# THE LINEARIZED EQUATION IN THE HÖLDER CLASSES

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## 0.1. The result.

**Proposition 0.1.** *Let $s,\ \tau$ be non-negative real numbers such that $s$ is not an integer.*
*If $\alpha \in \mathbb{T}^d$ belongs to $DC(\tau)$, and $\varphi$ belongs to $C_0^{s+d+\tau}(\mathbb{T}^d)$, then the equation*

$$\psi \circ R_\alpha - \psi = \varphi$$

*has a solution $\psi \in C_0^s(\mathbb{T}^d)$.*
*Moreover, with $\alpha \in DC(\gamma, \tau)$, one has*

$$||\psi||_{C^s} \leqslant C\gamma^{-1}||\varphi||_{C^{s+d+\tau}}.$$

I am not sure about the correct reference. I am essentially following Herman in his Asterisque book, Volume 1.

## 0.2. Smoothing operators and Hadamard convexity inequalities.

Let $\widehat{\chi} \in C^\infty(\mathbb{R}^d)$ be a non-negative even function with support in $[-1,1]^d$, equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]^d$. Let $\chi$ be the inverse Fourier transform of $\widehat{\chi}$ and define $\chi_t(x) = t^d\chi(tx)$ for $t > 0$. For $t \geqslant 1$, let $S(t)$ be the convolution operator $S(t)\varphi = \varphi * \chi_t$ from $C^0(\mathbb{T}^d)$ to $C^\infty(\mathbb{T}^d)$. Then, we have, for $n \in \mathbb{Z}^d$

$$\widehat{S(t)\varphi}(n) = \widehat{\varphi}(n)\widehat{\chi}(\frac{n}{t}).$$

One has the following estimates, for real numbers $s \leqslant r$ and $t \geqslant 1$:

$$||S(t)\varphi||_{C^r} \leqslant Ct^{r-s}||\varphi||_{C^s},$$

$$||S(t)\varphi - \varphi||_{C^s} \leqslant Ct^{s-r}||\varphi||_{C^r},$$

with constants $C$ depending on $r, s$ only.

From these estimates one gets *Hadamard convexity inequalities*: let $r_0 \leqslant r_1$, $u \in [0,1]$, $r_u = ur_1 + (1-u)r_0$. One has

$$||\varphi||_{C^{r_u}} \leqslant C||\varphi||_{C^{r_0}}^{1-u}||\varphi||_{C^{r_1}}^{u}.$$

## 0.3. Littlewood-Paley decomposition.

Let $\varphi \in C^0(\mathbb{T}^d)$. We define

$$\Delta_0\varphi := S(1)\varphi = \int_{\mathbb{T}} \varphi(x)\,dx,$$

$$\Delta_n\varphi = (S(2^n) - S(2^{n-1}))\varphi, \qquad \text{for}\ \ n > 0.$$

Observe that $\Delta_n\varphi$ is a trigonometric polynomial of degree [1] $< 2^n$, and that the series $\sum_n \Delta_n\varphi$ converge formally to $\varphi$. From the estimates above, we have, for $\varphi \in C^r(\mathbb{T}^d)$:

$$||\Delta_n\varphi||_{C^0} \leqslant C2^{-rn}||\varphi||_{C^r},$$

so the convergence is uniform as soon as $r > 0$. Conversely

[1] A trigonometric polynomial $\Phi$ has degree $< D$ if $\widehat{\Phi}(k) = 0$ for $||k||_\infty \geqslant D$

**Lemma 0.2.** *Let $r > 0$ be a real number which is not an integer, and $(\varphi_n)$ a sequence of trigonometric polynomials such that the degree of $\varphi_n$ is $< 2^n$. Assume that*

$$\sup_n 2^{rn}||\varphi_n||_{C^0} =: A < +\infty.$$

*Then the series $\sum_n \varphi_n$ converge uniformly to a function $\varphi$ which belongs to $C^r(\mathbb{T}^d)$ and we have*

$$||\varphi||_{C^r} \leqslant CA.$$

*Proof.* Clearly the series $\sum_n \varphi_n$ converge uniformly to a function $\varphi \in C^0(\mathbb{T}^d)$. We first deal with the case $0 < r < 1$. Observe that $\varphi_n = S(2^{n+1})\varphi_n$, hence

$$||\varphi_n||_{C^1} \leqslant C2^{(1-r)n}A.$$

Let $x, y \in \mathbb{T}^d$, and let $N \geqslant 0$ s.t. $2^{-N-1} \leqslant ||x-y|| \leqslant 2^{-N}$. For $n > N$, we just write

$$|\varphi_n(x) - \varphi_n(y)| \leqslant 2||\varphi_n||_{C^0} \leqslant 2^{1-rn}A.$$

For $n \leqslant N$, we write

$$|\varphi_n(x) - \varphi_n(y)| \leqslant ||x-y||||D\varphi_n||_{C^0} \leqslant C2^{-N}2^{(1-r)n}A.$$

Summing over $n$, we get

$$|\varphi(x) - \varphi(y)| \leqslant C2^{-rN}A \leqslant C||x-y||^r A,$$

which proves the result for $0 < r < 1$.

In the general case, we write $r = m + r'$ with an integer $m$ and $0 < r' < 1$. We have

$$||\varphi_n||_{C^m} \leqslant C2^{(n+1)m}2^{-rn} \leqslant C2^{-r'n}.$$

This proves that $\varphi \in C^m(\mathbb{T}^d)$. Then, as $0 < r' < 1$, the previous case shows that $D^m\varphi \in C^{r'}(\mathbb{T}^d)$. This concludes the proof. □

0.4. **Proof of proposition.** Let $r = s + d + \tau$, $\varphi \in C_0^r(\mathbb{T}^d)$. We write $\varphi = \sum_{n>0} \Delta_n\varphi$ and solve

$$\psi_n \circ R_\alpha - \psi_n = \Delta_n\varphi,$$

where $\psi_n$ is a trigonometric polynomial of mean value zero. We want to apply the lemma to show that $\psi = \sum_{n>0} \psi_n$ belongs to $C^s(\mathbb{T}^d)$. The Fourier coefficients of $\psi_n$ are given by:

$$\widehat{\psi}_n(k) = (\exp(2\pi i < k, \alpha >) - 1)^{-1}\widehat{\Delta_n\varphi}(k), \qquad \frac{1}{4}2^n < ||k||_\infty < 2^n.$$

From this, we get, by Cauchy-Schwartz inequality

$$||\psi_n||_{C^0} \leqslant \sum_k |\widehat{\psi}_n(k)| \leqslant \sqrt{S}||\Delta_n\varphi||_{L^2},$$

where

$$S := \sum_{2^{n-2} < ||k||_\infty < 2^n} |\exp(2\pi i k\alpha) - 1|^{-2}.$$

**Lemma 0.3.** *Assume that $\alpha \in DC(\gamma, \tau)$. Then*

$$S \leqslant C\gamma^{-2}2^{2n(d+\tau)}.$$

*Proof.* As $|\exp(2\pi i x) - 1| \geqslant 4||x||_{\mathbb{T}}$, it is sufficient to deal with

$$S' = \sum_{0<||k||_\infty<2^n} || < k, \alpha > ||_{\mathbb{T}}^{-2}.$$

Let $u := \gamma 2^{-(n+1)(d+\tau)}$. As $\alpha \in DC(\gamma, \tau)$, we have $|| < k, \alpha > ||_{\mathbb{T}} \geqslant u$ for $0 < ||k||_\infty < 2^{n+1}$; therefore, for each $j > 0$, there is at most one $k \in \mathbb{Z}^d$ with $0 < ||k||_\infty < 2^n$ such that $\{< k, \alpha >\} \in [ju, (j+1)u)$ and at most one such that $1 - \{< k, \alpha >\} \in [ju, (j+1)u)$. We have therefore

$$S' \leqslant 2u^{-2} \sum_{j>0} j^{-2} \leqslant cu^{-2}.$$

□

On the other hand, we have

$$||\Delta_n \varphi||_{L^2} \leqslant ||\Delta_n \varphi||_{C^0} \leqslant C 2^{-nr} ||\varphi||_{C^r},$$

hence we get from the lemma

$$||\psi_n||_{C^0} \leqslant C\gamma^{-1} 2^{n(1+\tau)} 2^{-nr} ||\varphi||_{C^r} \leqslant C\gamma^{-1} 2^{-ns} ||\varphi||_{C^r}.$$

Thus we obtain

$$\sup_n 2^{ns} ||\psi_n||_{C^0} \leqslant C\gamma^{-1} ||\varphi||_{C^r}.$$

Applying Lemma 0.2 allows to conclude that $\psi \in C^s(\mathbb{T}^d)$ with

$$||\psi||_{C^s} \leqslant C\gamma^{-1} ||\varphi||_{C^r}.$$

The proof is complete. □