}}.$$

As $\tau > n(n-2)$, we have $\frac{n+\tau}{n-1} > n$ and we can sum over $k \in \mathbb{Z}^n$, $k \neq 0$, the estimate above to get the inequality in Pyartli's theorem. □

We will now explain how to obtain the two corollaries.

[1]We used the sup norm on $\mathbb{Z}^n$ in the definition of the diophantine condition $HDC(\gamma, \tau)$. Here, it is more practical to use the Euclidean norm. This changes γ by a constant depending only on n and τ.

Corollary 0.6. *Let K be a compact subset of $\mathbb{R}^m$ and let α be a C^∞ map, defined in a neighborhood of K, taking values in $\mathbb{R}^n$. Let $\tau > n(n-2)$. Assume that there are constants $b > a > 0$ such that α is (a, b) weakly non degenerate at each point of K. Then, for each $\gamma > 0$, one has*

$$\mathrm{Leb}\{t \in K,\ \alpha(t) \notin HDC(\gamma, \tau)\} < C \frac{b}{a} \left(\frac{\gamma}{a}\right)^{1/(n-1)},$$

with a constant $C = C(K, n, \tau)$.

Proof. Let $x_0 \in K$. Let $\nu : (\mathbb{R}, 0) \to (\mathbb{R}^m, x_0)$ be a C^∞ map such that $\alpha \circ \nu$ is (a, b) non planar at 0. The vector $D\nu(0)$ is different from 0. Let $\ell : \mathbb{R}^{m-1} \to \mathbb{R}^m$ a linear map whose image supplements $\mathbb{R}D\nu(0)$ in $\mathbb{R}^m$. Then the differential at $(0, 0)$ of the map $g : (t, t') \mapsto \nu(t) + \ell(t')$ is invertible. Let $\varepsilon_0 > 0$ be small enough to have

- the map g is a diffeomorphism from $[-\varepsilon_0, \varepsilon_0]^m$ onto a neighborhood $V(x_0)$ of x_0;
- for each $t' \in [-\varepsilon_0, \varepsilon_0]^{m-1}$, the map $t \mapsto \alpha \circ g(t, t')$ is $(\frac{a}{2}, 2b)$ non planar at each $t \in [-\varepsilon_0, \varepsilon_0]$.

From Pyartli's theorem, there exists $C_0 = C(x_0, n, \tau)$ such that, for each $t' \in [-\varepsilon_0, \varepsilon_0]^{m-1}$, each $\gamma > 0$, one has

$$\mathrm{Leb}\{t \in [-\varepsilon_0, \varepsilon_0],\ \alpha \circ g(t, t') \notin HDC(\gamma, \tau)\} < C \frac{b}{a} \left(\frac{\gamma}{a}\right)^{1/(n-1)}.$$

From Fubini's theorem, one gets

$$\mathrm{Leb}\{x \in V(x_0),\ \alpha(x) \notin HDC(\gamma, \tau)\} < C_1 \frac{b}{a} \left(\frac{\gamma}{a}\right)^{1/(n-1)}.$$

One concludes observing that the compact subset K is contained in a finite union of neighborhoods $V(x_i)$. □

Corollary 0.7. *Let $\gamma > 0$, $\tau_0 \geq 0$, $M > m$. There exists τ_1, depending only on n, M, τ_0, such that, for any germ $\alpha : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ which is weakly non degenerate at 0 and satisfies $\alpha(0) \in HDC(2\gamma, \tau_0)$, one has, for small $\varepsilon > 0$*

$$\mathrm{Leb}\{x \in \mathbb{R}^m,\ \|x\| < \varepsilon,\ \alpha(x) \notin HDC(\gamma, \tau_1)\} = O(\varepsilon^M).$$

Proof. Let $b > a > 0$ be constants such that α is (a, b) weakly non degenerate at 0. Let $\nu : (\mathbb{R}, 0) \to (\mathbb{R}^m, 0)$, $\ell : \mathbb{R}^{m-1} \to \mathbb{R}^m$, $g(t, t') := \nu(t) + \ell(t')$, $\varepsilon_0 > 0$ be as in the proof of Corollary 0.6. Let $\varepsilon_1 > 0$ be small enough so that the ball $\{\|x\| < \varepsilon_1\}$ is contained in $g([-\varepsilon_0, \varepsilon_0]^m)$. For $k \in \mathbb{Z}^n$, $k \neq 0$, denote $\alpha_k(x) :=< \frac{k}{\|k\|}, \alpha(x) >$. With $\tau_1 \geq \tau_0$ to be chosen later, define

$$E_k := \{x \in \mathbb{R}^m, |\alpha_k(x)| \leq \gamma \|k\|^{-n-\tau_1}\}.$$

Let $\varepsilon < \varepsilon_1$. For $\|x\| < \varepsilon$, one has $|\alpha_k(x) - \alpha_k(0)| \leq C\varepsilon$ and $|\alpha_k(0)| \geq 2\gamma \|k\|^{-n-\tau_0}$. This implies $|\alpha_k(x)| \geq \gamma \|k\|^{-n-\tau_0}$ if $C\varepsilon < \gamma \|k\|^{-n-\tau_0}$. Therefore E_k does not intersect the ball $\{\|x\| < \varepsilon\}$ when $\|k\| < \rho_0 \varepsilon^{-\frac{1}{n+\tau_0}}$, with $\rho_0 := \left(\frac{\gamma}{C}\right)^{\frac{1}{n+\tau_0}}$.

On the other hand, one has from Proposition 0.2, as in the proof of Pyartli's theorem, for every $t' \in [-\varepsilon_0, \varepsilon_0]^{m-1}$

$$\mathrm{Leb}\{t \in [-\varepsilon_0, \varepsilon_0],\ g(t, t') \in E_k\} \leq \rho_1 \|k\|^{-\frac{n+\tau_1}{n-1}},$$

with ρ_1 depending on γ, a, b, ν. By Fubini's theorem, one gets

$$\mathrm{Leb}(E_k \cap \{\|x\| < \varepsilon\}) < \rho_2 \|k\|^{-\frac{n+\tau_1}{n-1}}.$$

Summing over $\|k\| \geq \rho_0 \varepsilon^{-\frac{1}{n+\tau_0}}$ gives

$$\mathrm{Leb}\{x \in \mathbb{R}^m,\ \|x\| < \varepsilon,\ \alpha(x) \notin HDC(\gamma, \tau_1)\} \leqslant \rho_3\, \varepsilon^{\frac{1}{n+\tau_0}(\frac{n+\tau_1}{n-1}-n)}.$$

When τ_1 is large enough, the exponent of ε is $> M$. $\square$