Triangulating manifolds

Ramsay Dyer

MSP and INRIA

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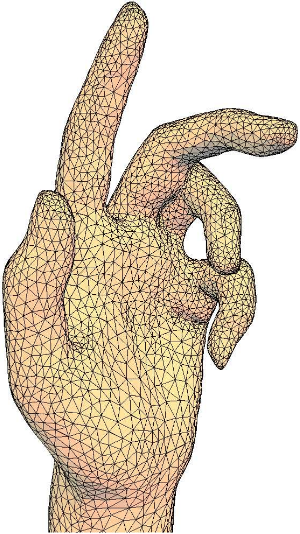
Collaborators

- Torsten Möller, Hao Zhang
- Jean-Daniel Boissonnat, Arijit Ghosh, Mathijs Wintraecken
- Gert Vegter, Nikolay Martynchuk

Triangulating manifolds



We want

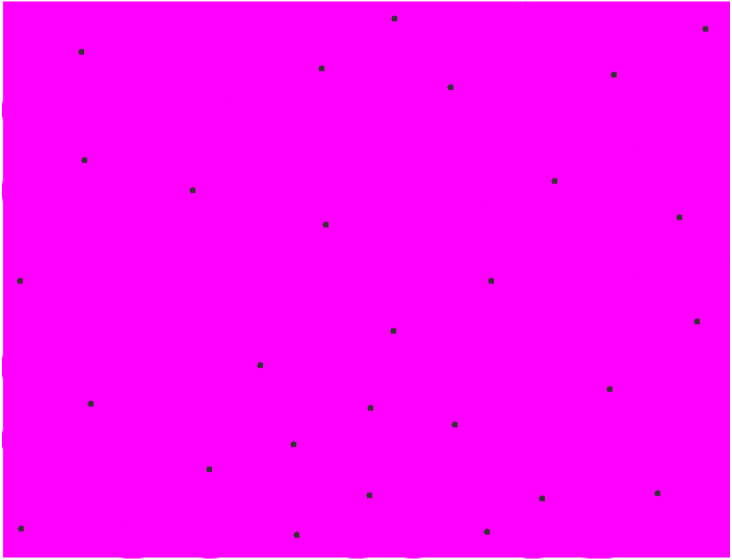
1. homeomorphic simplicial representation
2. algorithmically realizable
3. geometric fidelity
4. sampling criteria

Foundational work for 1

- Cairns (1934), Whitehead (1940)
- Whitney (1957)

The Delaunay triangulation

and the Voronoi diagram



The Delaunay triangulation
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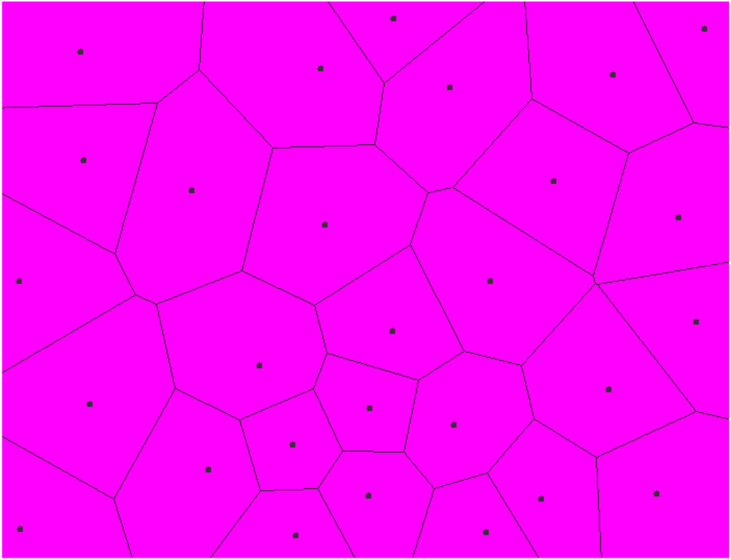

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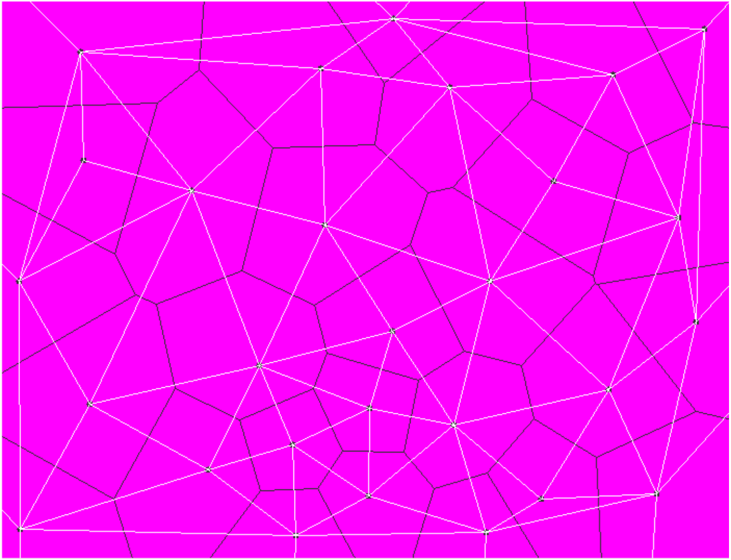

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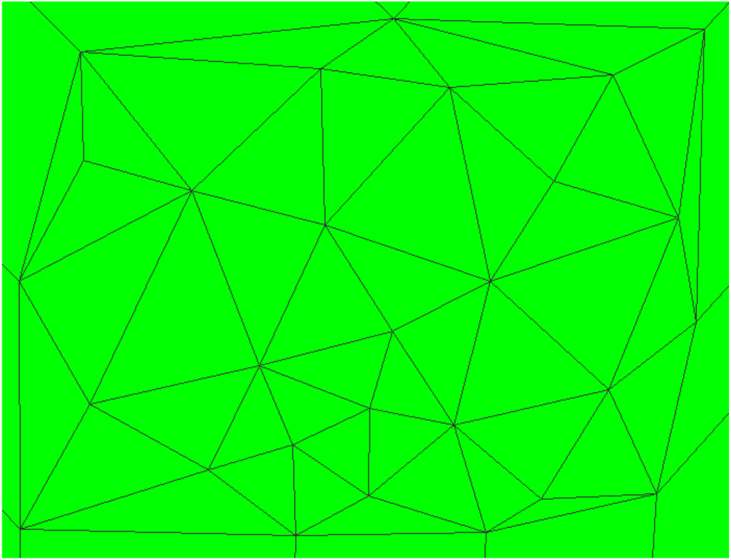

The Delaunay triangulation

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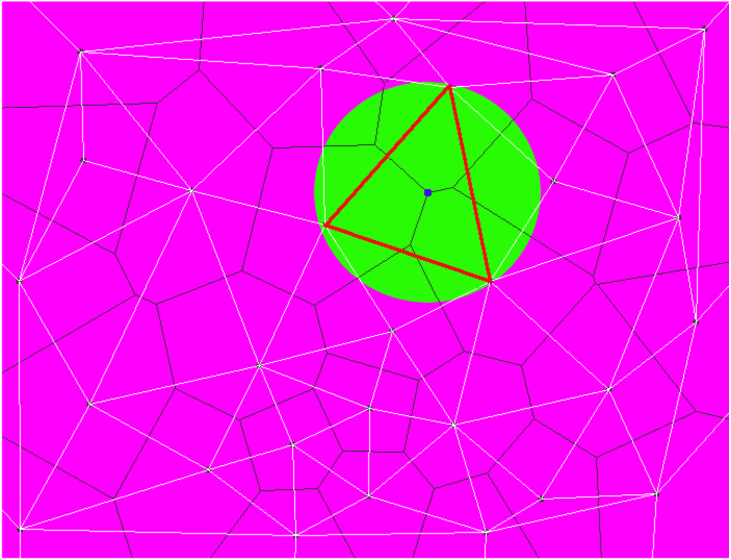
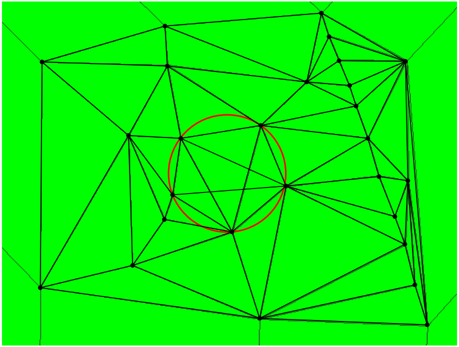
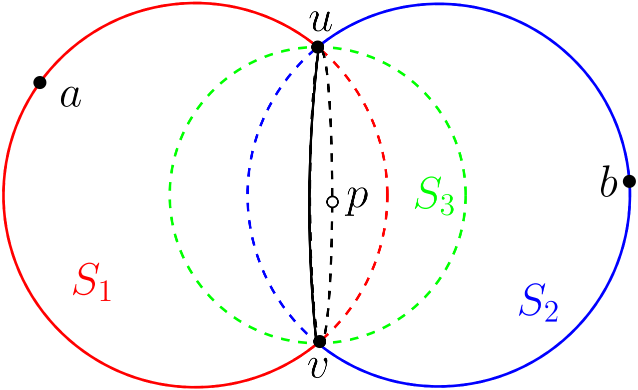


Degenerate configurations



- $P \subset \mathbb{R}^m$ is *degenerate* if there are more than $m+1$ points on the boundary of an empty ball.
- If P is not degenerate, the Delaunay complex is a triangulation (Delaunay 1934).

Delaunay's triangulation proof



Three spheres intersect nicely

- If an $(m-1)$-simplex is on the boundary of three spheres, one of them is contained by the other two.
- Exactly two cofaces to an $(m-1)$-simplex.

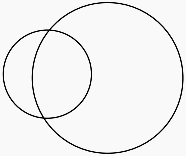
Extension to surfaces

Pseudodisks (Boissonnat and Oudot 2005)

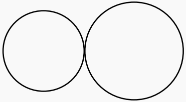
Pseudodisks

Boundaries intersect

transversely twice:

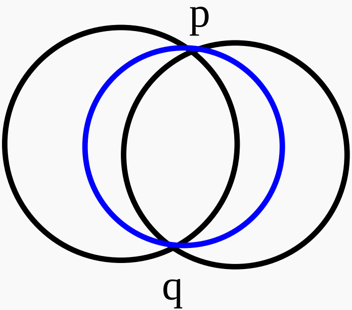


tangentially once:



or not at all.

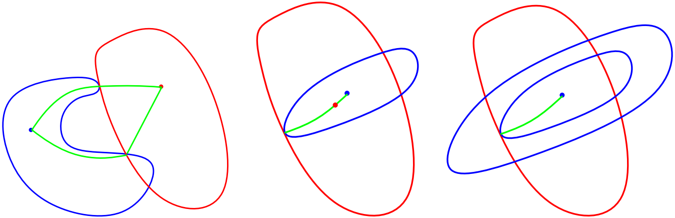
Pseudo-disks suffice



If three circles contain p and q, one is contained in the other two.

Geodesic pseudoballs

(D., Möller, Zhang 2008)



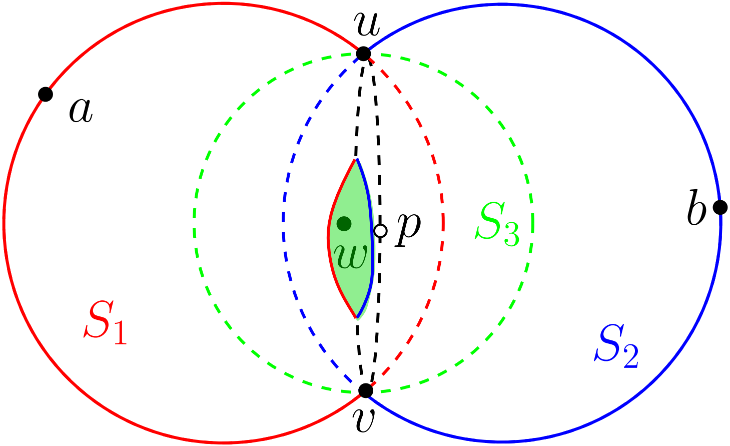
X-radius: maximum radius of ...

- sampling radius, $r(x)$: empty disk centered at x
- convexity radius, $\mathrm{cr}(x)$: convex disk centered at x
- injectivity radius, $\iota(x)$: disk with nonintersecting radial geodesics

Theorem (Sampling density criterion)

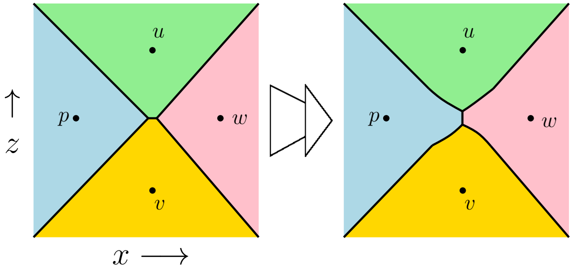
If $r(x) < \min\{\mathrm{cr}(x), \frac{1}{2}\iota(x)\}$, *then the Delaunay complex triangulates the surface.*

Problems in higher dimension



An obstruction

(Boissonnat, D., Ghosh, Martynchuk 2016)



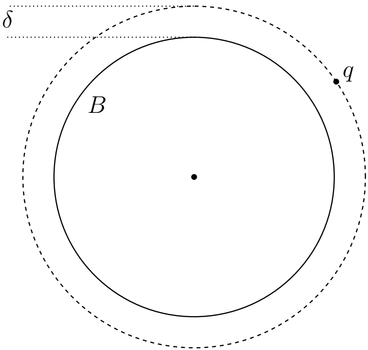
A smooth and almost Euclidean metric

$$g(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + f(y(q)) \end{pmatrix}, \quad \text{where } y(q) \text{ is the } y\text{-coordinate of } q$$

- f a bump function; does not require compact support
- presents an obstruction to Delaunay triangulation at all scales

Protection

In Euclidean space



Definition (protected)

A simplex σ is *protected* if it has a Delaunay ball B whose boundary contains no other points from P.
We say σ is *δ-protected* if $d_{\mathbb{R}^m}(q, \partial B) > \delta$ for all $q \in P \setminus \sigma$.

δ-protected point set

A point set $P \subset \mathbb{R}^m$ is δ-*generic* if the Delaunay m-simplices are all δ-protected.

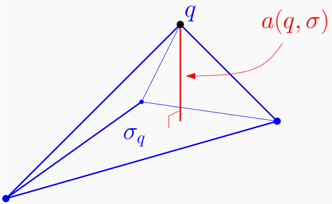
- t_0: lower bound on thickness
- μ_0: shortest edge length / largest diameter

Delaunay stability (Boissonnat, D., Ghosh 2013)

- Delaunay complex doesn't change with small perturbation of the points or of the metric ($\sim \mu_0 t_0 \delta$)
- in the presence of a sampling radius ϵ: lower bound on quality of the Delaunay simplices ($\sim (\delta/\epsilon)^2/m$)

Simplex quality

Altitude



The *altitude* of q in σ is its distance to the affine hull of σ_q, the opposite face:

$$a(q, \sigma) = d_{\mathbb{R}^m}(\mathrm{aff}(q, \sigma_q)).$$

Thickness

The *thickness* of a j-simplex σ with diameter $L(\sigma)$ is

$$t(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p \in \sigma} \frac{a(p,\sigma)}{jL(\sigma)} & \text{otherwise.} \end{cases}$$

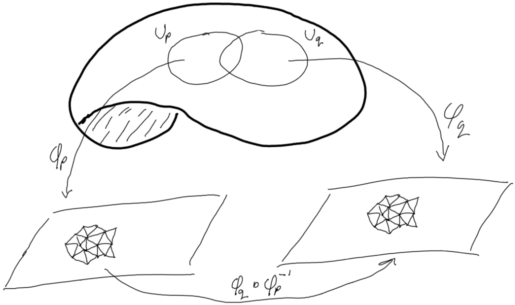
Protection

Algorithmically attainable

Algorithms designed to improve simplex quality in Delaunay triangulations can be adapted to provide protection.

- weighting
- refinement
- perturbation
  - The Moser–Tardos algorithm (algorithmic Lovász local lemma) considerably simplifies the analysis, and improves the provided protection ($\sim (\mu_0/2)^{m^2}$) (Boissonnat, D., Ghosh 2015)

Delaunay triangulation of manifolds

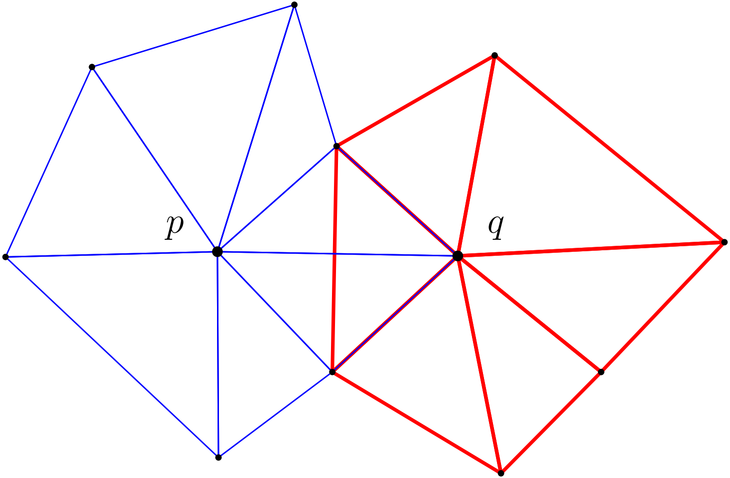


Local approach

- obtain protection in local coordinate charts
- use the local Euclidean metric
- the metric is close to that on the manifold
- in fact, only transition functions are used

Delaunay triangulation of manifolds

Inconsistent configurations



Manifold Delaunay complex

(Boissonnat, D., Ghosh 2017)

$F\colon (X, d_X) \to (Y, d_Y)$ is a ξ-*distortion map* if

$$|d_Y(F(x), F(y)) - d_X(x, y)| \le \xi d_X(x, y).$$

Definition ((μ_0, ϵ)-net)

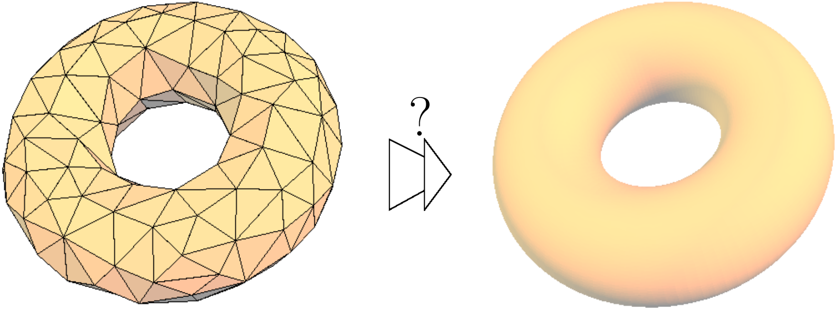
- ϵ a sampling radius (for each $x \in M,\ d_M(x, P) < \epsilon$)
- for each $p, q \in P,\ d_M(p, q) \ge \mu_0 \epsilon$

Theorem (manifold Delaunay complex via perturbation)

- *$P \subset M$ a (μ_0, ϵ) net in each coordinate chart*
- *ϵ a local sampling radius*
- *each ϕ_p is a ξ-distortion map, $\xi \sim (\mu_0/2)^{m^3} \rho_0^m$,*
- *$\rho_0 = \rho/\epsilon < \mu_0/4$ bounds the magnitude of the perturbation ρ*

Then the perturbation algorithm produces a manifold Delaunay complex $\mathrm{Del}(P')$ *for M.*

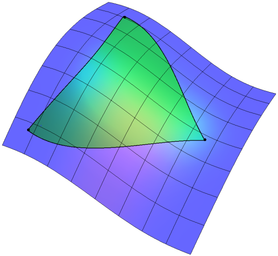
Homeomorphism problem



- What is a good map from the complex to the manifold?
- How do we show that this map is a homeomorphism?

Riemannian simplices

via the barycentric coordinate map



- M an n-dimensional Riemannian manifold
- $B \subset M$ a convex set (restricts size)
- $\sigma^j = \{p_0, \ldots, p_j\} \subset B$ a finite set of vertices
- Δ^j the standard Euclidean j-simplex

The Barycentric coordinate map

$$\mathcal{B}_{\sigma^j} : \boldsymbol{\Delta}^j \to M$$

The Riemannian simplex $\boldsymbol{\sigma}_M^j$ is the image of this map.

The barycentric coordinate map, $\mathcal{B}_\sigma$

Riemannian centre of mass

Energy function

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, p_i)^2$$

barycentric coordinates: $\lambda_i \geq 0$; $\sum_{i=0}^{j} \lambda_i = 1$

$$\mathcal{B}_{\sigma^j} : \mathbf{\Delta}^j \to M$$
$$\lambda \mapsto \operatorname*{argmin}_{x \in \overline{B}_r} \mathcal{E}_\lambda(x)$$

- ι_M: injectivity radius
- Λ_+: upper bound on sectonal curvatures

Theorem (Karcher 1977)

If $\{p_0, \ldots, p_j\} \subset B_r \subset M$, and B_r is an open ball of radius r with

$$r < \min\left\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda_+}}\right\},$$

then $\mathcal{E}_\lambda$ is convex and has a unique minimum in B_r.

Nondegenerate Riemannian simplices

(D., Vegter, Wintraecken 2015)

Definition

A Riemannian simplex $\boldsymbol{\sigma}_M$ is *nondegenerate* if the barycentric coordinate map $\mathcal{B}_{\sigma^j} \to M$ is an embedding.

Notation

$$v_i(x) = \exp_x^{-1}(p_i) \quad \text{and} \quad \sigma(x) = \{v_0(x), \ldots, v_j(x)\} \subset T_x M$$

Proposition

A Riemannian simplex $\boldsymbol{\sigma}_M \subset M$ is nondegenerate if and only if $\sigma(x) \subset T_x M$ is nondegenerate for every $x \in \boldsymbol{\sigma}_M$.

Nondegenerate Riemannian simplices

(D., Vegter, Wintraecken 2015)

Theorem (Nondegeneracy criteria)

If

- *sectional curvatures* K *bounded by* $|K| \leq \Lambda$
- $\boldsymbol{\sigma}_M \subset B_r \subset M$
- B_r *is an open geodesic ball of radius* r *with*

$$r < r_0 = \min\left\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda}}\right\}$$

Then $\boldsymbol{\sigma}_M$ *is nondegenerate if*

$$t(\boldsymbol{\sigma}_{\mathbb{E}}) > 3\sqrt{\Lambda}L(\boldsymbol{\sigma}_{\mathbb{E}}),$$

where $\boldsymbol{\sigma}_{\mathbb{E}}$ *is the Euclidean simplex with the same edge lengths as* $\boldsymbol{\sigma}_M$ (geodesic distances).

Riemannian Delaunay triangulation

(D., V., W., 2015); (B., D., G. 2017)

Theorem (Riemannian DT)

If $P \subset M$ is a (μ_0, ϵ)-net with

$$\epsilon \leq \min\left\{ \tfrac{1}{4}\iota_M,\ \sim \Lambda^{-\frac{1}{2}}(\mu_0/2)^{m^3}\rho_0^m \right\},$$

then

- $\mathrm{Del}(P')$ *is a Delaunay triangulation*
- *it admits a piecewise flat metric defined by geodesic edge lengths*
- *the barycentric coordinate map* $H\colon |\mathrm{Del}(P')| \to M$ *is a* ξ*-distortion map with* $\xi \sim (\mu_0/2)^{m^3}\rho_0^m \Lambda \epsilon^2$ (they're Gromov–Hausdorff close)

Local metric criteria for triangulation

Problem

Given a compact manifold M, a simplicial complex $\mathcal{A}$, and a map $H\colon |\mathcal{A}| \to M$, show that H is a homeomorphism.

Approach

- work locally in a *compatible* coordinate chart for both M and $\mathcal{A}$
- use only the local Euclidean metric
- no differentiablility assumption (but strong bi-Lipschitz constraint)

Two steps

Working in the local Euclidean domain

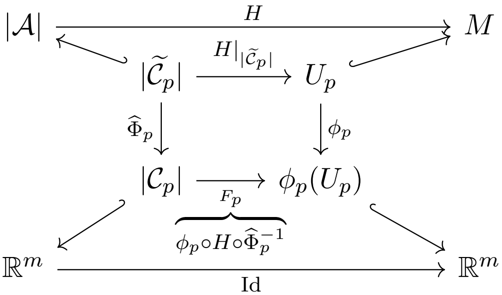
- show that H is a local homeomorphism, and thus a covering map
- establish injectivity

Local metric criteria for triangulation

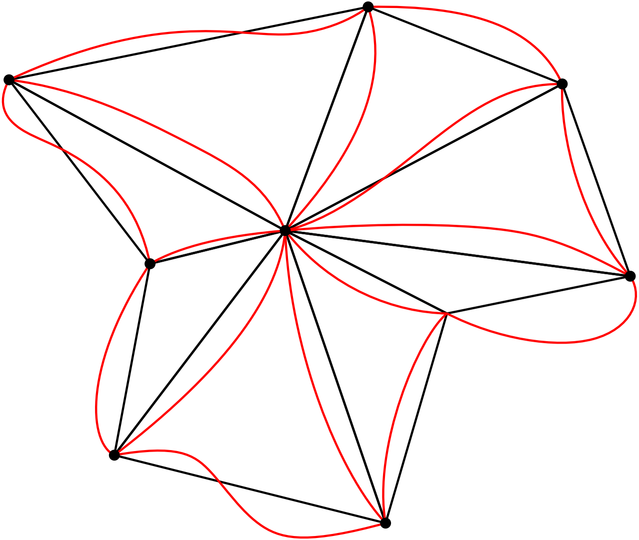
The setting

Compatible atlases

1. P vertices of $\mathcal{A}$, and $\{(U_p, \phi_p)\}_{p \in P}$ coordinate atlas for M
2. $\{(\widetilde{\mathcal{C}}_p, \widehat{\Phi}_p)\}_{p \in P}$ a PL-coordinate atlas for $\mathcal{A}$, where $H(\widetilde{\mathcal{C}}_p) \subset U_p$, and $\widehat{\Phi}_p$ is the secant map of $\phi_p \circ H$



Our focus is on the map $F_p = \phi_p \circ H \circ \widehat{\Phi}_p^{-1} \colon |\mathcal{C}_p| \subset \mathbb{R}^m \to \mathbb{R}^m$.



$$F_p \colon |\underline{\mathrm{St}}(\hat{p})| \to \phi_p \circ H(|\underline{\mathrm{St}}(p)|)$$

$$\hat{x} = \widehat{\Phi}_p(x)$$

Local metric criteria for triangulation

Local homeomorphism

Definition

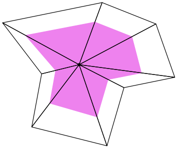
$F\colon |\mathcal{C}| \to \mathbb{R}^m$ is *simplexwise positive* if its restriction to each simplex is an orientation preserving embedding. (defined via degree theory)

Lemma (Whitney)

- $F\colon |\mathcal{C}| \subset \mathbb{R}^m \to \mathbb{R}^m$ *simplexwise positive*
- $V \subset |\mathcal{C}|$ *open, connected, and* $F(V) \cap F(|\partial\mathcal{C}|) = \emptyset$

If there is a $y \in F(V) \setminus F(|\mathcal{C}^{m-1}|)$ *such that* $F^{-1}(y)$ *is a single point, then* $F|_V$ *is an embedding.*

- $\mathcal{C}_p = \underline{\mathrm{St}}(\hat{p})$
- $V_p = \{x \in \underline{\mathrm{St}}(\hat{p}) \mid \lambda_{\hat{p},\boldsymbol{\sigma}}(x) > \frac{1}{(m+1)} - \delta\}$
- $\{\widehat{\Phi}_p^{-1}(V_p)\}$ is a cover of $|\mathcal{A}|$.



Local homeomorphism

Ingredients to apply Whitney's lemma

Require: $F_p|_{\boldsymbol{\sigma}}$ a ξ-distortion map ($\boldsymbol{\sigma}$ any m-simplex)

Lemma (trilateration)

If $F\colon \boldsymbol{\sigma} \subset \mathbb{R}^m \to \mathbb{R}^m$ is a ξ-distortion map that leaves the vertices fixed, and $\xi < 1$, then

$$|x - F(x)| \leq \frac{3\xi L}{t}.$$

Lemma (point covered once)

If $\quad \xi \leq \frac{1}{6}\frac{m}{m+1}t_0^2, \quad$ *then* $F^{-1}(F(b)) = \{b\}$.

Lemma (barycentric boundary separation)

$x \in \boldsymbol{\sigma} \in \underline{\mathrm{St}}(\hat{p})$*; barycentric coordinate* $\lambda_{\boldsymbol{\sigma},\hat{p}}(x) \geq \alpha$.
Then

$$d_{\mathbb{R}^m}(x, |\partial \underline{\mathrm{St}}(\hat{p})|) \geq \alpha a_0, \quad d_{\mathbb{R}^m}(x, |\partial \underline{\mathrm{St}}(\hat{p})|) \geq \alpha m t_0 s_0$$

Local metric criteria for triangulation

Injectivity

1. H is a covering map
2. $\Longrightarrow$ injective if $\exists x \in M,\ H^{-1}(H(x)) = \{x\}$
3. e.g., if $q \in P,\ H(q) \in H(\boldsymbol{\sigma}) \Longrightarrow q$ a vertex of $\boldsymbol{\sigma}$

Sufficient requirement for injectivity

$p, q \in P$, $H(q) \in U_p$:

$$\phi_p \circ H(q) \in \widehat{\Phi}_p(|\underline{\mathrm{St}}(p)|) \Longrightarrow q \text{ is a vertex of } \underline{\mathrm{St}}(p) \qquad (*)$$

Proof that $(*) \Rightarrow$ 3 .

- suppose $x \in \boldsymbol{\sigma},\ H(x) = H(q)$
- barycentric boundary separation $\Rightarrow \lambda_{\hat{p},\hat{\boldsymbol{\sigma}}}(\hat{x}) < \frac{1}{m+1}$
- but $\lambda_{\hat{p},\hat{\boldsymbol{\sigma}}}(\hat{x}) = \lambda_{p,\boldsymbol{\sigma}}(x) \qquad (\hat{x} = \widehat{\Phi}_p(x))$
- true for all vertices of $\boldsymbol{\sigma}$, but need $\sum_{p\in\boldsymbol{\sigma}} \lambda_{p,\boldsymbol{\sigma}}(x) = 1$

□

Local metric criteria for triangulation

Theorem (triangulation)

$H\colon |\mathcal{A}| \to M$ is a homeomorphism if we have (for all $p \in P$):

1. **compatible atlases**
2. **simplex quality** *Every simplex $\boldsymbol{\sigma} \in \underline{\mathrm{St}}(\hat{p}) = \widehat{\Phi}_p(\underline{\mathrm{St}}(p))$ satisfies $s_0 \leq L(\boldsymbol{\sigma}) \leq L_0$ and $t(\boldsymbol{\sigma}) \geq t_0$.*
3. **distortion control** $F_p = \phi_p \circ H \circ \widehat{\Phi}_p^{-1}\colon |\underline{\mathrm{St}}(\hat{p})| \to \mathbb{R}^m$, *when restricted to any m-simplex in $\underline{\mathrm{St}}(\hat{p})$, is an orientation-preserving ξ-distortion map with*

$$\xi < \frac{s_0 t_0^2}{12 L_0} = \tfrac{1}{12}\mu_0 t_0^2.$$

4. **vertex sanity** *For all other vertices $q \in P$, if $\phi_p \circ H(q) \in |\underline{\mathrm{St}}(\hat{p})|$, then q is a vertex of $\underline{\mathrm{St}}(p)$.*

Closing thoughts

Exploiting the differential

- suppose T a linear isometry, and $\|(dF_p|_{\boldsymbol{\sigma}})_u - T\| < \xi$ for all m-simplices $\boldsymbol{\sigma} \in \underline{\mathrm{St}}(\hat{p})$ and $u \in \boldsymbol{\sigma}$
- then

$$\xi < \tfrac{1}{2}\mu_0 t_0 \quad \text{instead of} \quad \xi < \tfrac{1}{12}\mu_0 t_0^2$$

  suffices for triangulation
- because we can avoid trilateration

Challenges and directions

- Actual implementation in higher dimensions
- What is the best simplex quality we can acheive?
  - Siargey Kachanovich; Aruni Choudhary
- Structured manifolds

Thank You.