EXACT RESULTS FOR DISORDERED SYSTEMS II: Superfluid to Bose Glass transition in 1D

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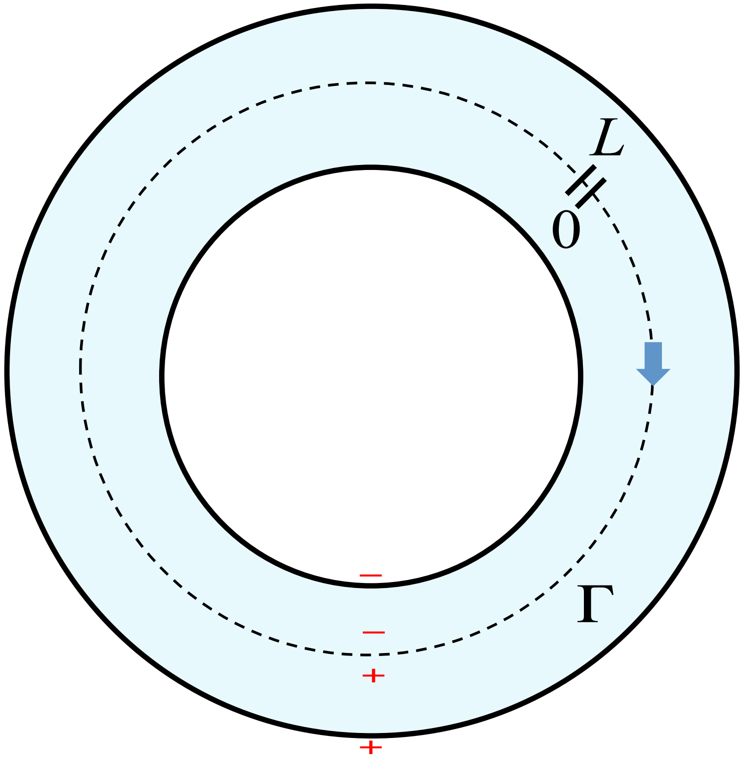
- **Superfluid hydrodynamics & vortex-instanton analogy**
  - Nelson-Kosterlitz criterion
  - mapping 1D superfluids at T=0 to classical 2D systems
  - 2D vortexes *vs* 1D instanton
- **Self-averaging in SF and at the SF-BG transition line**
  - Proof of self-averaging for superfluid stiffness
  - Giamarchi-Schultz line $K = 3/2$

- **Scratched-XY (strong disorder) criticality**
  - Exponentially rare – exponentially weak distribution (ς- exponent)
  - $K = \varsigma^{-1}$ criterion for scratched-XY universality
  - asymptotically exact RG equations for sXY transition

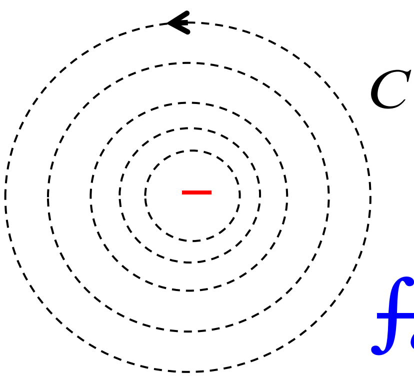
Superfluid transition in 2D. Nelson-Kosterlitz criterion

Superfluidity = impossibility to "undo" phase windings:

superfluid current $J = \Lambda \nabla \varphi \neq 0$ as long as $\varphi(L) = \varphi(0) + 2\pi I$



The least energetic way to change the topological invariant $2\pi I = \oint_\Gamma \nabla \varphi \, d\vec{l}$ is to have a vertex-pair crossing the stream



$$\oint_C \nabla \varphi \, d\vec{l} = -2\pi$$

kinetic energy of the flow

$$E = \int d^2 r \; \frac{\Lambda}{2} |\nabla \varphi|^2 \; : \; 2\pi\Lambda \int \frac{dr}{r} \sim 2\pi\Lambda \ln(L/\xi) >> T$$

Superfluid transition in 2D. Nelson-Kosterlitz criterion

Energy vs Entropy of a macroscopic (size L) pair:

$$\begin{cases} E_{pair} = 2\pi\Lambda \ln(L/\xi) \\ S_{pair} = 2\ln(L/\xi)^2 = 4\ln(L/\xi) \\ F_{pair} = (2\pi\Lambda - 4T)\ln(L/\xi) \end{cases}$$

$$F_{pair} < 0 \quad \rightarrow \quad \frac{\Lambda(T_C)}{T_C} = \frac{2}{\pi}$$

Thermodynamic def. of superfluid density from the response to the phase twist φ_0

$$\psi(L) = \psi(0)\, e^{i\varphi_0}$$

$$f(\varphi) = \frac{\Lambda_S}{2}\left(\frac{\varphi_0}{L}\right)^2$$

$\Lambda_S \neq \Lambda$ because

$$\nabla\varphi = \varphi_0 / L,\ (\varphi_0 + 2\pi)/L,\ \ldots$$

$$\frac{\Lambda_S(T_C)}{T_C} = \frac{2}{\pi}[1 - 16\pi e^{-4\pi} + \ldots] = \frac{1.99965\ldots}{\pi}$$

Superfluid hydrodynamics in 1D & vortex-instanton analogy

$$S = \int d^2 r \left\{ i\bar{n}(x) \frac{\partial \varphi}{\partial y} + \frac{K}{2\pi} \left| \nabla \varphi \right|^2 \right\}$$

$r = (x, y)$

$y = c\,\tau$

velocity of sound

Same as classical XY

Hamiltonian dynamics with complex-number canonical variables (ψ, ψ^*) :

Given $H[\psi^*, \psi]$, the Equation of motion is $i\dot{\psi} = \dfrac{\delta H[\psi^*, \psi]}{\delta \psi^*}$

Same as the minimal action principle for $S = \int dx dt \left\{ i\psi^* \dot{\psi} - E[\psi^*, \psi] \right\}$

For $E[\psi^*, \psi] = \dfrac{1}{2m} \left| \dfrac{\partial}{\partial x} \psi \right|^2 + \dfrac{g}{2} |\psi|^4 - \mu |\psi|^2$

$$\frac{\delta S}{\delta \psi^*(x, \tau)} = 0 \quad \rightarrow \quad i\dot{\psi} = -\frac{1}{2m} \Delta \psi + g |\psi|^2 \psi - \mu \psi \qquad (GPE)$$

$$S = \int dx d\tau \left\{ \psi^* \dot{\psi} + \frac{1}{2m}\left|\frac{\partial \psi}{\partial x}\right|^2 + \frac{g}{2}|\psi|^4 - \mu|\psi|^2 \right\}$$

$$\boxed{\tau = it}$$

Static homogeneous solution $\bar{\psi}^2 = \mu / g = \bar{n}$. Substitute $\psi = \sqrt{\bar{n} + \delta n(x,\tau)}\; e^{i\varphi(x,\tau)}$ and expand in δn and phase derivatives [full time derivatives can be dropped, except for $\dot{\varphi}$ because $\int d\tau \dot{\varphi} = 2\pi j$]

$$S = \int dx d\tau \left\{ i(\bar{n} + \delta n)\dot{\varphi} + \frac{\bar{n}}{2m}\left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{g}{2}\delta n^2 \right\} + const$$

Take Gaussian integrals over δn in $\int D\psi\, e^{-S}$ (complete the square)

$$S = \int dx d\tau \left\{ i\bar{n}\dot{\varphi} + \frac{\bar{n}}{2m}\left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{1}{2g}\dot{\varphi}^2 \right\}$$

Introduce $y = c\tau$ with $c^2 = \bar{n}g / m$

$$S = \int d^2 r \left\{ i\bar{n}\frac{\partial \varphi}{\partial y} + \frac{K}{2\pi}(\nabla \varphi)^2 \right\}$$

with $K = \pi\sqrt{\bar{n} / mg}$

If the system is not weakly interacting

$$S = \int dx d\tau \left\{ i(\bar{n} + \delta n)\dot{\varphi} + \frac{\Lambda}{2}\left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{1}{2\kappa}\delta n^2 \right\}$$

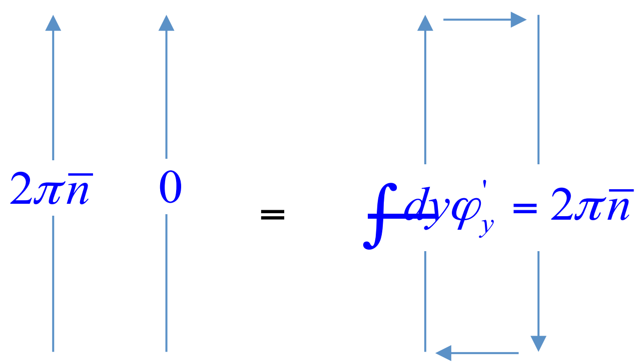
Free-energy response coefficients: superfluid stiffness and compressibility

φ **- Phase of the SF order-parameter field**

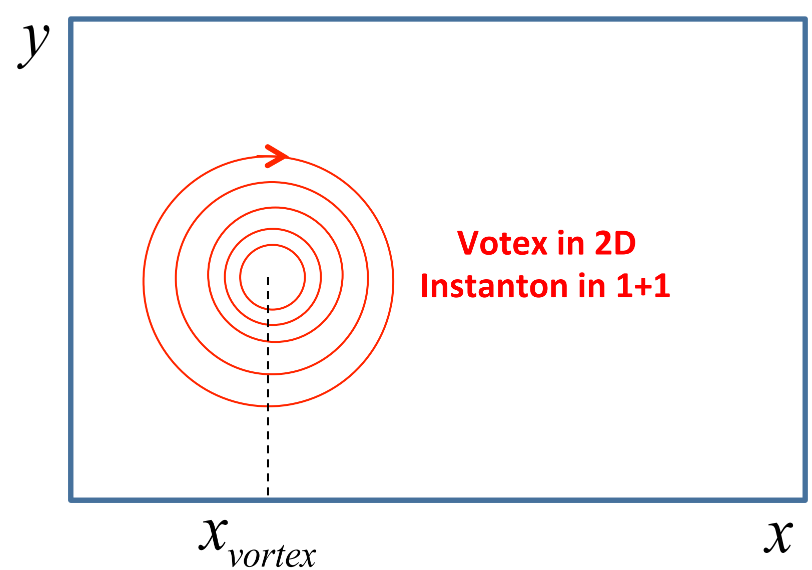
$$S = \int d^2 r \left\{ i\bar{n}\frac{\partial \varphi}{\partial y} + \frac{K}{2\pi}|\nabla \varphi|^2 \right\}$$

with $K = \pi\sqrt{\Lambda\kappa}$ and $c^2 = \Lambda / \kappa$

Quantum 1+1 action = classical 2D XY-model at $\Lambda^{(2D)} / T = K / \pi$ with the vortex phase term



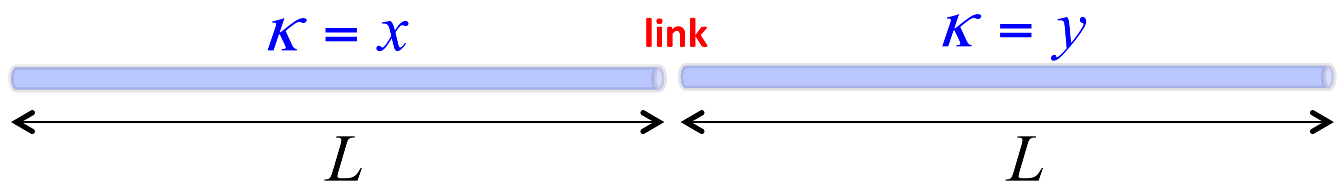
$$\mathrm{Im}\, S = \sum_v 2\pi j_v \int_0^{x_v} dx\, \bar{n}$$



Self-averaging in the SF phase & at the SF-BG transition

Composition laws:

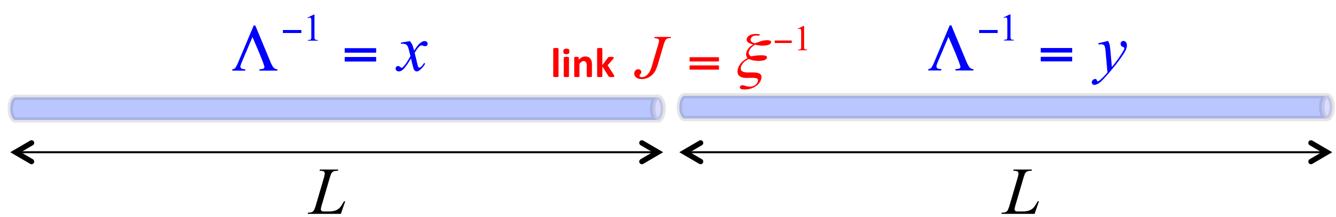
Compressibility:



Minimize $E = \frac{1}{2}\left[\frac{N_1^2}{Lx} + \frac{N_1^2}{Ly}\right]$ **with** $N_1 + N_2 = N$ **to get** $E = \frac{1}{2}\frac{N^2}{(2L)z}$ **with** $z = (x+y)/2$

$$\kappa(2L) = (\kappa_{left} + \kappa_{right})/2$$

Superfluid stiffness:

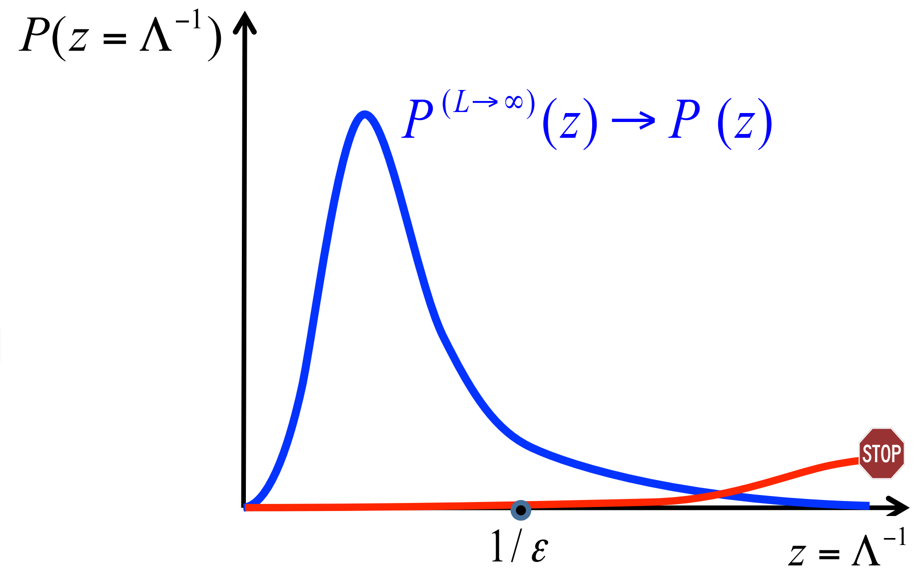


Minimize $E = \frac{1}{2}\left[\frac{\varphi_1^2}{Lx} + \frac{\varphi_2^2}{Ly} + \frac{\varphi_J^2}{\xi}\right]$ **with** $\varphi_1 + \varphi_2 + \varphi_J = \varphi$ **to get** $E = \frac{1}{2}\frac{\varphi^2}{(2L)z}$ **with** $z = (x + y + \xi/L)/2$

$$\Lambda^{-1}(2L) = [\Lambda^{-1}_{left} + \Lambda^{-1}_{right} + 1/(JL)]/2$$

Self-averaging in the SF phase & at the SF-BG transition

Minimal def. of the superfluid state: Λ^{median} **is finite**



[probability to find $\Lambda > \varepsilon$ is 50%

$\Lambda_S < (3/2\pi)^2/\kappa$ **is not even possible(!), see below]**

Why so fancy through the median ?

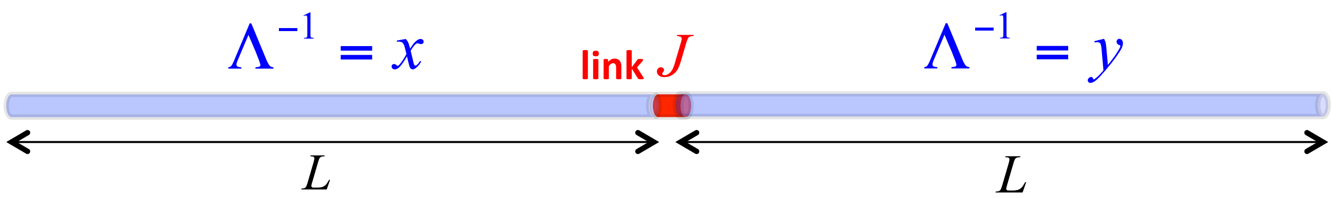
Weak links may result in a $P(z)$ tail such that $\langle z^2 \rangle \to \infty$

We do NOT require that even $\int zP(z)dz$ is finite

rare but disastrous event for SF!

Self-averaging in the SF phase & at the SF-BG transition

Composition law:



$$\Lambda^{-1}(2L) = [\Lambda^{-1}_{left} + \Lambda^{-1}_{right} + 1/\bar{J}L]/2$$

$$P^{(2L)}(z) = \int_0^{\infty} dxdydJ\ P^{(L)}(x)P^{(L)}(y)\, Q_{x,y}(J)\, \delta\left[z - \frac{x+y+1/\bar{J}L}{2}\right]$$

Normalizable, microscopic, L-independent

In the thermodynamic limit $L \to \infty$

$$P(z) = \int_0^{\infty} dxdydJ\ P(x)P(y)\, Q_{x,y}(J)\, \delta\left[z - \frac{x+y+1/\bar{J}L}{2}\right]$$

Define $f = \int_0^{\infty} dz\, P(z) \leq 1$ **Integrate** **to get** $f = f^2$ ⇒ $f = 1$

(recall that it cannot be

Self-averaging in the SF phase & at the SF-BG transition

In Fourier space: $P(k) = \int_0^\infty dxdy\, P(x)P(y)\, e^{ik(x+y)/2} \underbrace{\int_0^\infty dJ\, Q_{x,y}(J)\, e^{ik/(2\bar{J}L)}}_{=1 \text{ for } L\to\infty}$

$$P(k) = P^2(k/2)$$

$$\boxed{P(k) = e^{ik/\Lambda_S} \quad \rightarrow \quad P(x) = \delta(x - \Lambda_S^{-1})}$$

Self-averaging

Generalized function $\delta(x-a)$ is compatible with divergent $\langle x^2 \rangle$: $\delta(x-a) = \lim_{\varepsilon\to 0} \dfrac{\varepsilon/\pi}{(x-a)^2 + \varepsilon^2}$

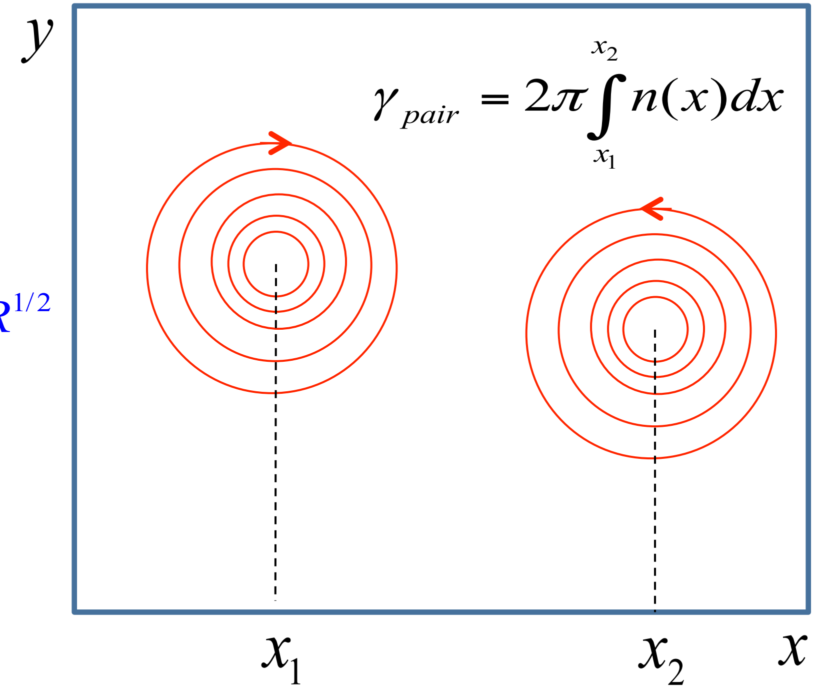
Thermodynamic limit: $L \to \infty$ first, while the number of experiments remains finite!

One can use the action $S = \int dr \left\{ i\bar{n}(x) \dfrac{\partial \varphi}{\partial y} + \dfrac{K}{2\pi} |\nabla \varphi|^2 \right\}$

Generic Giamarchi-Shultz K=3/2 criterion in 1D. Same status as BKT theory in 2D.

$$\gamma = \mathrm{Im}\, S = \sum_{v} 2\pi q_v \int_0^{x_v} \bar{n}(x)dx$$

$$\gamma_{pair\ R} = 2\pi \int_{x_1}^{x_2 = x_1 + R} \bar{n}(x)dx \rightarrow 2\pi\kappa \int_0^R \varepsilon(x)dx \propto \left(\frac{\Delta}{U}\right) R^{1/2}$$



Vortex pairs have to pile up in imaginary time!

Standard Nelson-Kosterlitz *energy vs entropy* argument:

Energy (action) of L-pair: $E/T = \int dr \frac{K}{2\pi} |\nabla\varphi|^2 = 2K \ln L$

Entropy of a "vertical" pair: $S = \ln L^3 = 3 \ln L$

Vertical Vortex pairs proliferate when
$E/T - S < 0 \quad \rightarrow \quad K < 3/2$

[Vertical pairs renormalize Λ and C but keep K the same;
E&M analogy: vertical polarization does not screen horizontal fields)

In the superfluid state Λ cannot be arbitrary small (need $K = \pi\sqrt{\Lambda\kappa} \geq 3/2$)

Weak-disorder limit near the SF-MI tip:

KT superfluid-Mott transition → Energy gap $\Delta_{M\,I} \propto \exp\left\{-\#\sqrt{t/(U-U_C)}\right\}$

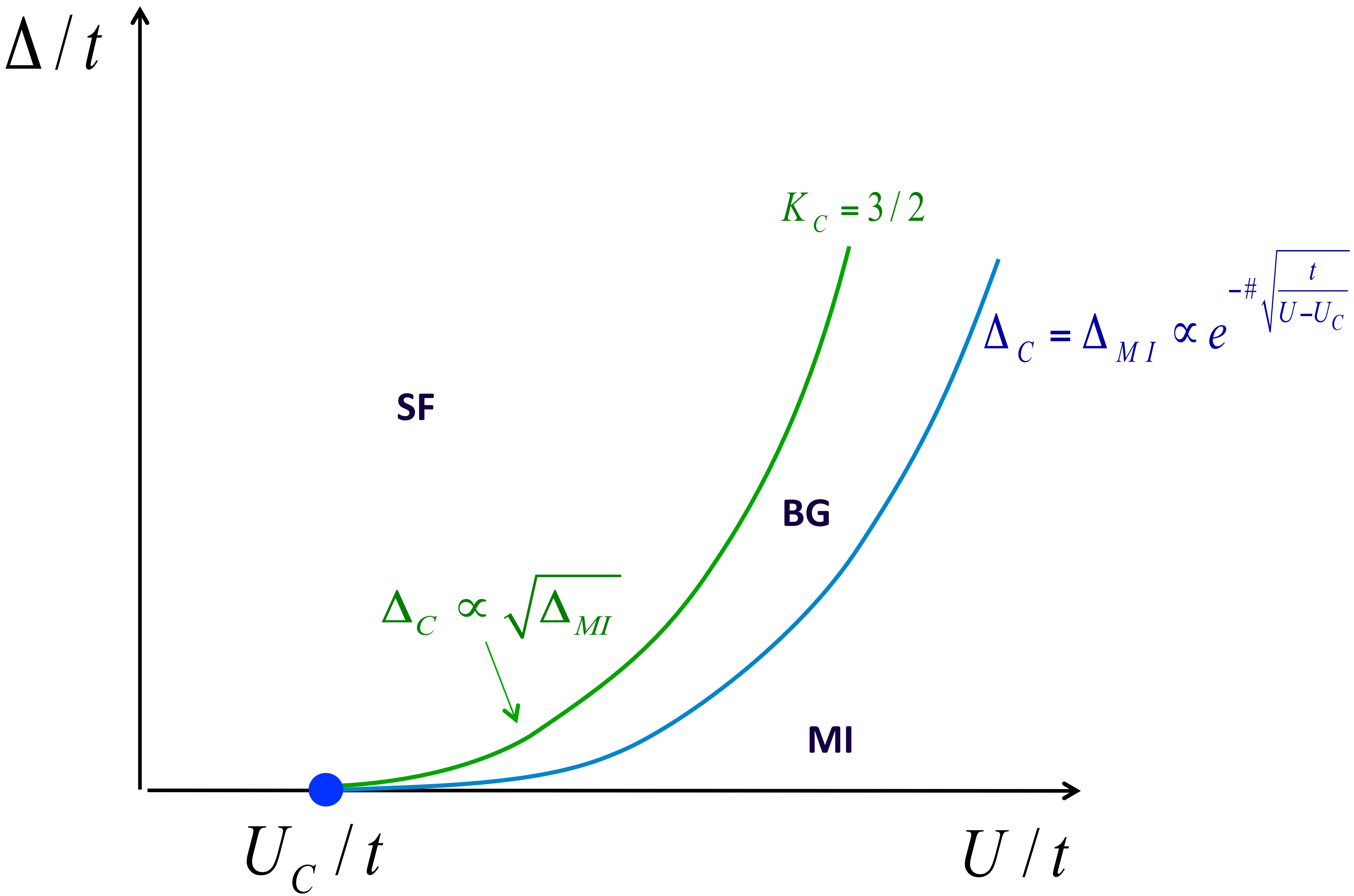
correlation length $\xi \propto 1/\Delta_{M\,I}$

Griffiths type BG-MI transition → $\Delta_C^{(MI-BG)} = \Delta_{M\,I}$ **(never to be seen numerically or experimentally !)**

Vortex phase argument → $\gamma_{pair\ R} \propto \left(\frac{\Delta}{U}\right)\xi^{1/2} \sim 1$ **leads to**

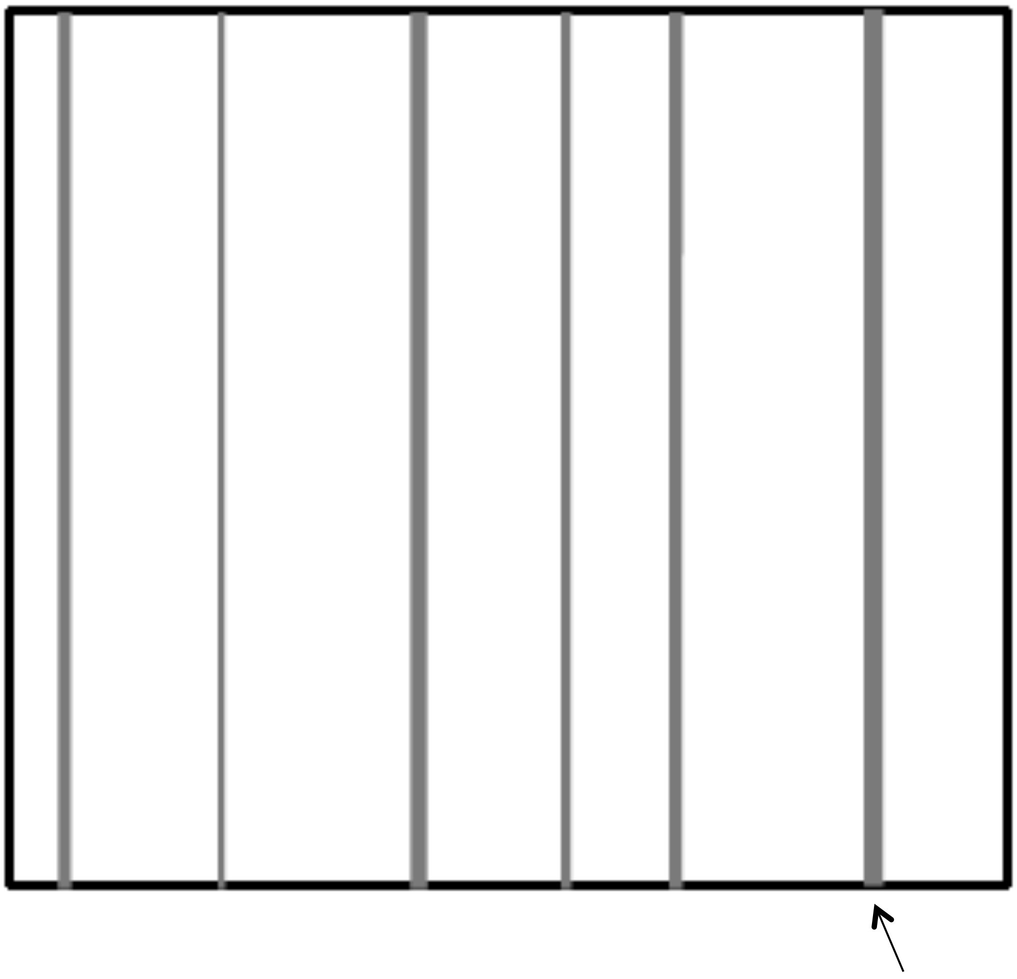
$$\Delta_C^{(BG-SF)} \propto \sqrt{\Delta_{M\,I}} \propto \exp\left\{-\frac{\#}{2}\sqrt{t/(U-U_C)}\right\}$$

One-dimensional diagram tip



Weak-link (scratched XY) criticality

2D Classical analog: XY-model with "scratches"



Exponentially-rare exponentially-weak links

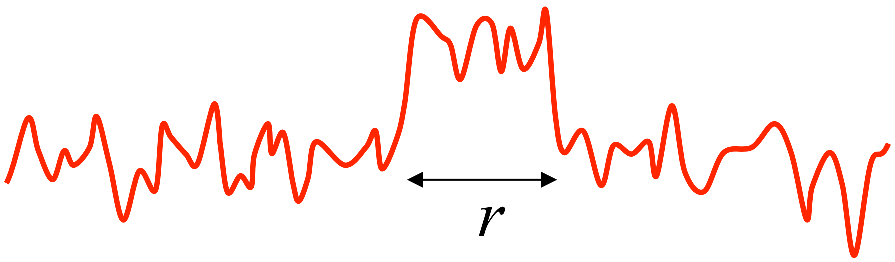
Can a new universality class preempt BKT transition?

$? : \Lambda_C > (2/\pi)T$ **(classical)**

$? : K_C > 3/2$ **(quantum)**

Exponentially-rare exponentially-weak links

Consider rare statistical fluctuations when locally disorder is creating and insulating region or large (atypical) barrier of length r



Probability to have it in a system of size L

$$P(r) \propto L e^{-c_1 r}$$

This leads to a Josephson-type weak link connecting superfluid regions

$$J(r) \propto e^{-c_2 r}$$

Typical weakest link (probability of order unity) in a system of size L

$$P(r) \sim 1 \quad \rightarrow \quad r \propto c_1^{-1} \ln L \quad \rightarrow \quad J_{weakest} \propto 1 / L^{c_2/c_1}$$

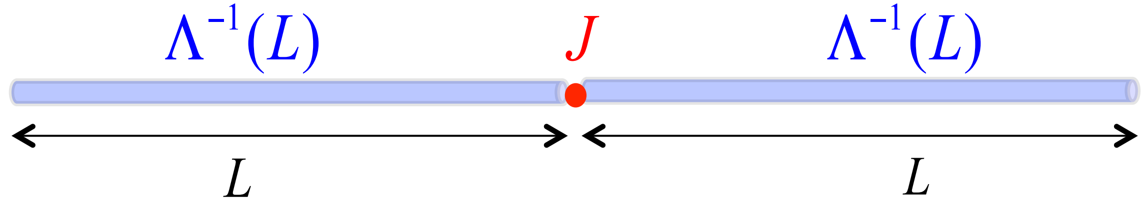
Convention: microscopic (irrenormalizable) parameter $\varsigma = 1 - c_2 / c_1$

$$\boxed{J_{typical\ weak} \propto 1 / L^{1-\varsigma}}$$

Classical-field argument:

[Alexander, Bernasconi, Schneider, and Orbach RMP 81]

$$\Lambda^{-1}(2L) - \Lambda^{-1}(L) = \frac{1}{2JL} = w$$



RG flow equation (using $l = \ln L$): $\dfrac{d\Lambda^{-1}}{dl} = w$ with $w = w_0(\lambda_0 / L)^{\varsigma}$

[can be formulated as the solution of $\dfrac{dw}{dl} = -\varsigma w$

If $\varsigma > 0$ the answer saturates to some finite value for Λ ,

for $\varsigma < 0$ we end up with $\Lambda(\infty) = 0$, i.e. the state is NOT superfluid ⇒ $\varsigma_C = 0$

$$J_{typical\ weakest} \propto 1 / L^{1-\varsigma}$$

Quantum-field effects.

1. If $K(L) < 3/2$ [or $\Lambda_S(L) < 9/4\pi^2\kappa$], the flow is always to BG due to vertical vortex pairs

2. Kane-Fisher renormalization of weak links

For one link in SF: $J(L) = J \times (\lambda_0 / L)^{1/K}$ or $\dfrac{1}{LJ(L)} \propto \dfrac{1}{J}(L/\lambda_0)^{1/K-1}$ [Kane, Fisher 92]

Since we consider only K>3/2 one link in an infinite system is always an irrelevant perturbation. Kane-Fisher renormalization stops when

$$J(\lambda)\lambda \sim J \times (\lambda_0/\lambda)^{1/K-1} \sim 1$$

Kane-Fisher renormalization is always making weak links weaker (in absolute terms) !

\+ we are dealing with a collection of progressively weaker links as L is increased!

Asymptotically: $J_{typical\ weakest}(L) \propto (1/L)^{1-\varsigma+1/K}$ → $K_C = \varsigma^{-1} > 3/2$

Scratched XY universality

More precisely, need to solve RG equations since K flows with L

$$J(\lambda) \propto \lambda^{\varsigma - 1} \times \exp\left\{ - \int_{\ln \lambda_0}^{\ln \lambda} \frac{dl}{K(l)} \right\} \rightarrow$$

$$\frac{dw}{dl} = \frac{1/K - \varsigma}{1 - 1/K} w$$

$$\frac{dK}{dl} = -w$$

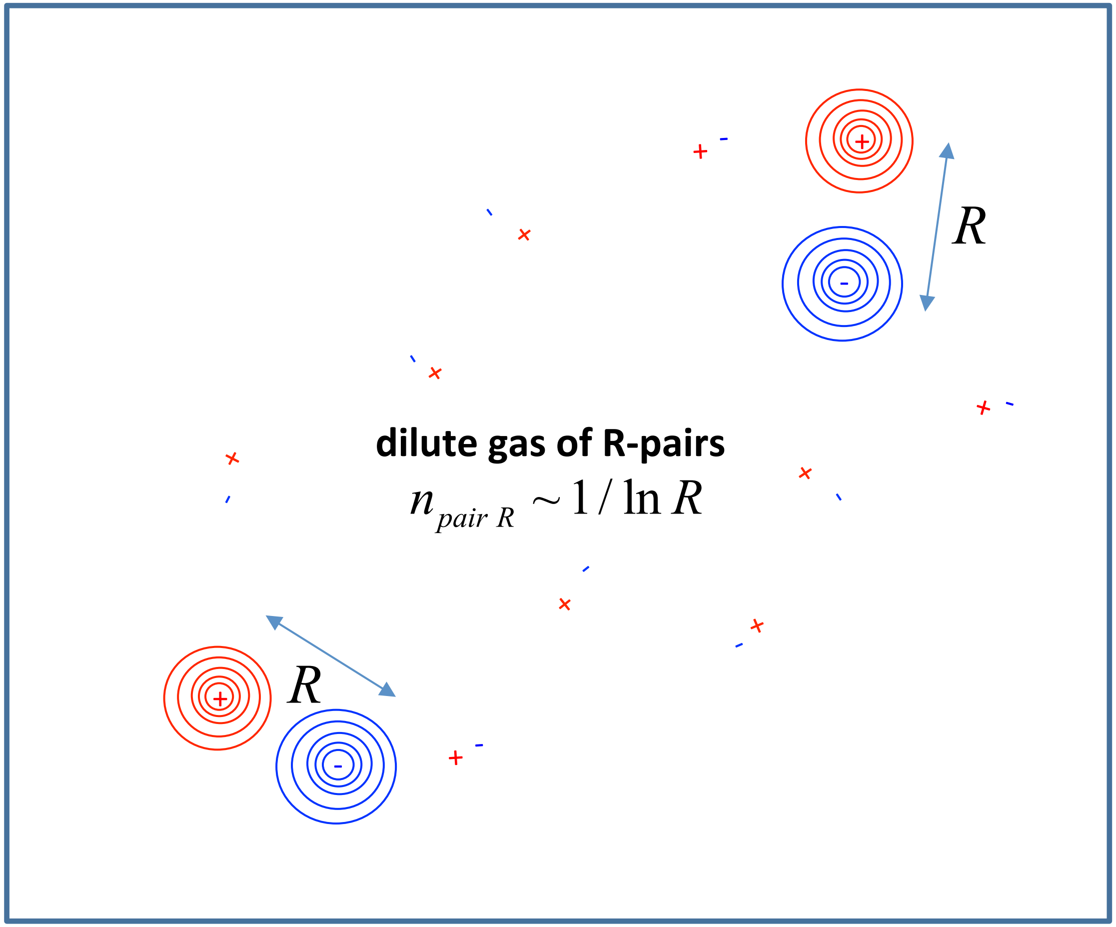
Linearized version near criticality : $(K = \varsigma^{-1} + x)$

$$\begin{cases} \dfrac{dw}{dl} = -\dfrac{\varsigma^2}{1-\varsigma} xw \\ \dfrac{dx}{dl} = -w \end{cases}$$

BKT type behavior: $x(\infty) \propto \sqrt{|U - U_C|}$ **and** $x(L) = \frac{1-\varsigma}{\varsigma^2} \frac{1}{\ln L}$ **at sXY criticality**

Linearized Equations $$\frac{dw}{dl} = -\frac{\varsigma^2}{1-\varsigma} xw \,, \qquad \frac{dx}{dl} = -w \qquad (x = K - 1/\varsigma)$$

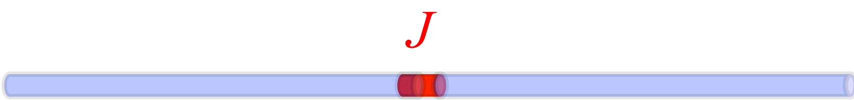
are asymptotically exact at criticality (and in the SF phase). Composite weak-links are statistically irrelevant in $1/\ln L \to 0$



Classical vortex-pair BKT case

Linearized Equations $\frac{dw}{dl} = -\frac{\varsigma^2}{1-\varsigma} xw \, , \qquad \frac{dx}{dl} = -w \qquad (x = K - 1/\varsigma)$

are asymptotically exact at criticality (and in the SF phase). Composite weak-links are statistically irrelevant in $1/\ln L \to 0$



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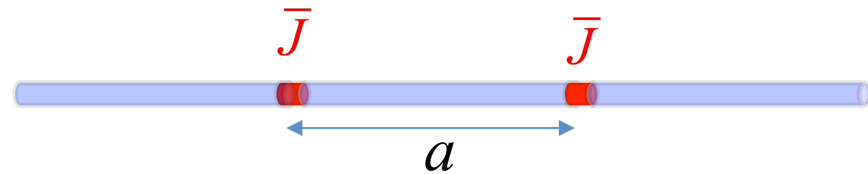
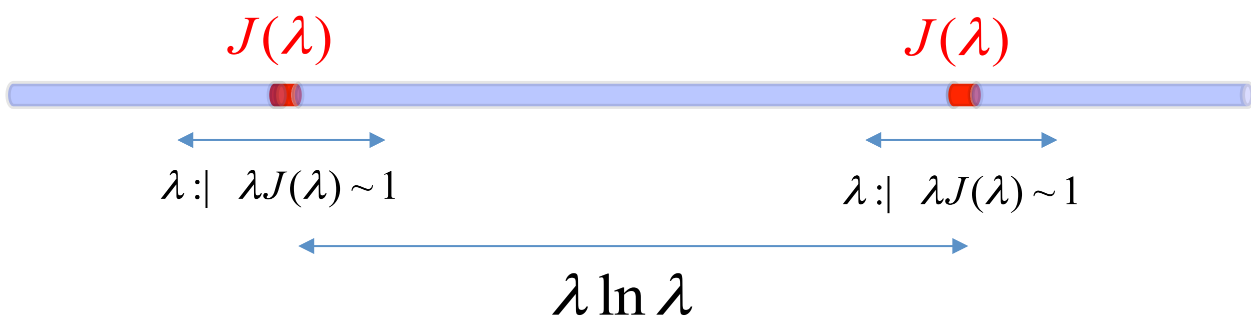


“Coulomb blockade” composite with $J_{comp} \sim \bar{J}^2 a\kappa$

Even at criticality $\int_{\lambda_0} \frac{d \ln a}{\ln^2 a} \to \lambda_0$ (mirco scale)



dilute gas of weak links

All effects together:

$$\frac{dK}{dl} = -w - y; \quad \frac{dw}{dl} = \frac{1/K - \varsigma}{1 - 1/K} w; \quad \frac{dy}{dl} = (3 - 2K) y$$

weak links — vortex-pairs

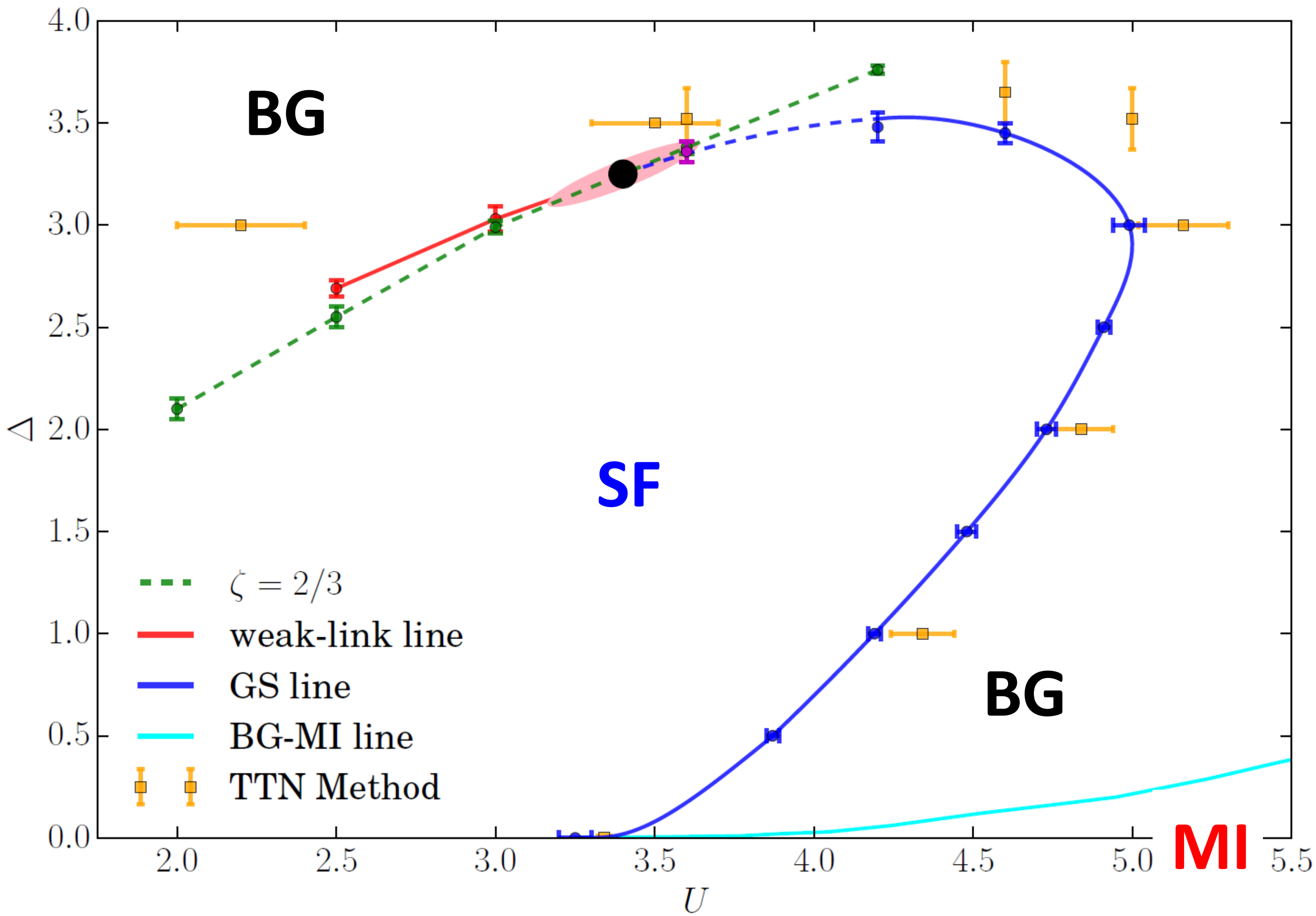
Linearized version for $\left[K = 3/2 + x, \quad \varsigma = 2/3 + \delta \right]$

$$\frac{dK}{dl} = -w - y; \quad \frac{dw}{dl} = -\left(\frac{4}{3} x + 3\delta \right) w; \quad \frac{dy}{dl} = -2xy$$

⇩

Smooth crossover between sXY and GS criticalities [continuous first derivative]

$$H = -t\sum_{<ij>} a_i\ a_j^{\dagger} + \frac{U}{2}\sum_i n_i^2 - \sum_i (\mu - \varepsilon_i) n_i$$ **(in 1D)**



● Tri-critical point separating sXY and GS lines

Procedure for determining exponent ς .

1. **Fix system size $L >> 1$, run $N >> L$ disorder realization $[N << e^L]$, and measure superfluid responses**

$\left\langle \Lambda^{-1} \right\rangle_N$ **- property of the SF self-averaging system**

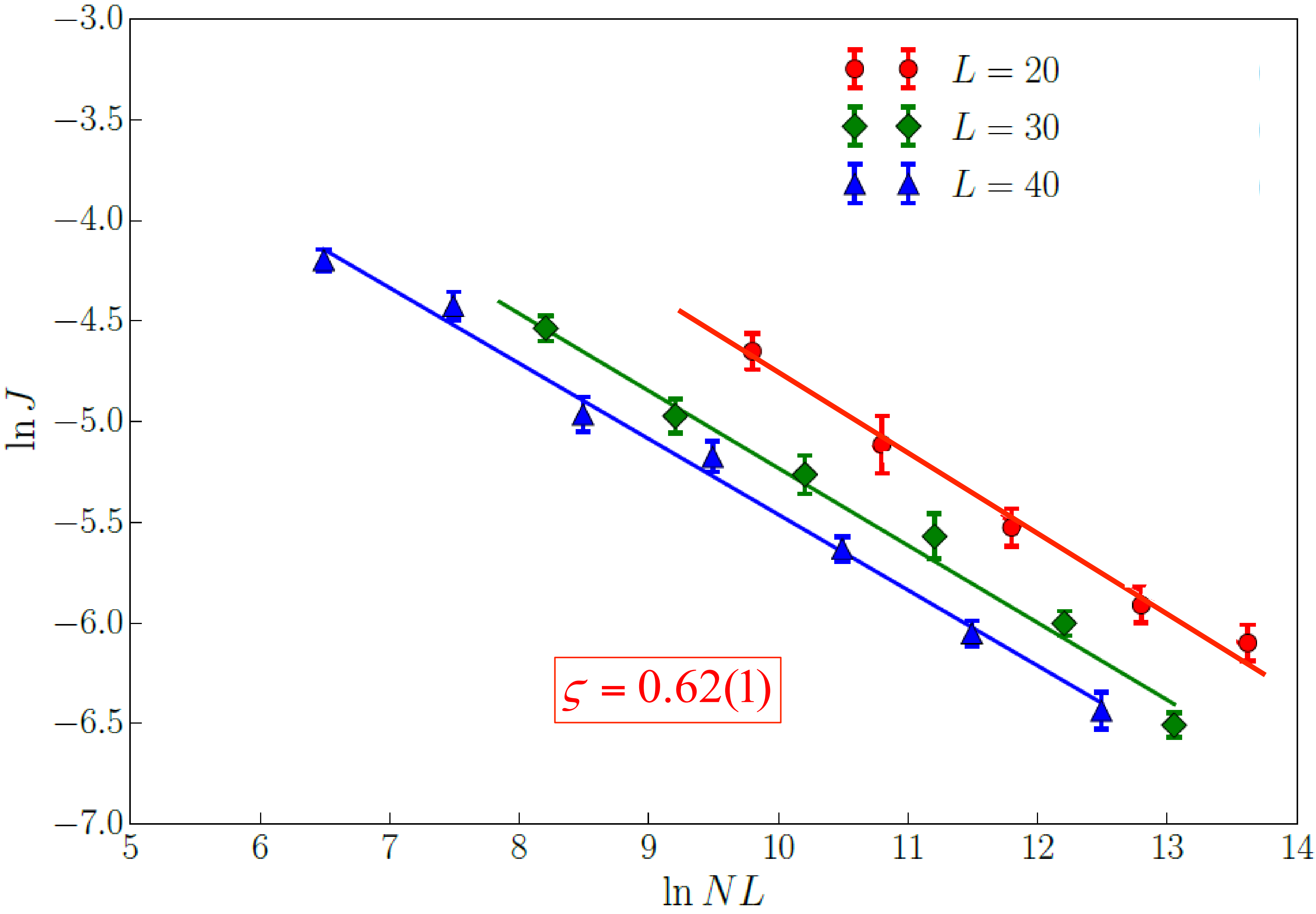
J_j **- weak-link Josephson coupling; using** $1/JL = \Lambda^{-1} - \left\langle \Lambda^{-1} \right\rangle$

2. **Determine the typical weakest link generated in n disorder realizations**

$$J(n) \equiv \left\langle J \right\rangle_{N/n}$$

and examine scaling $\ln J(n) = (\varsigma - 1)\ln(NL) + const$

3. **Check that ς is system-size independent doubling system size**



L = 20
L = 30
L = 40
ln J
ln NL
ς = 0.62(1)

Verifying sXY experimentally is not easy.

He-4 films with Ce-scratches?

Josephson Junction arrays?