Computational geometry, optimal transport and applications

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Joint works with Thomas Gallouët, Jun Kitagawa, Pedro Machado, Jocelyn Meyron, Jean-Marie Mirebeau, Boris Thibert

Geometric Understand in Higher Dimension / 8 Juin 2017 / Collège de France

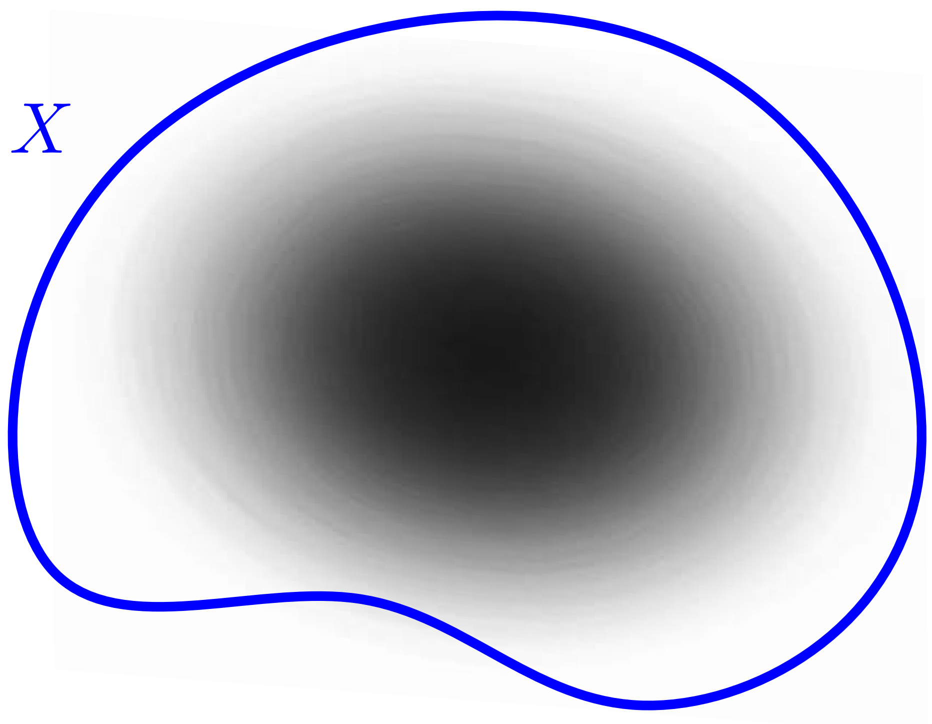
Overview

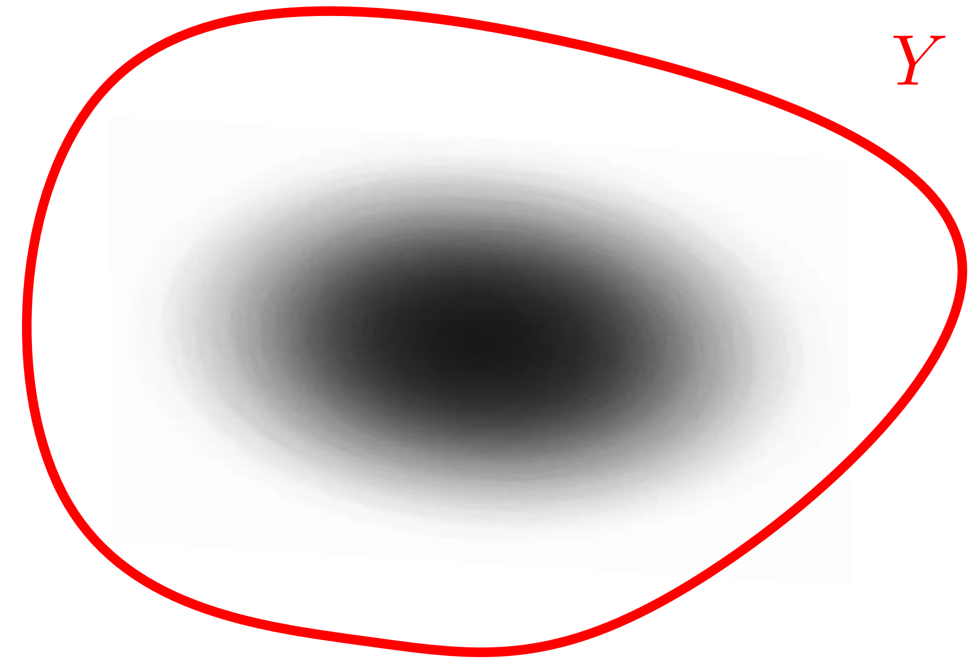
1. Optimal transport & Laguerre diagrams
2. First application: non-imaging optics
3. Second application: enforcing incompressibility

1. Optimal transport & Laguerre diagrams

Optimal transport

Data: $\rho =$ prob density on X ν probability meas. on Y





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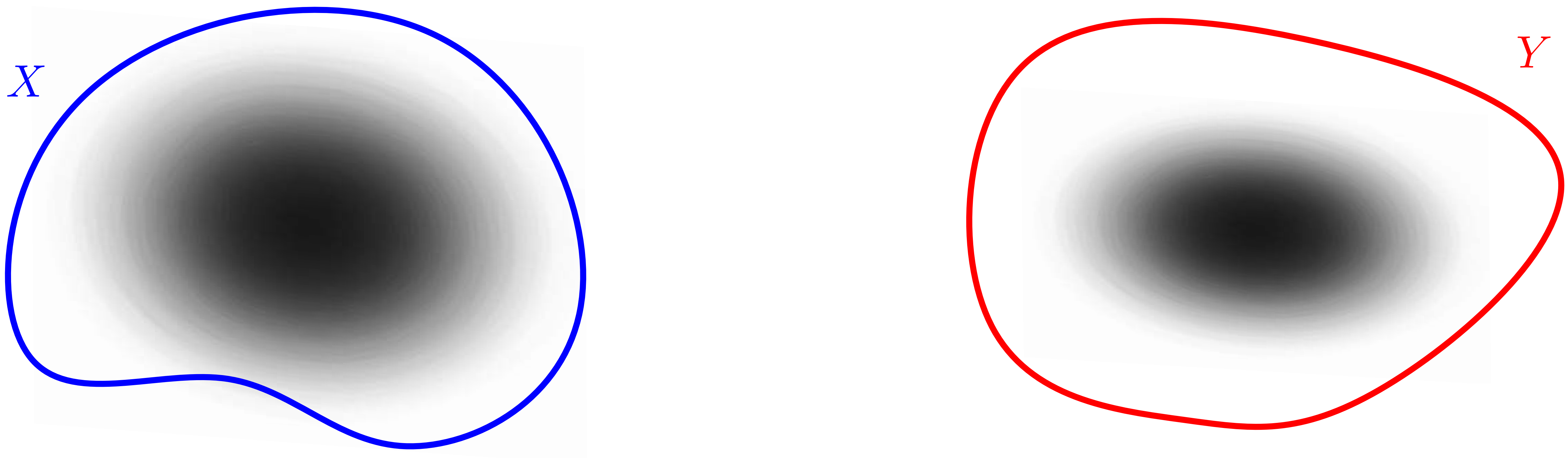


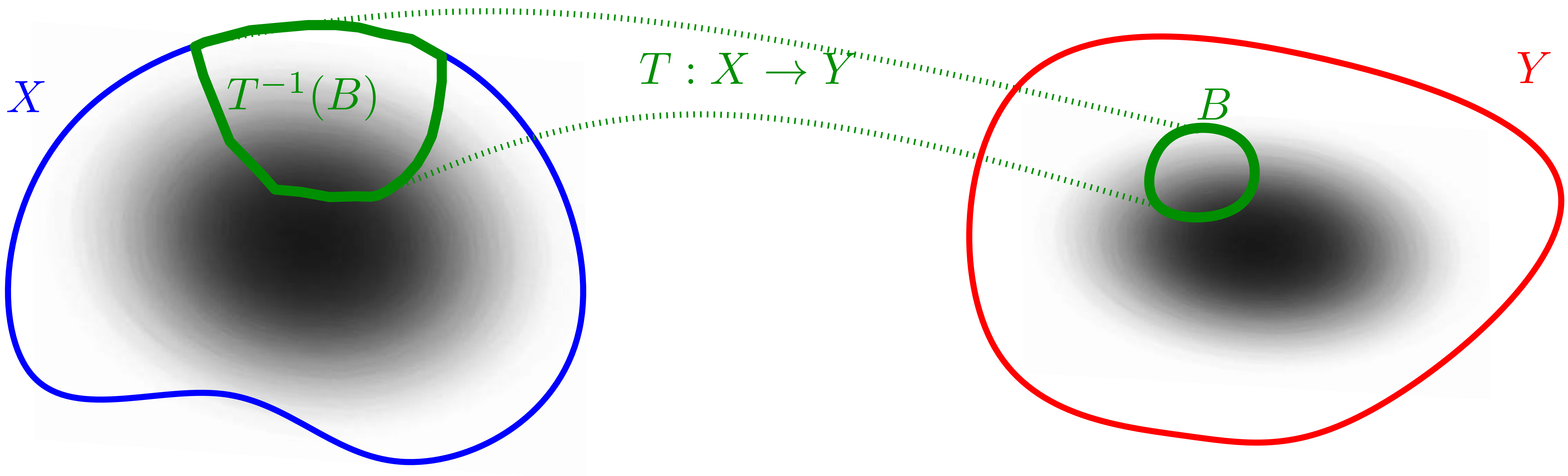
Think of ρ, ν as describing piles of sand, made of many grains.

Assume that moving a grain with mass dm from x to y costs $c(x, y)dm$.

Optimal transport problem: what is the cheapest way of moving ρ to ν ?

Optimal transport

Data: $\rho =$ prob density on X $\qquad$ ν probability meas. on Y



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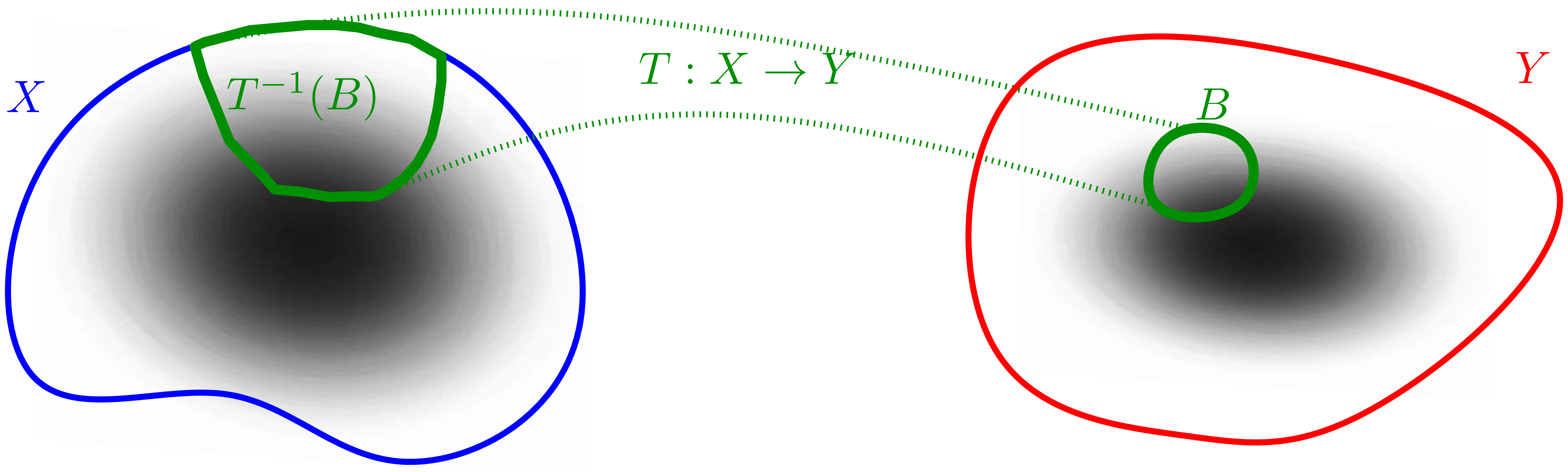

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Optimal transport problem: minimize $\int_X c(x, T(x))\,\mathrm{d}\,\rho(x)$ where $T_{\#}\rho = \nu$

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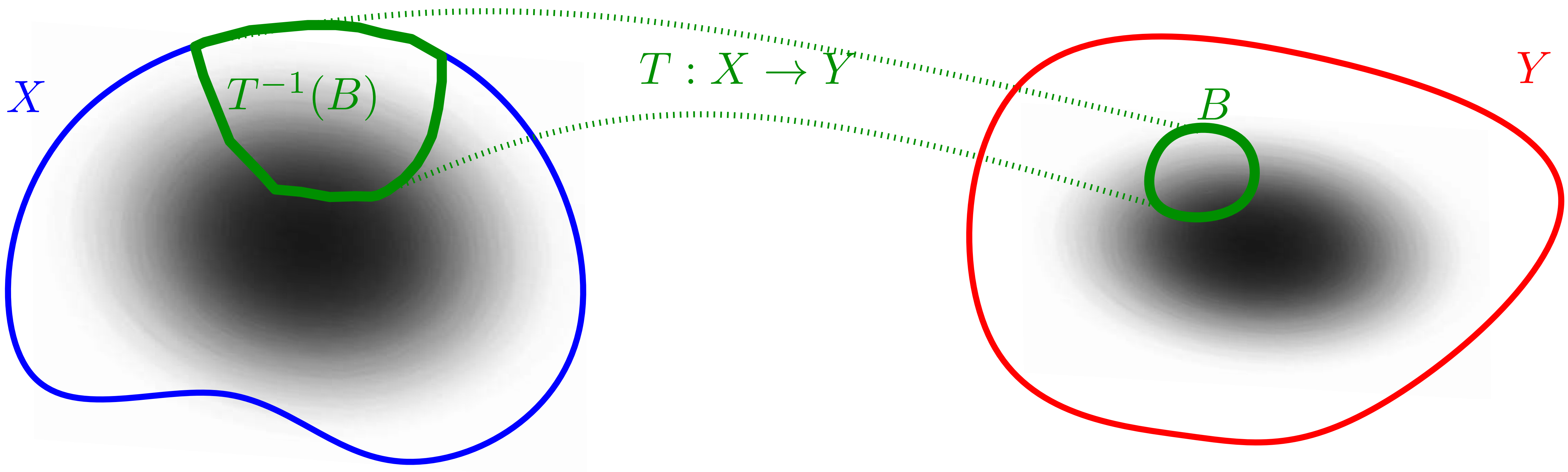

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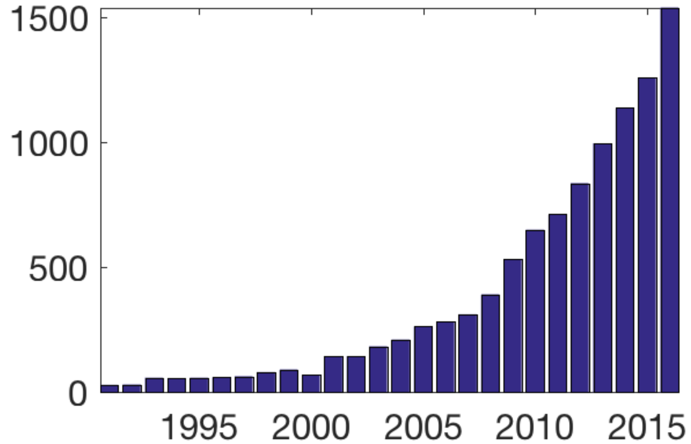
Many applications:

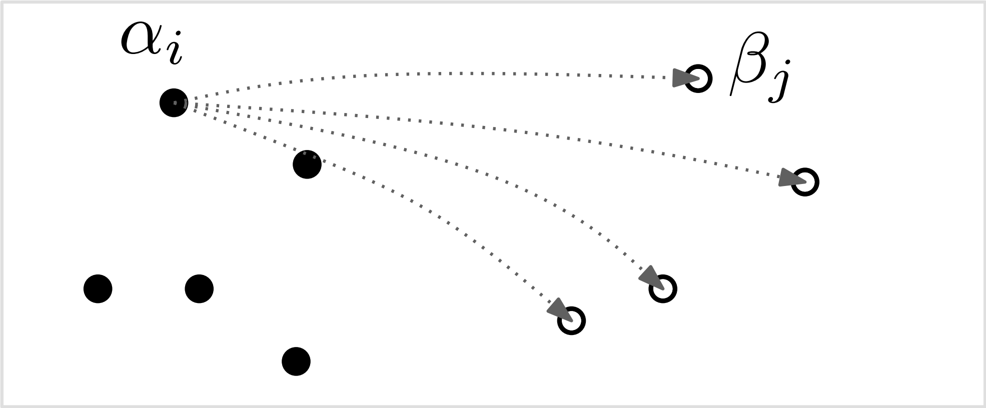
PDEs, functional inequalities, probabilities,

computer graphics, machine learning, inverse problems, etc.



articles containing "OT"

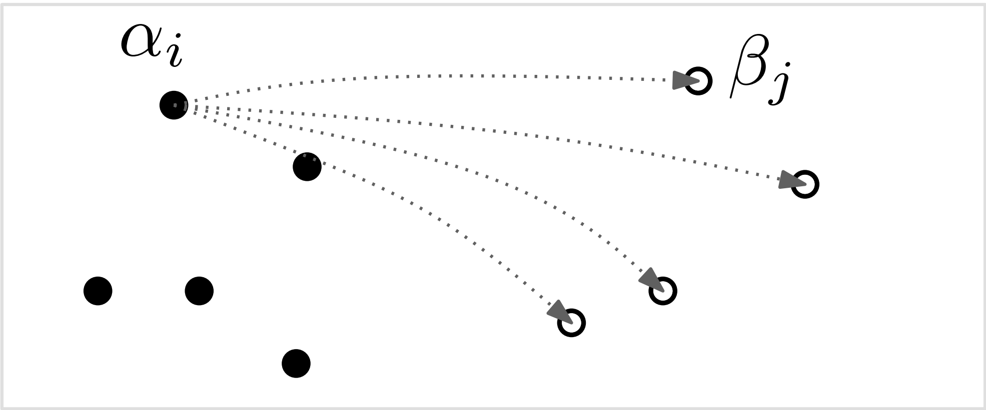
Computational optimal transport



Discrete source and target

- linear programming
- Hungarian algorithm
- Sinkhorn/IPFP

Computational optimal transport



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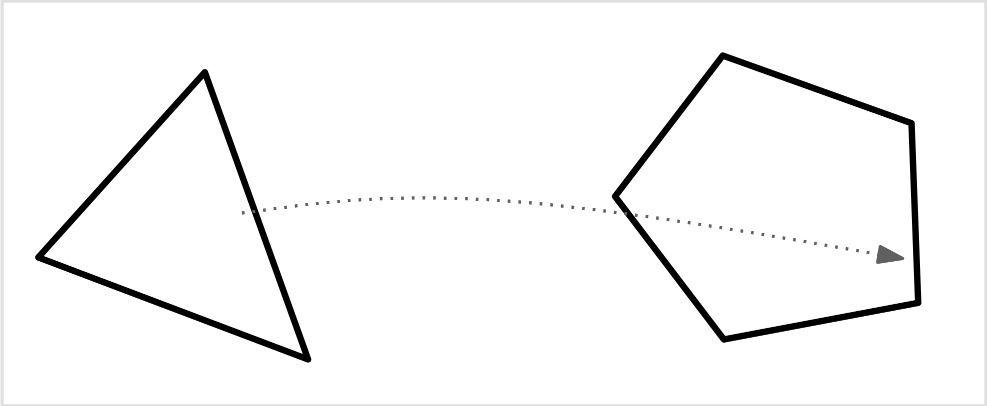
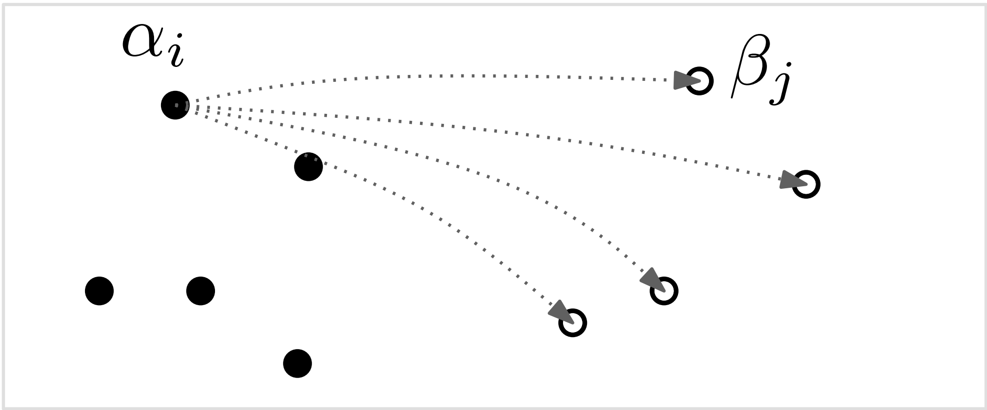


Source and target with density:

dynamic (Benamou-Brenier) formulation

finite-differences for Monge-Ampère

Computational optimal transport



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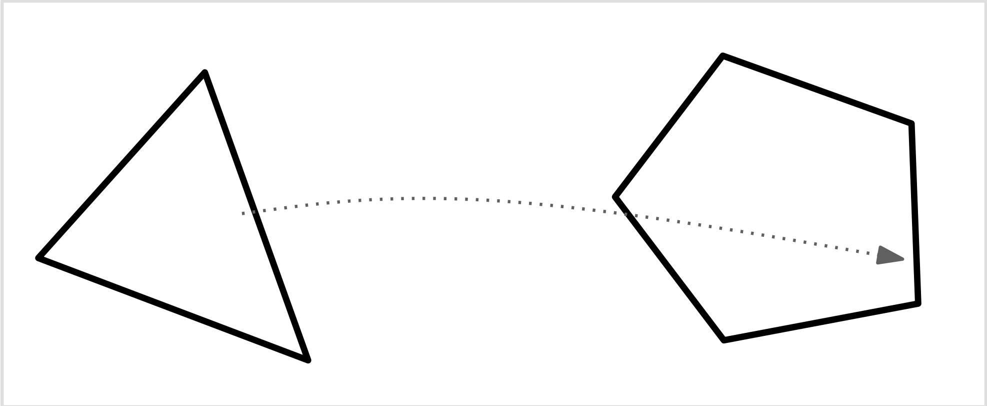

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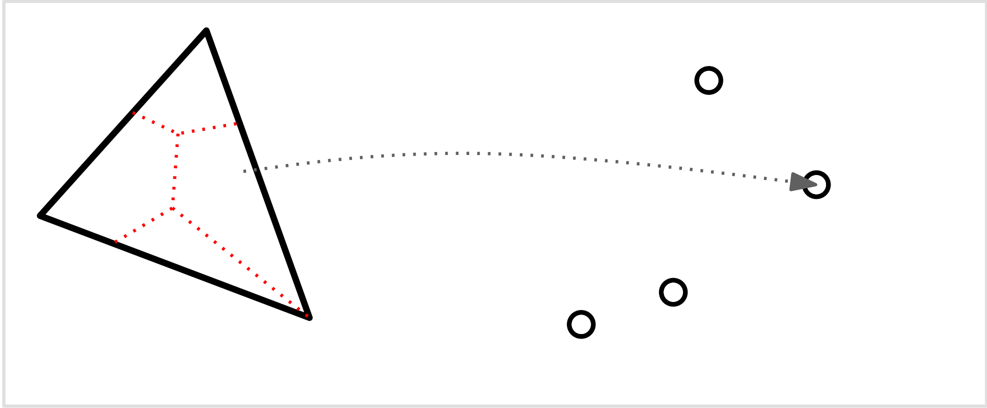
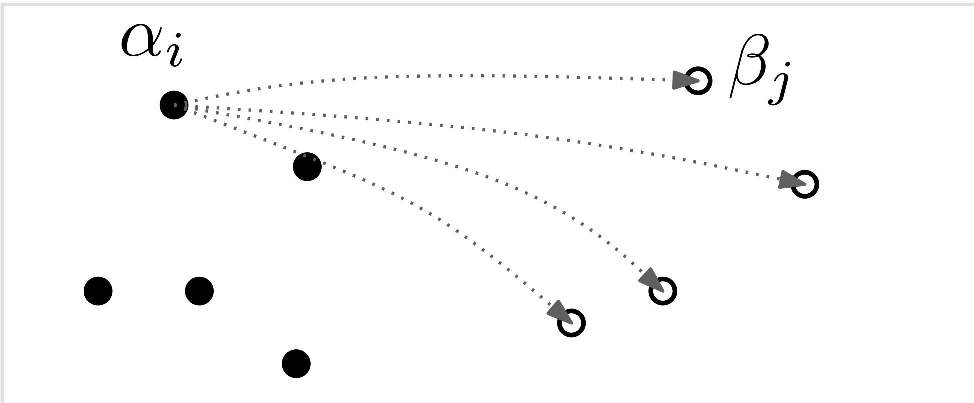


Source with density, discrete target:

Minkowski, Alexandrov, etc.

Computational optimal transport

Flexibility for the cost function **but** computationally expensive



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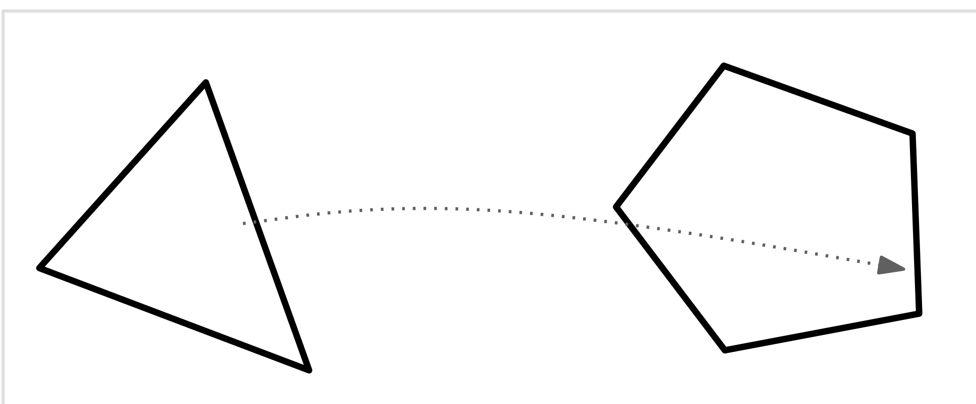

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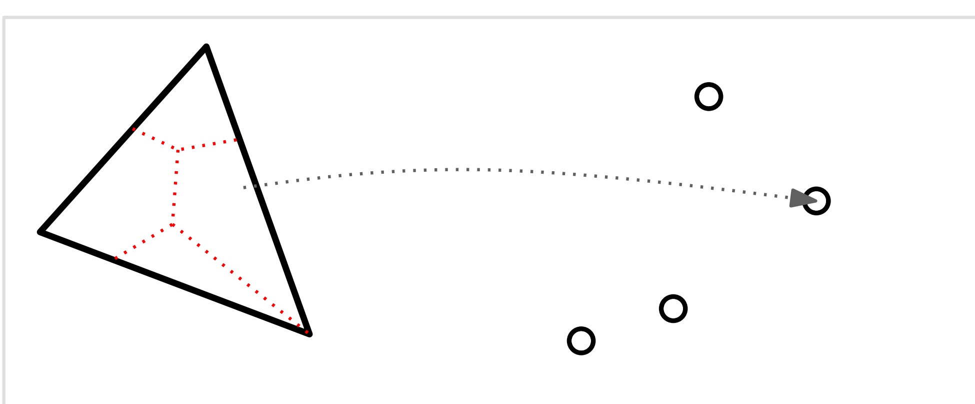
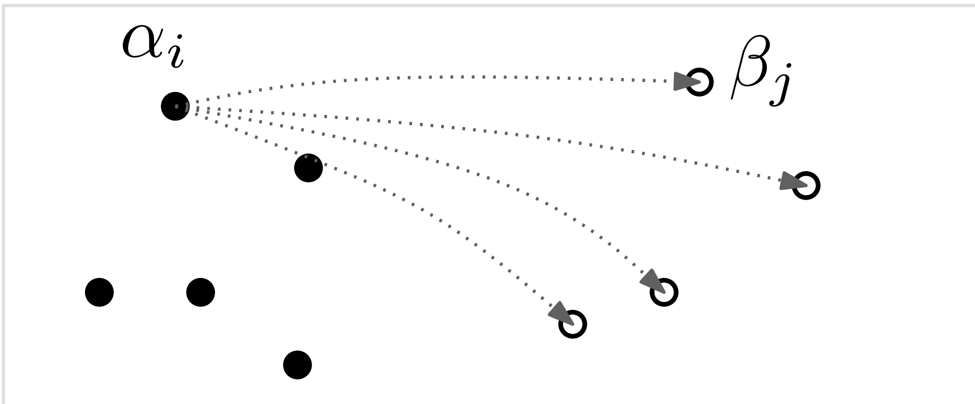


Source with density, discrete target:

- Minkowski, Alexandrov, etc.

Computationally efficient **but** restricted to "geometric" cost functions.

Computational optimal transport



Discrete source and target

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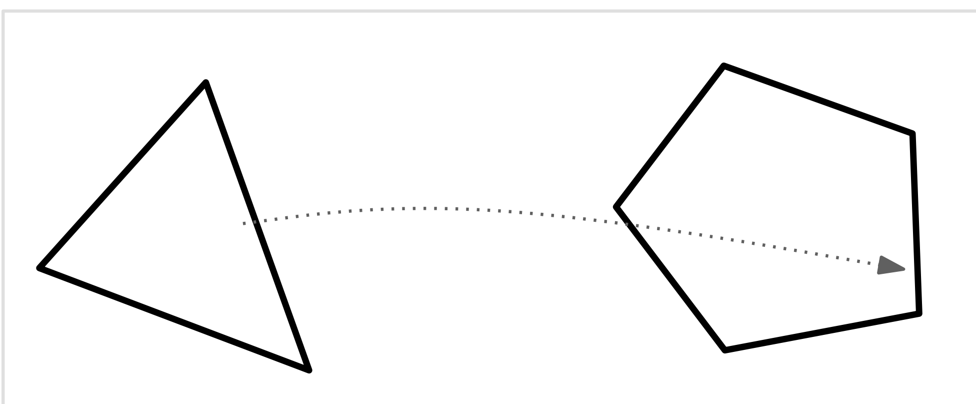

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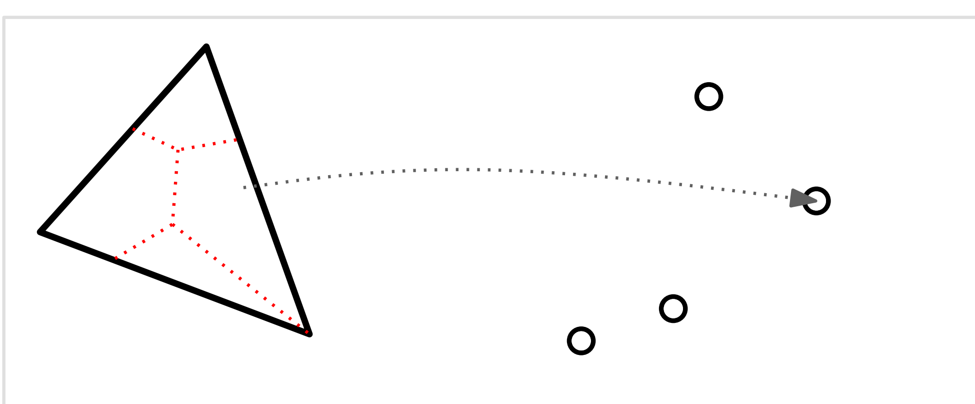

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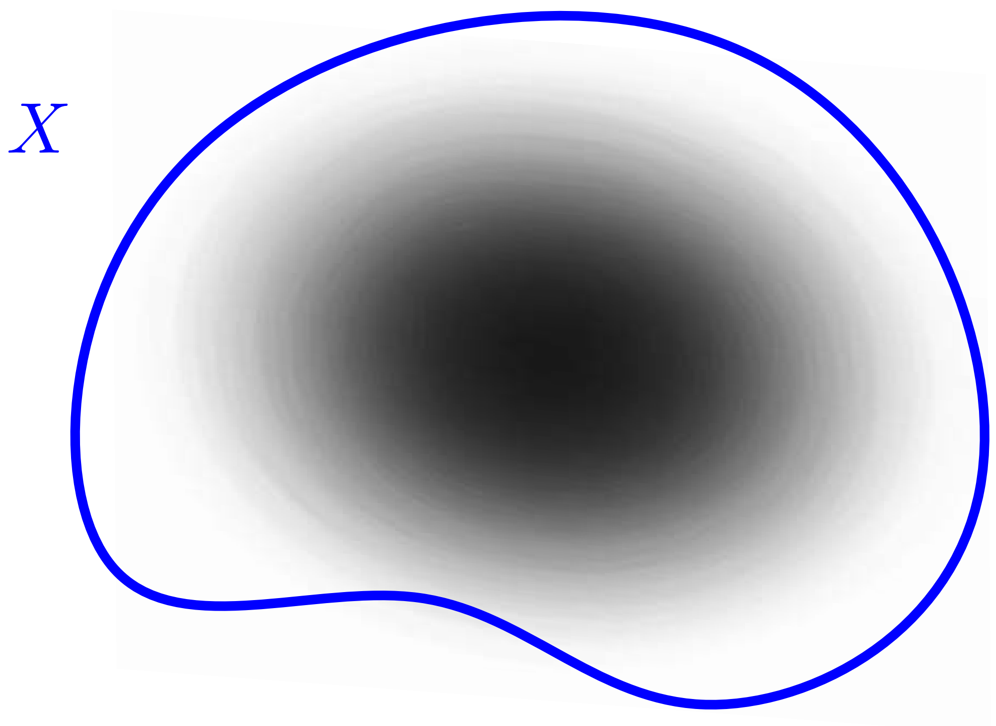
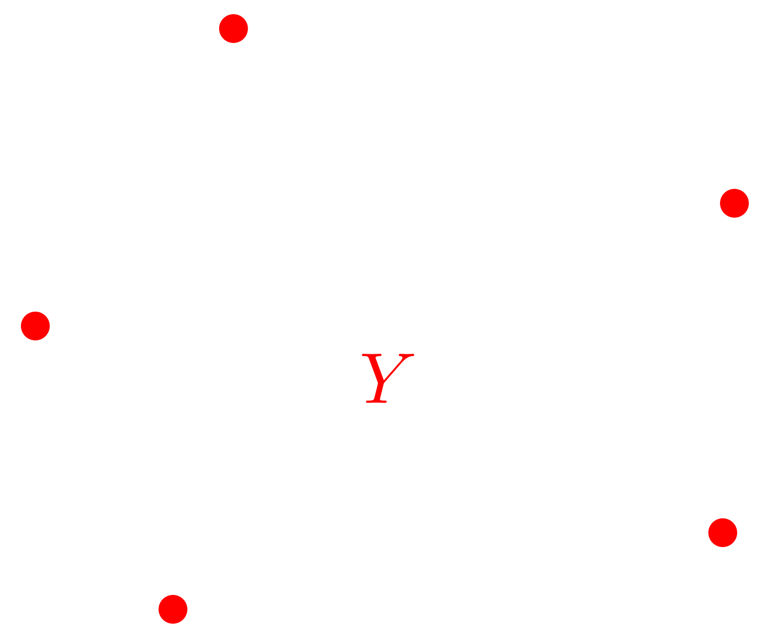
"semi-discrete optimal transport"

Semi-discrete optimal transport

Data: $\rho =$ prob density on X $\qquad$ $\nu = \sum_{y \in Y} \nu_y \delta_y$ prob. on finite Y





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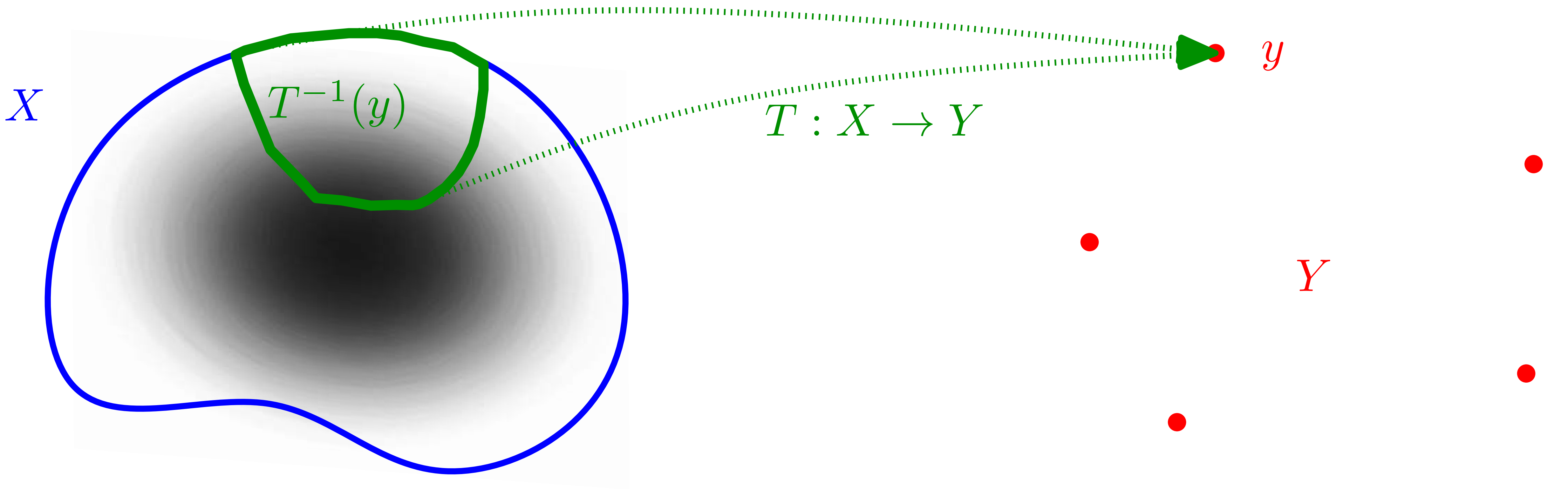

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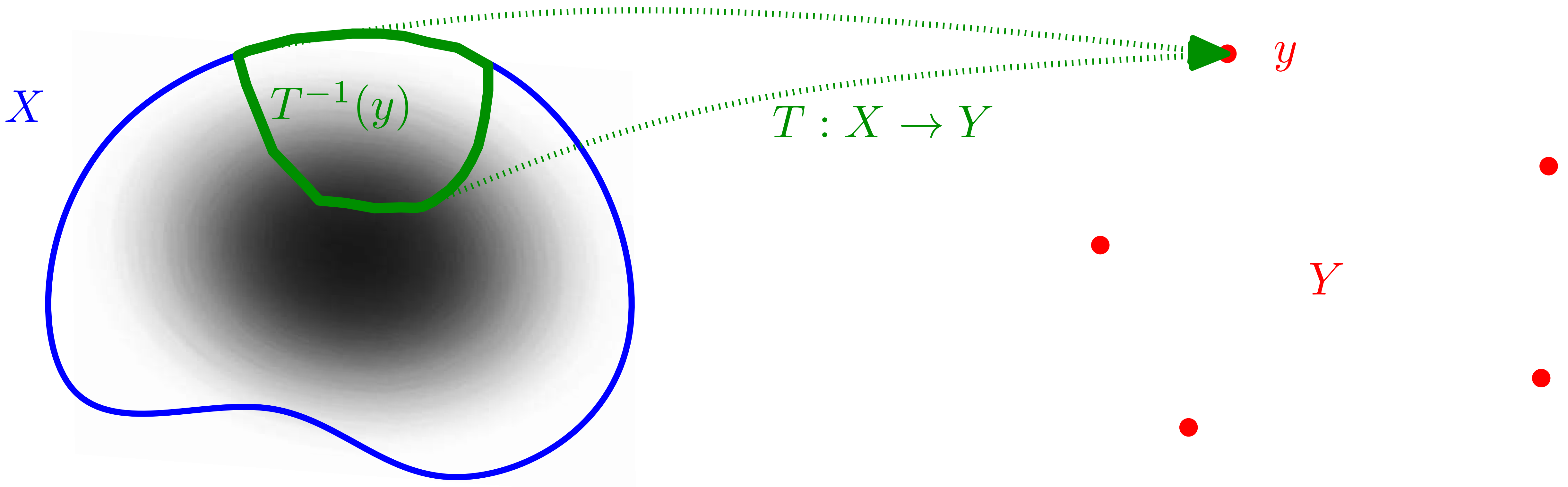

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The set of transport maps is **huge** ($\subseteq$ measurable partitions of X) ...

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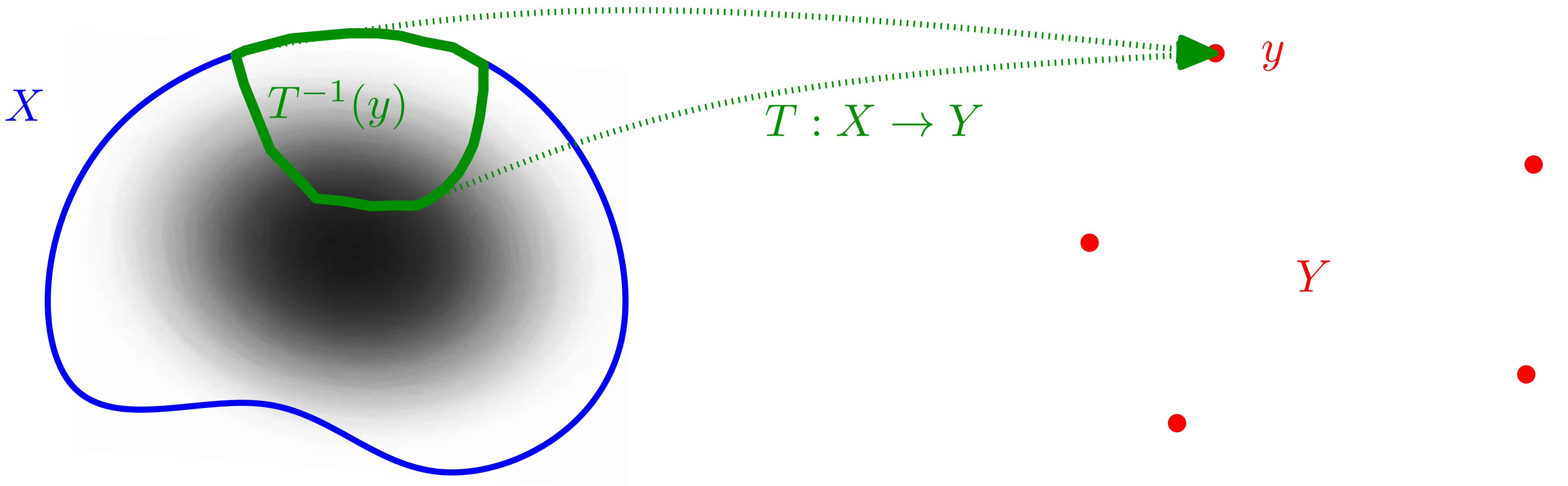

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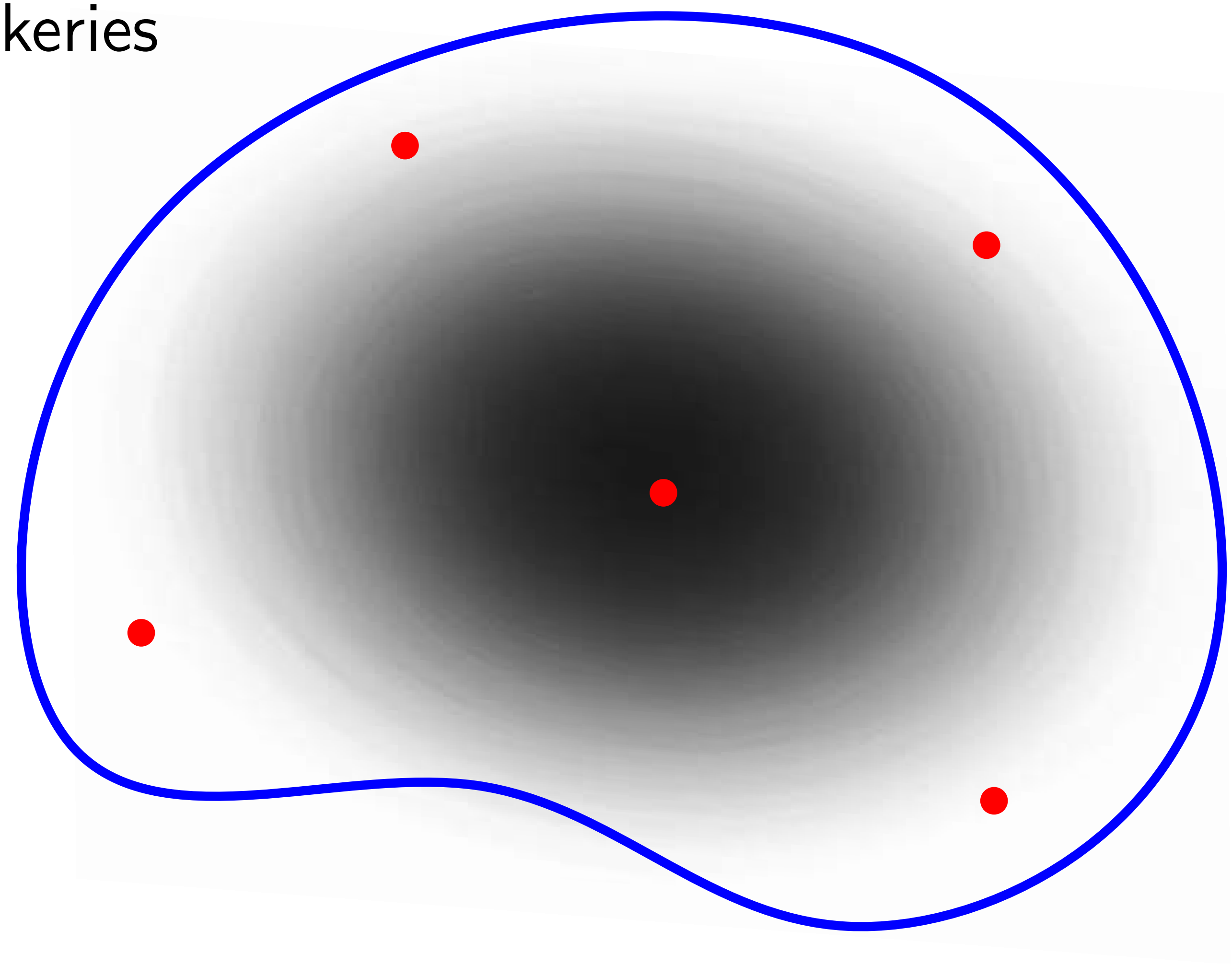
. . . but fortunately optimal maps form a much smaller (finite-dimensional) set.

Semi-discrete OT and Laguerre diagrams

$\rho : X \to \mathbb{R}$ density of population $\qquad c(x, y) := \|x - y\|^2$ cost of walking from x to y

$Y =$ location of bakeries



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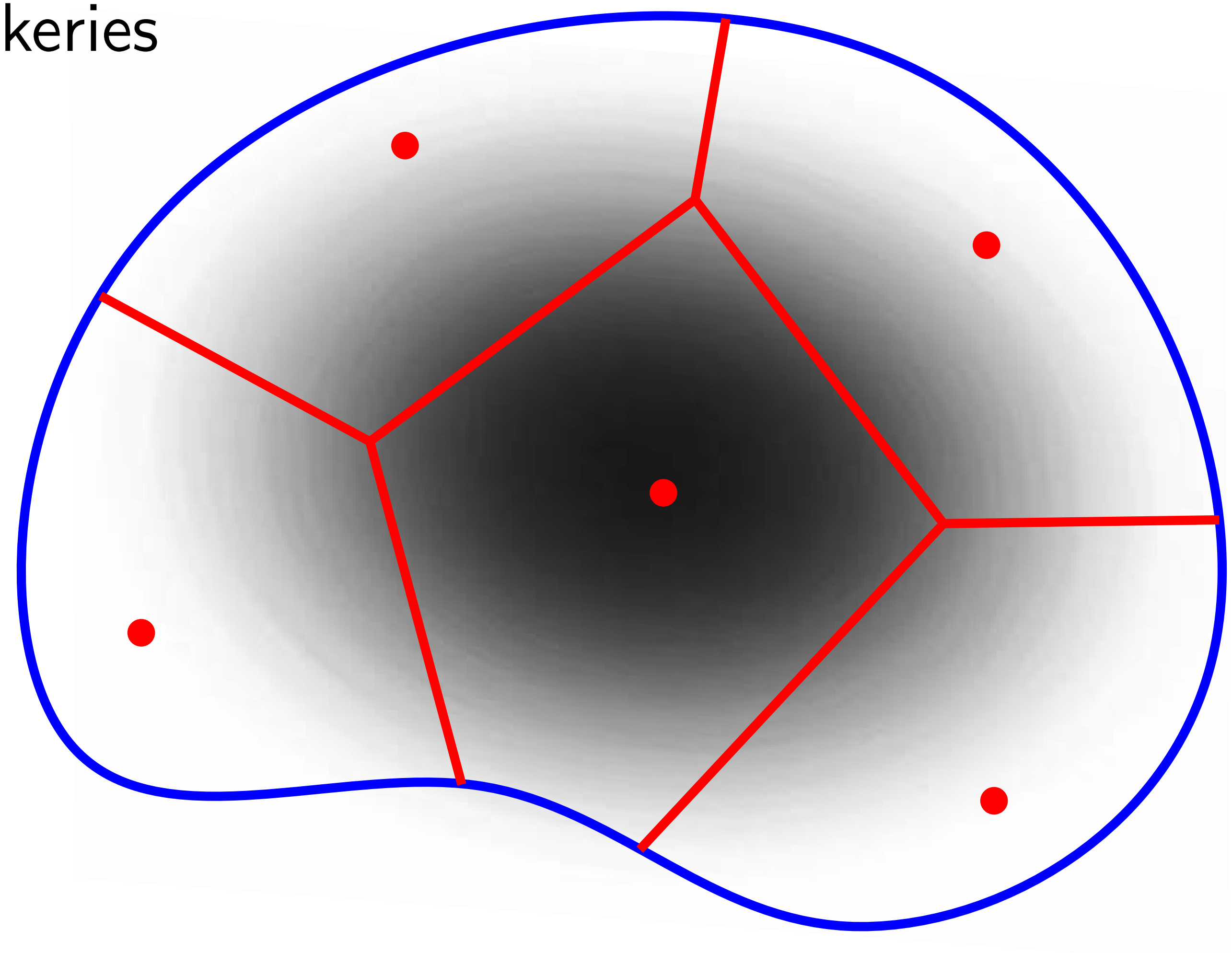

- If the price of bread is uniform, people go the closest bakery:

$$\boxed{\mathrm{Vor}(y) = \{x \in X; \forall z \in Y,\ c(x, y) \leq c(x, z)\}}$$

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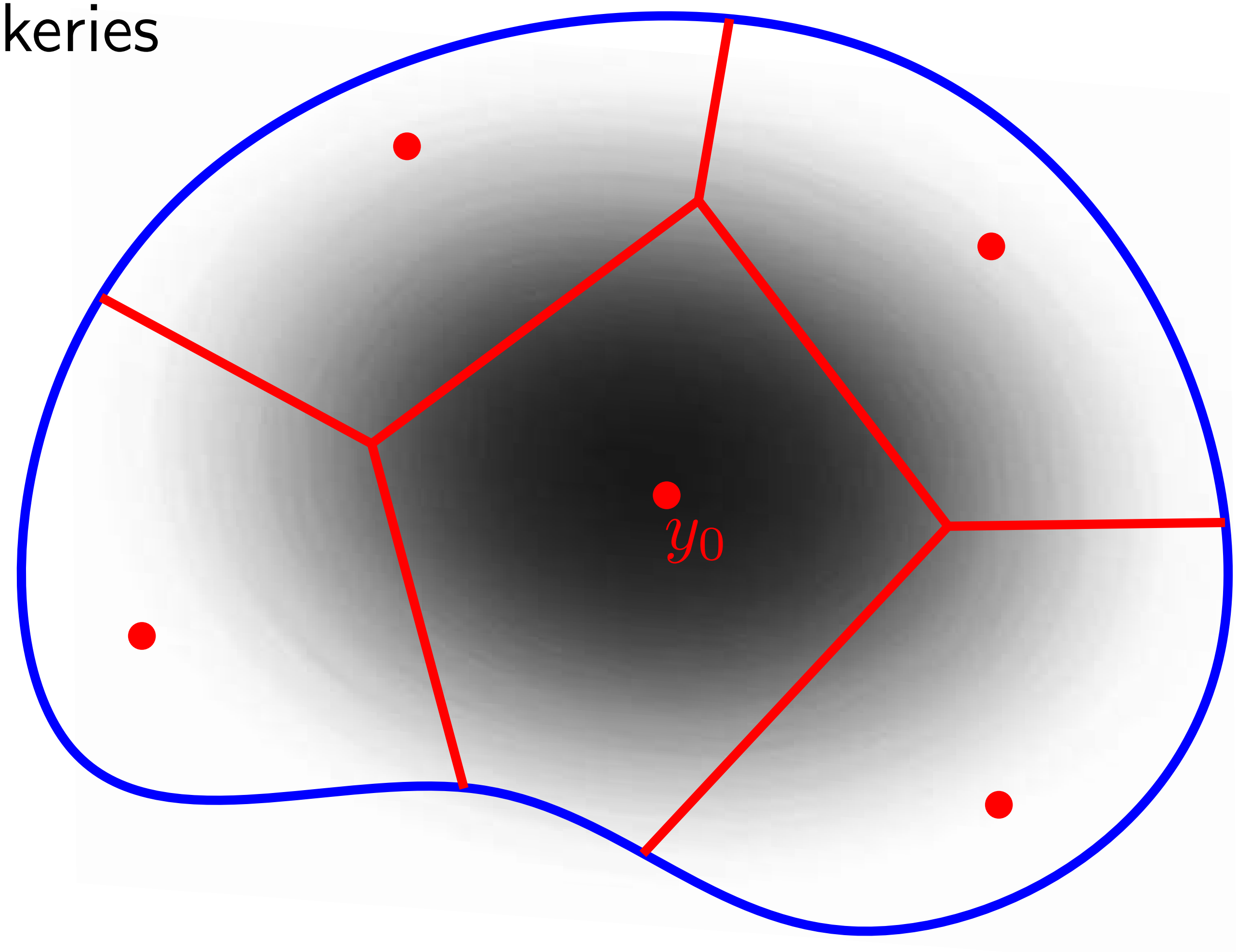

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Minimizes total distance walked ... **but** might exceed the capacity of bakery y_0!

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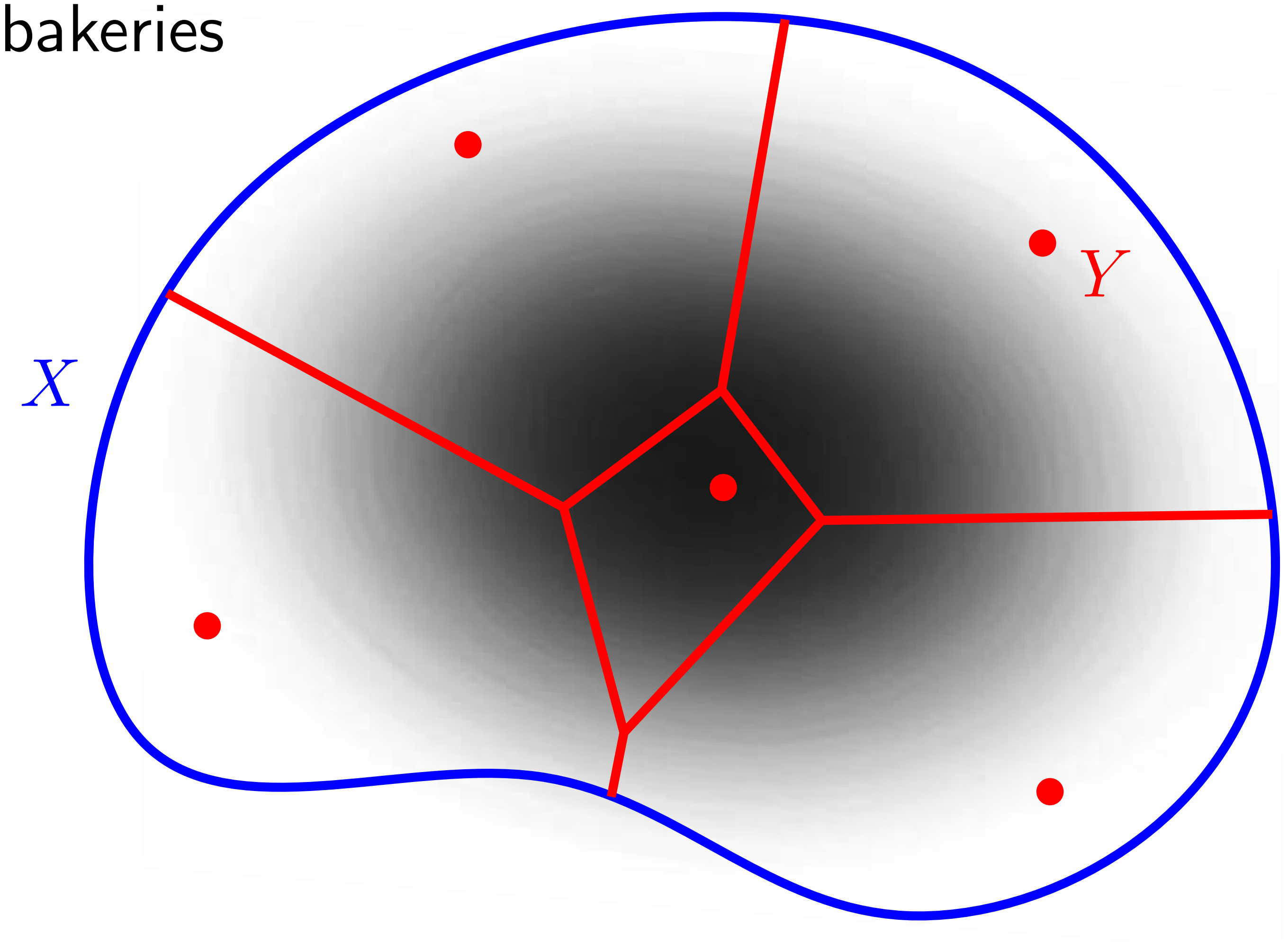

- If prices are given by $\psi : Y \to \mathbb{R}$, people make a compromise:

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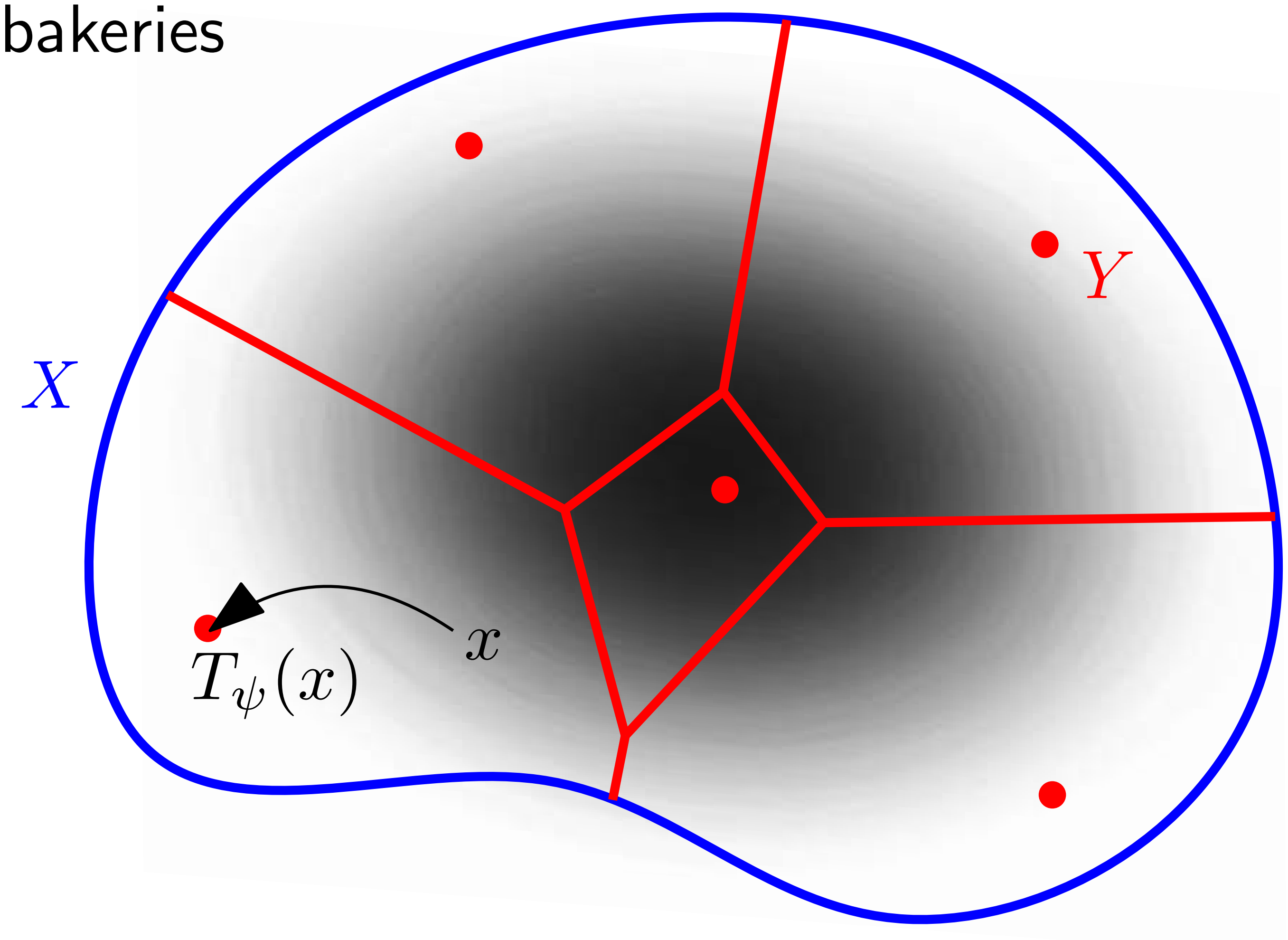

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Lemma: The Laguerre diagram induces an **optimal transport** between ρ and
$$\nu_\psi := \sum_{y \in Y} \rho(\mathrm{Lag}_y(\psi))\delta_y$$

Semi-discrete OT as Concave Maximization

Theorem: Finding an **optimal transport** between ρ and $\nu = \sum_Y \nu_y \delta_y$

$\iff$ finding **prices** ψ on Y such that $\nu_\psi = \nu$ [Gangbo McCann '96]

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- Coordinate-wise increments $\mathrm{O}(\frac{N^3}{\varepsilon}\log(N))$. [Oliker–Prussner '99]

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$\iff$ maximizing the **concave** function Φ [Aurenhammer, Hoffman, Aronov '98]

$$\Phi(\psi) := \sum_y \int_{\mathrm{Lag}_y(\psi)} [c(x,y) + \psi(y)] \,\mathrm{d}\,\rho(x) - \sum_y \psi(y) \nu_y$$

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  [M. 11], [de Goes *et al* 12], [Lévy 15]

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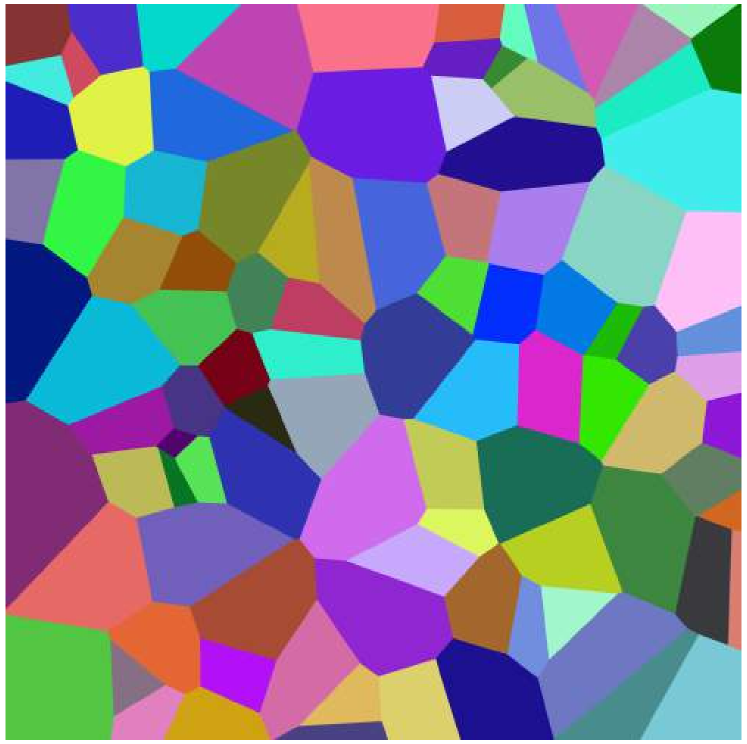
- First variational approaches, without convergence analysis [M. 11], [de Goes *et al* 12], [Lévy 15]
- Damped Newton's algorithm, with global linear convergence, under (rather) general assumptions on ρ and c. [Kitagawa, M., Thibert 16]

Numerical example 1

Source: $\rho =$ uniform on $[0,1]^2$,

Target: $\nu = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{y_i}$ with y_i uniform i.i.d. in $[0,1]^2$

Voronoi diagram



$$\psi_0 = 0$$

$$\varepsilon_0 \simeq 0.05$$

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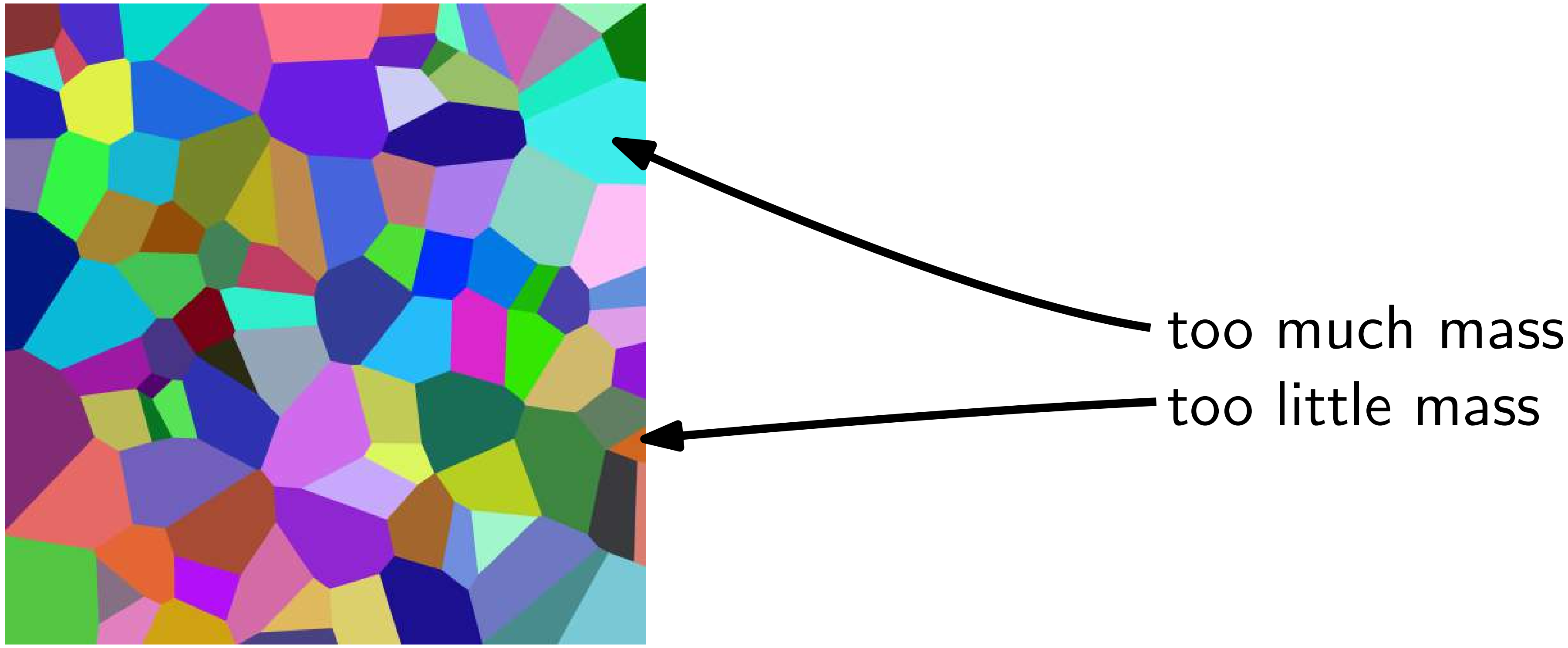

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Where $\varepsilon_k := \sum_i |\rho(\mathrm{Lag}_i(\psi_k)) - \frac{1}{N}|$ is the amount of misallocated mass.

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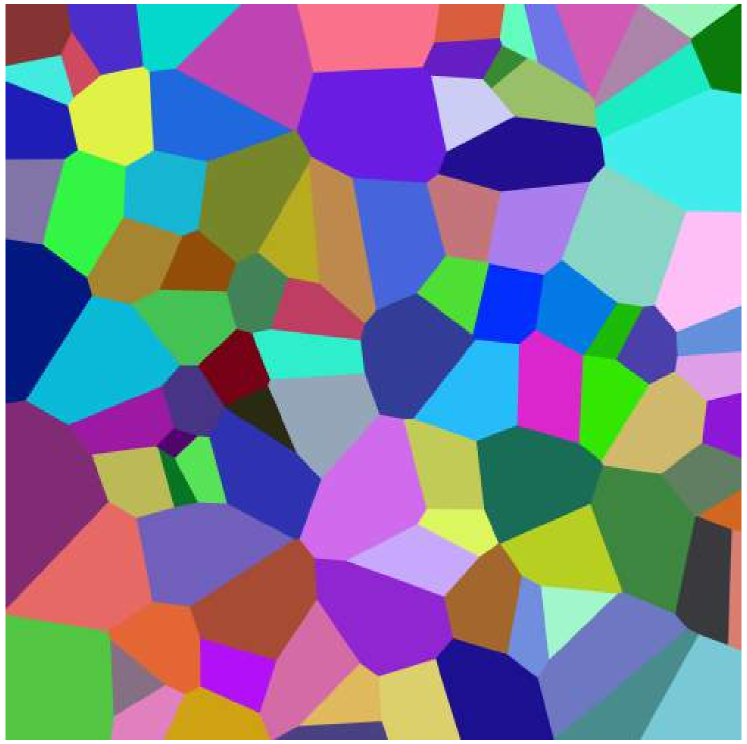

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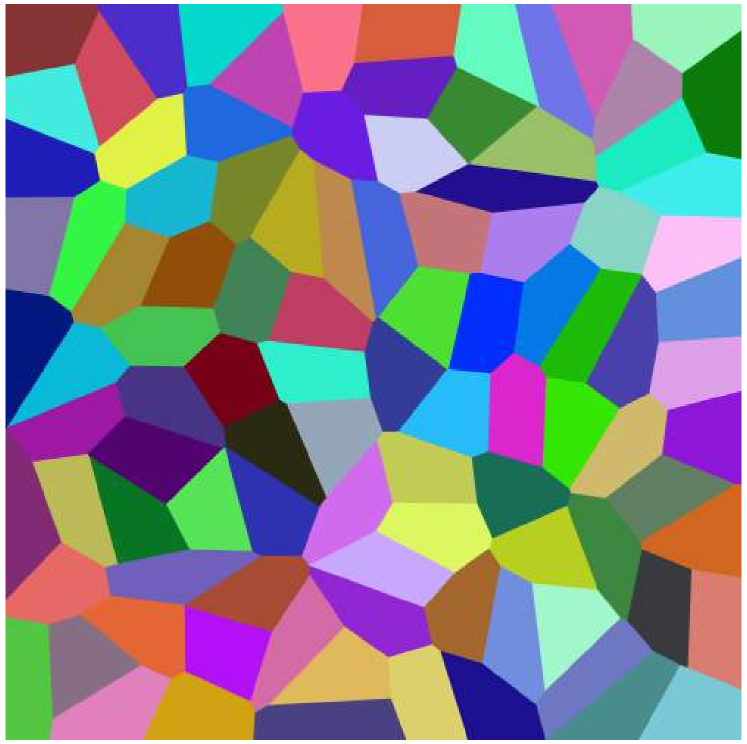
Laguerre diagram



$\psi_1 = \mathrm{Newt}(\psi_0)$

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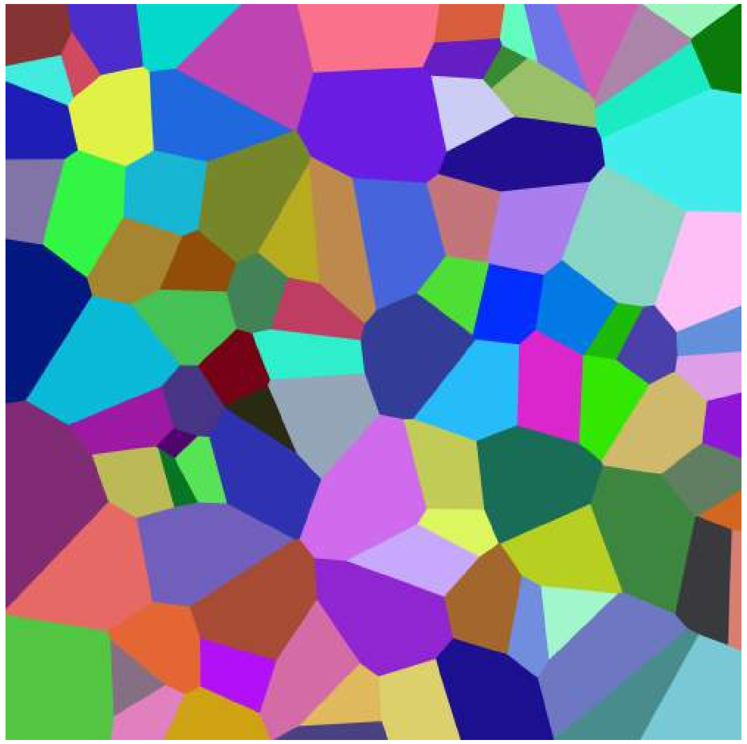

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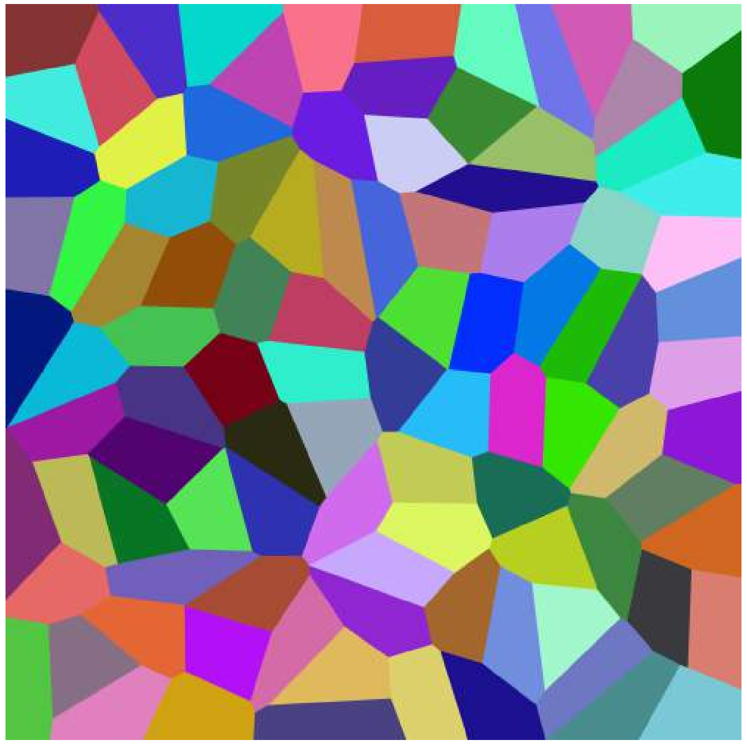

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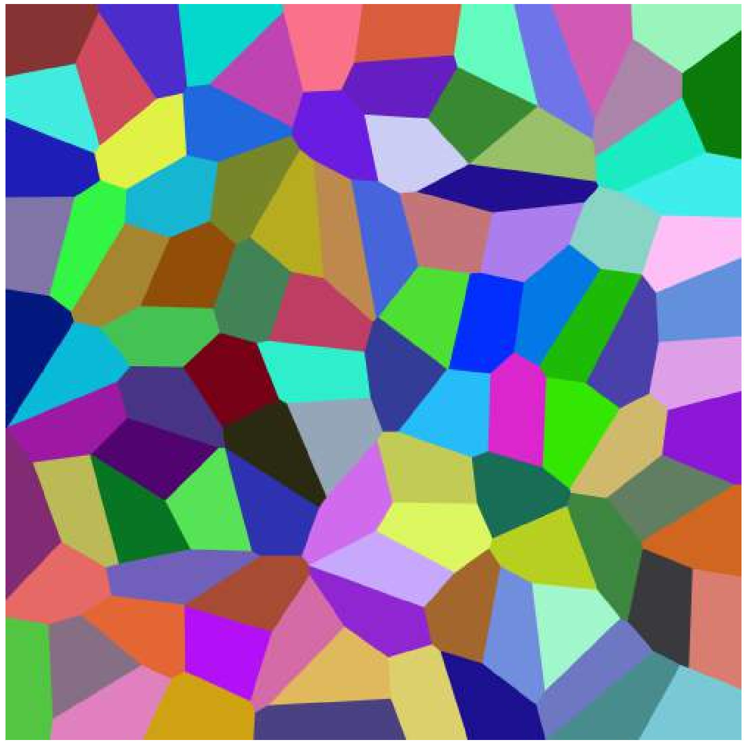

$\psi_2 = \mathrm{Newt}(\psi_1)$

$\varepsilon_2 \simeq 10^{-9}$

Where $\varepsilon_k := \sum_i |\rho(\mathrm{Lag}_i(\psi_k)) - \frac{1}{N}|$ is the amount of misallocated mass.

Numerical example 2

Source: $\rho =$ uniform on $[0,1]^2$, **Cost:** $c(x,y) = \|x-y\|^2$

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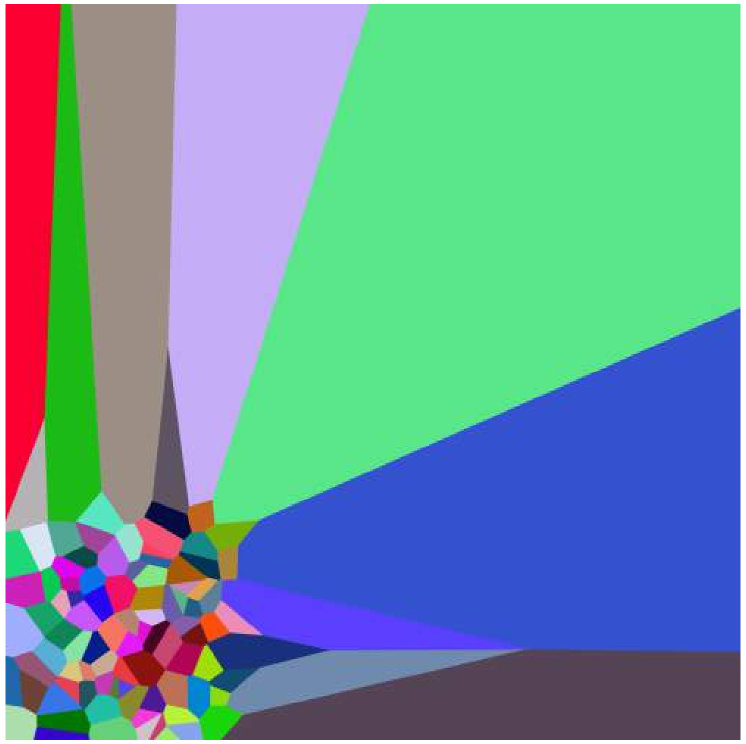
Voronoi diagram



$\psi_0 = 0$

$\varepsilon_0 \simeq 0.48$

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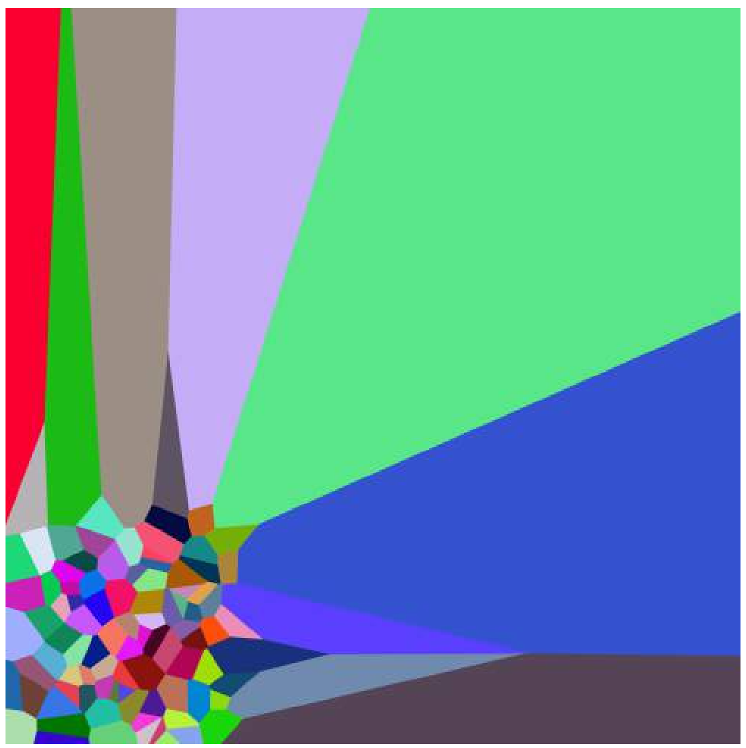

$\psi_0 = 0$

$\varepsilon_0 \simeq 0.48$

Laguerre diagram

$\psi_1 = \mathrm{Newt}(\psi_0)$

$\varepsilon_1 \simeq 0.024$

NB: The points do **not** move.

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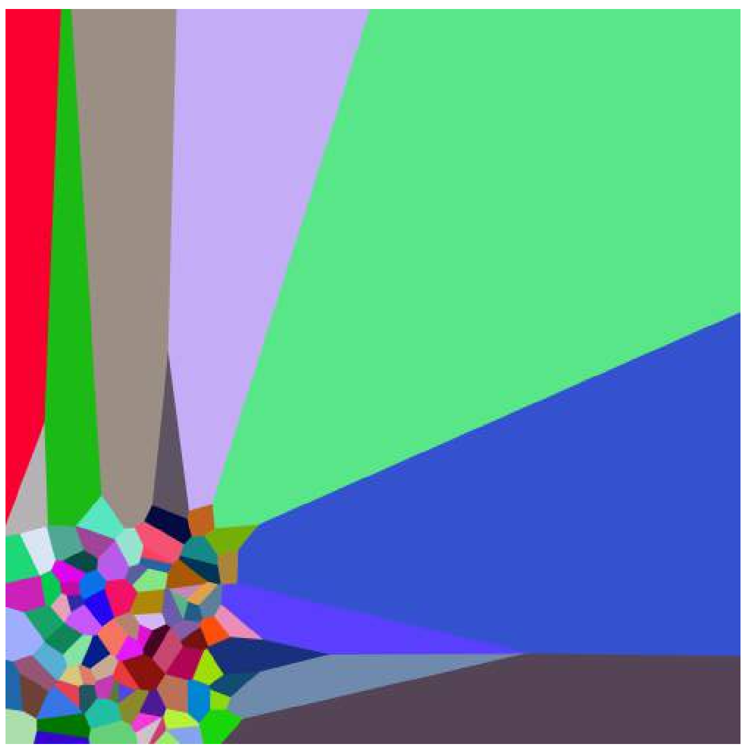

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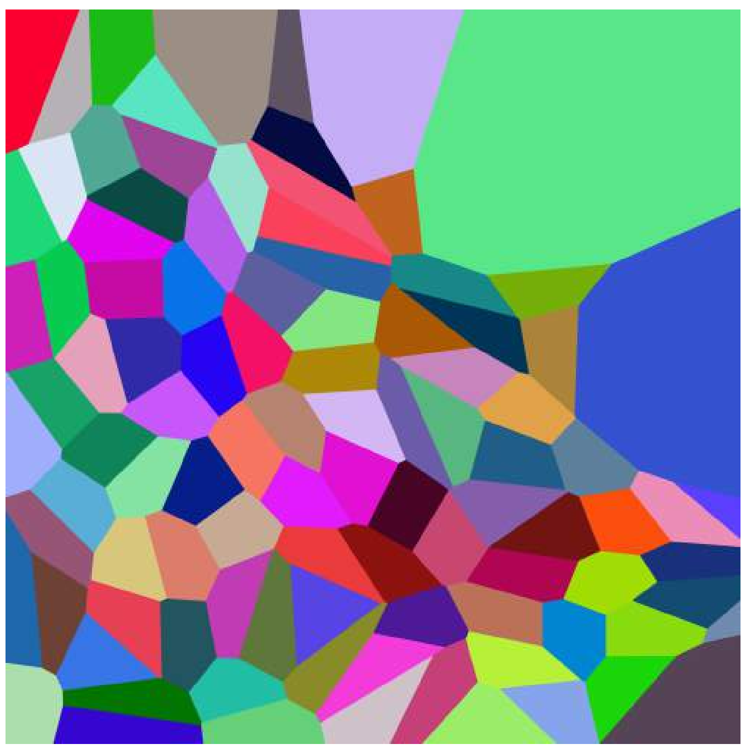

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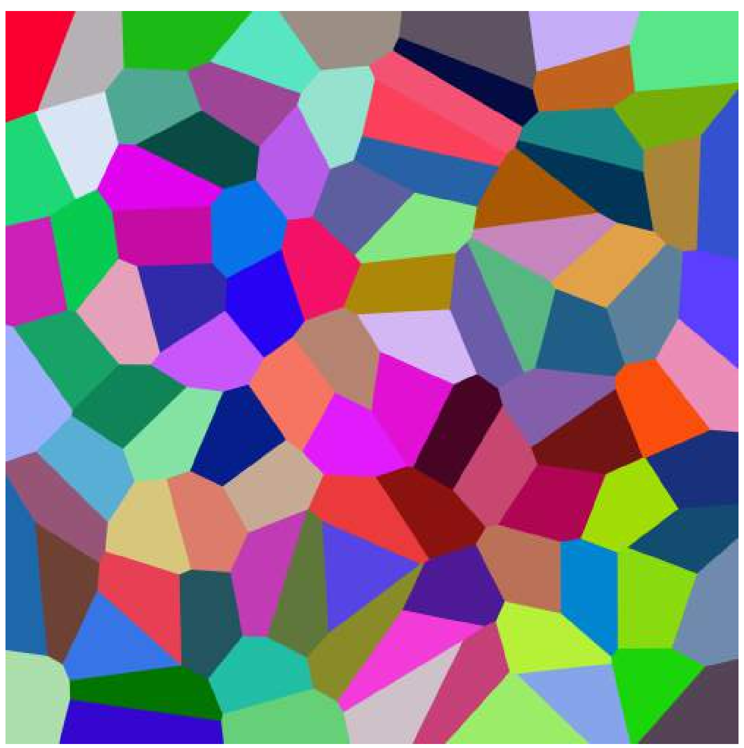
Laguerre diagram



$\psi_2 = \mathrm{Newt}(\psi_1)$

$\varepsilon_2 \simeq 10^{-6}$

NB: The points do **not** move.

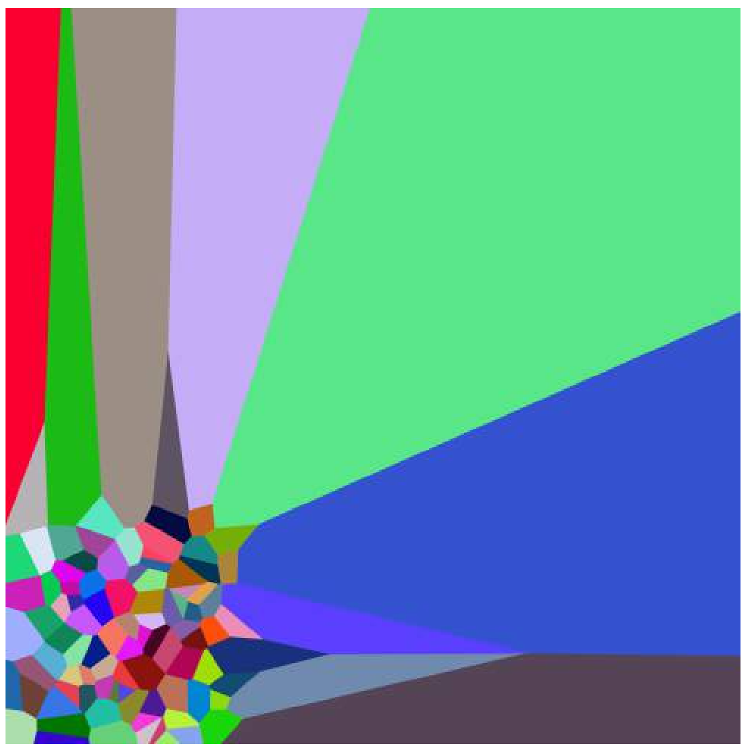
Numerical example 2

Source: $\rho =$ uniform on $[0,1]^2$,

Cost: $c(x,y) = \|x-y\|^2$

Target: $\nu = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{y_i}$ with y_i uniform i.i.d. in $[0,\frac{1}{3}]^2$

Voronoi diagram



$\psi_0 = 0$

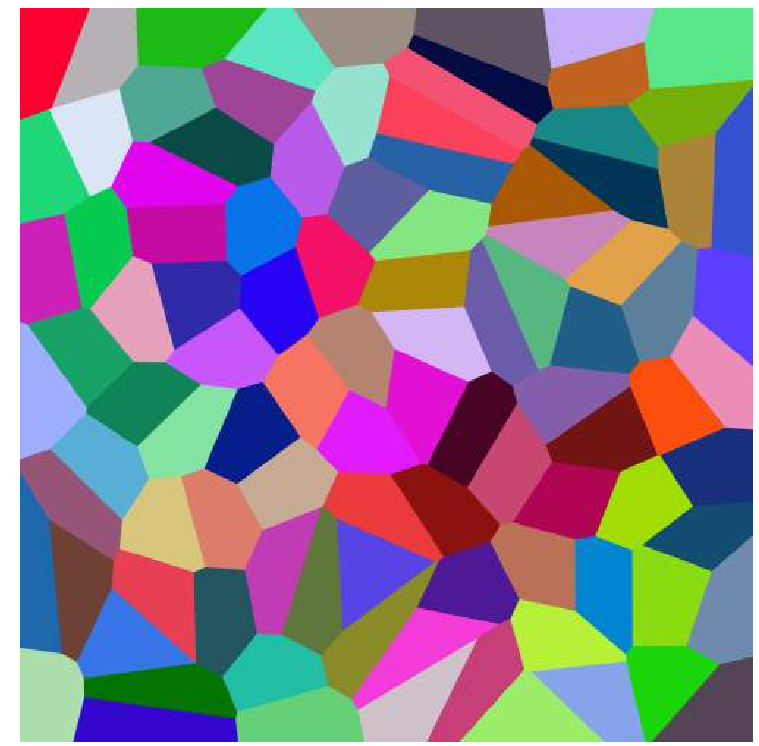
$\varepsilon_0 \simeq 0.48$

Laguerre diagram

$\psi_1 = \mathrm{Newt}(\psi_0)$

$\varepsilon_1 \simeq 0.024$

Laguerre diagram



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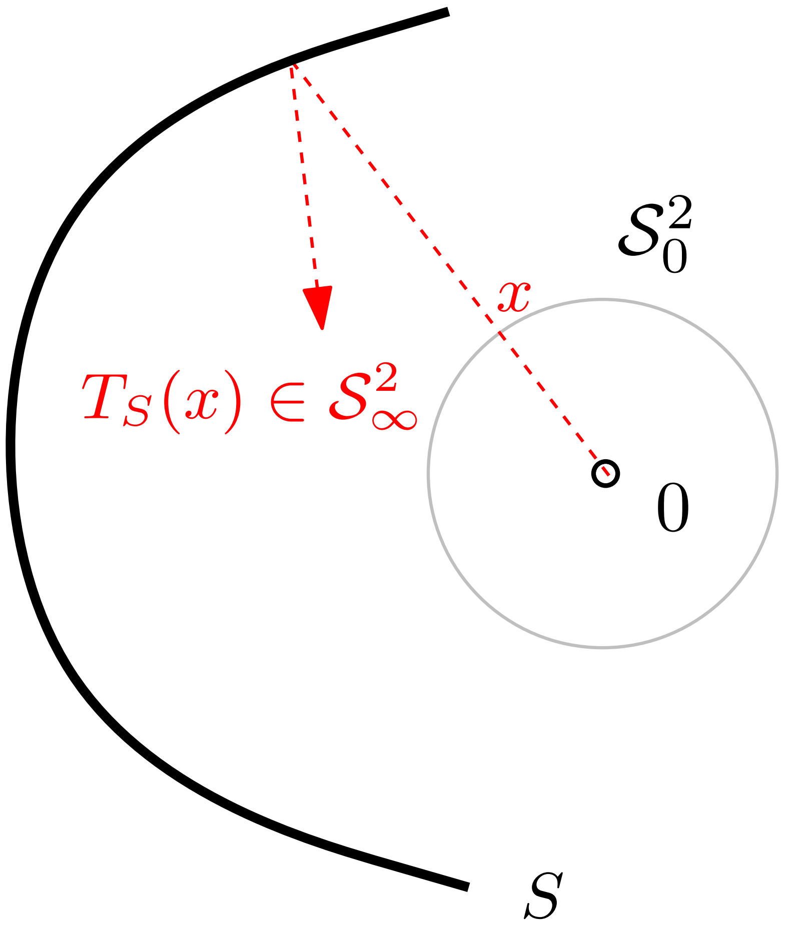
NB: The points do **not** move.

Laguerre diagrams are able to encode an actual *transport* of mass (large movement).

2. First application: non-imaging optics

Joint works with J. Kitawaga, P. Machado, J. Meyron and B. Thibert

(Point source) Inverse Reflector Problem



Forward problem:

point light source $:= \rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathcal{S}^2_0)$
surface S

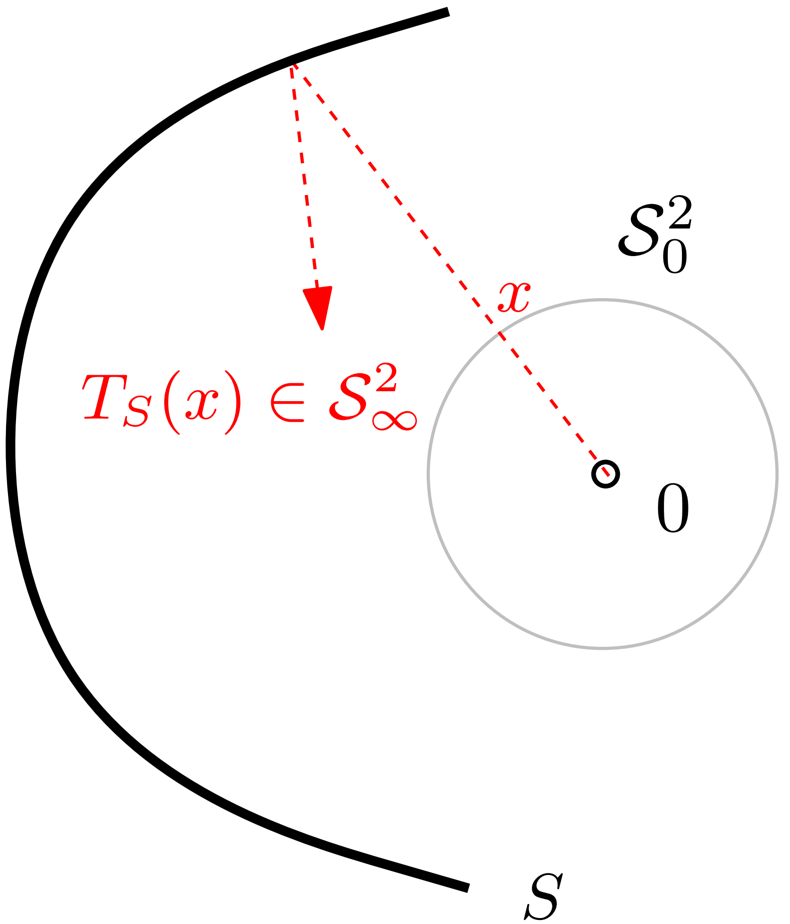
INPUT

raytracing

light distribution after reflection : $T_{S\#}\rho \in \mathrm{Prob}(\mathcal{S}^2_\infty)$

OUTPUT

(Point source) Inverse Reflector Problem



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surface S

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light distribution after reflection : $T_{S\#}\rho \in \mathrm{Prob}(\mathcal{S}_\infty^2)$

OUTPUT

Inverse problem:

source: $\rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathcal{S}_0^2)$
target: $\nu \in \mathrm{Prob}(\mathcal{S}_\infty^2)$

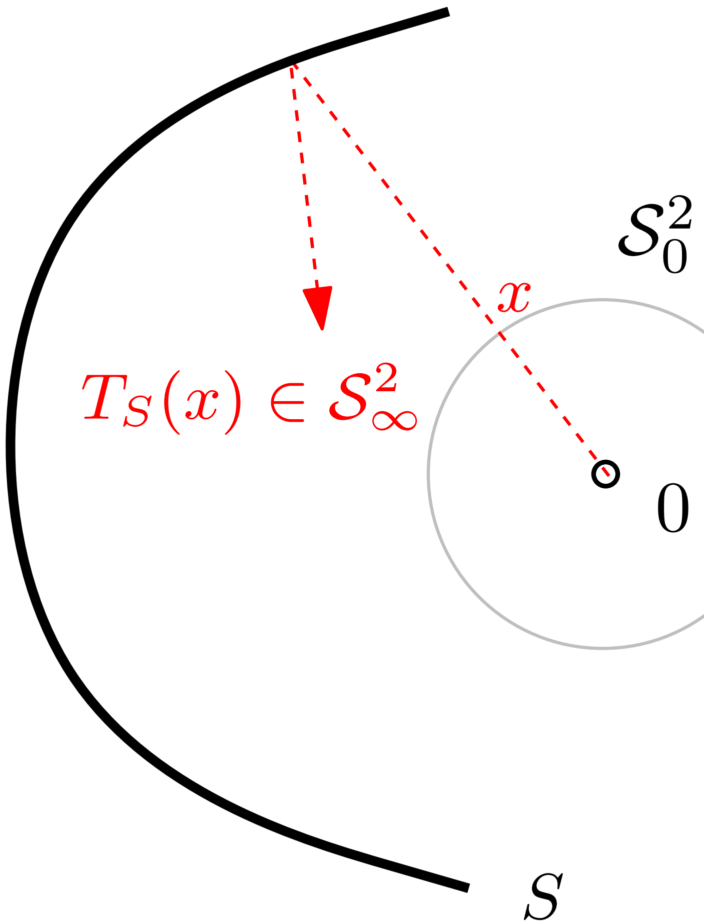
INPUT

??

surface S s.t. $T_{S\#}\rho = \nu$

OUTPUT

(Point source) Inverse Reflector Problem



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surface S

raytracing

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OUTPUT

$\longrightarrow$ Optical components for **car beams, public lighting, hydroponic agriculture**

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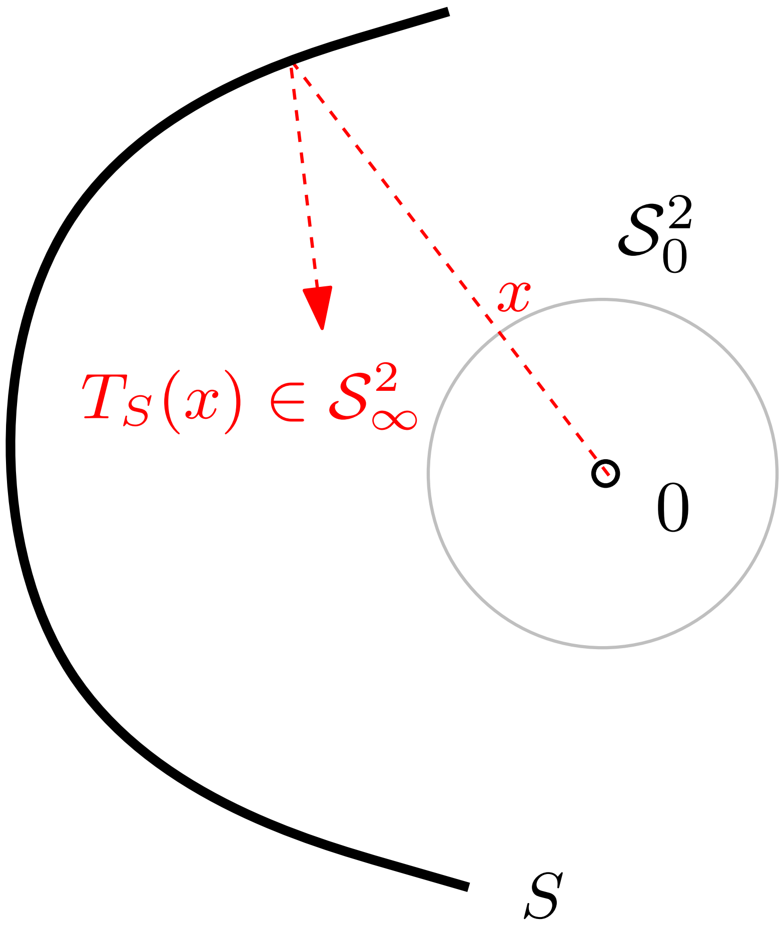

Forward problem:

$\boxed{\begin{array}{l}\text{point light source} := \rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathcal{S}_0^2) \\ \text{surface } S\end{array}}$ Input

$\Big\downarrow$ **raytracing**

$\boxed{\text{light distribution after reflection} : T_{S\#}\rho \in \mathrm{Prob}(\mathcal{S}_\infty^2)}$

Output

Inverse problem:

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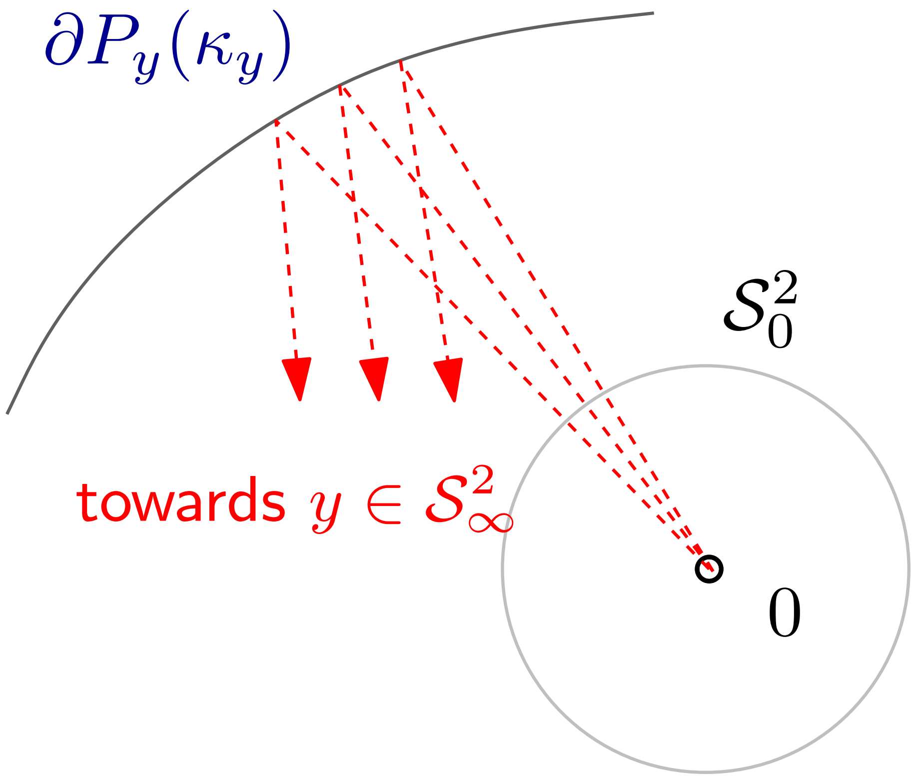
Output

$\longrightarrow$ Optical components for **car beams, public lighting, hydroponic agriculture**

$\longrightarrow$ Zoology of similar optics problems: collimated source, lenses, near field targed...

Semidiscrete Inverse Reflector Problem

Assume $\nu := \sum_{y \in Y} \nu_y \delta_y$, and let $P_y(\kappa_y) :=$ solid paraboloid of revolution with focal 0, direction y and focal distance κ_y



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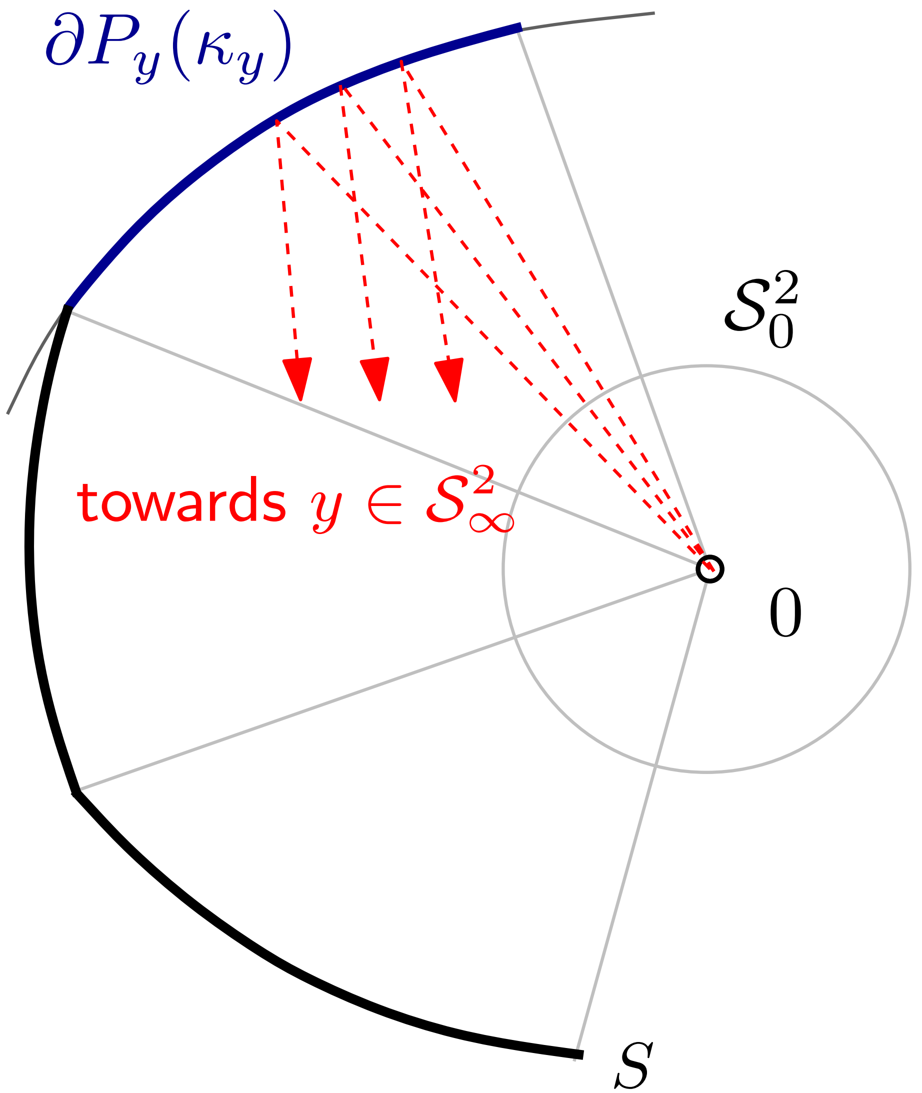
$S :=$ surface $= \partial \left(\cap_y P_y(\kappa_y) \right)$



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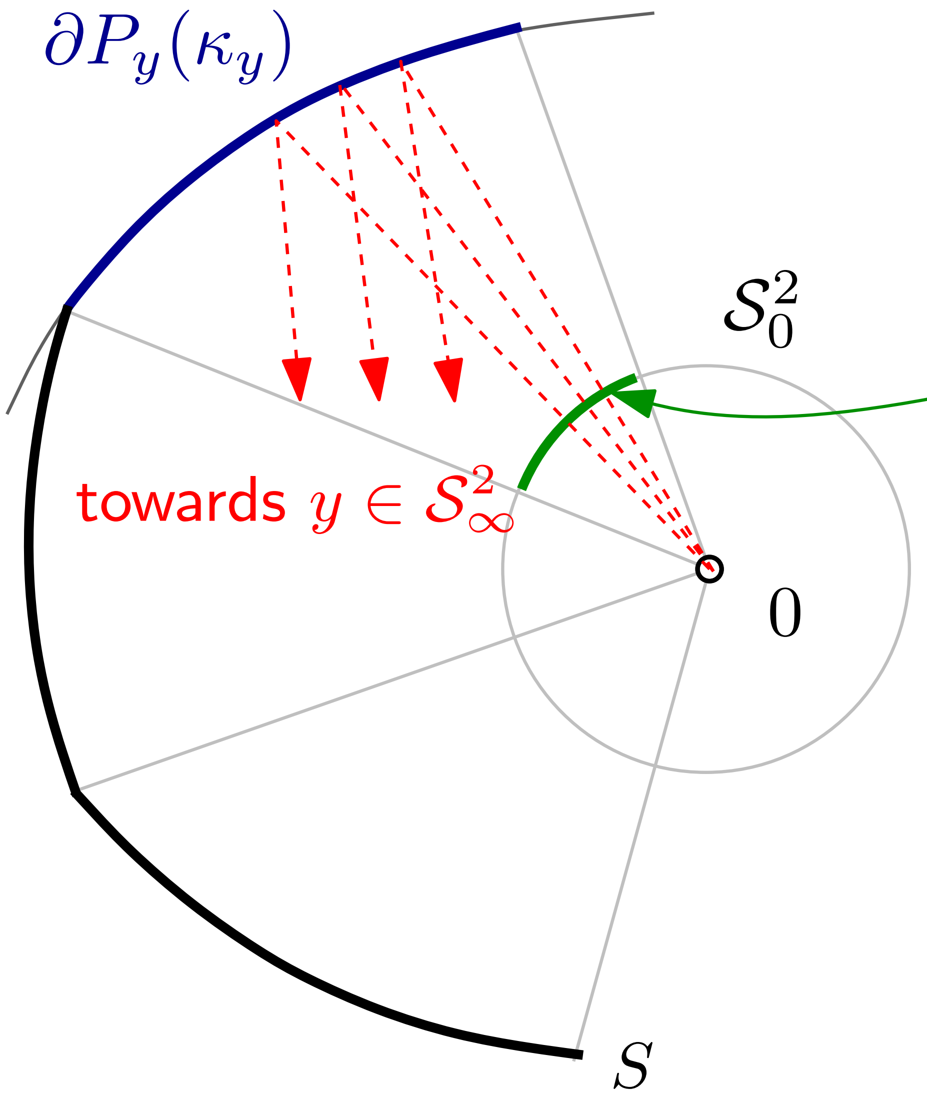

$\rho(V_y(\kappa)) =$ amount of light reflected towards $y \in \mathcal{S}_\infty^2$.

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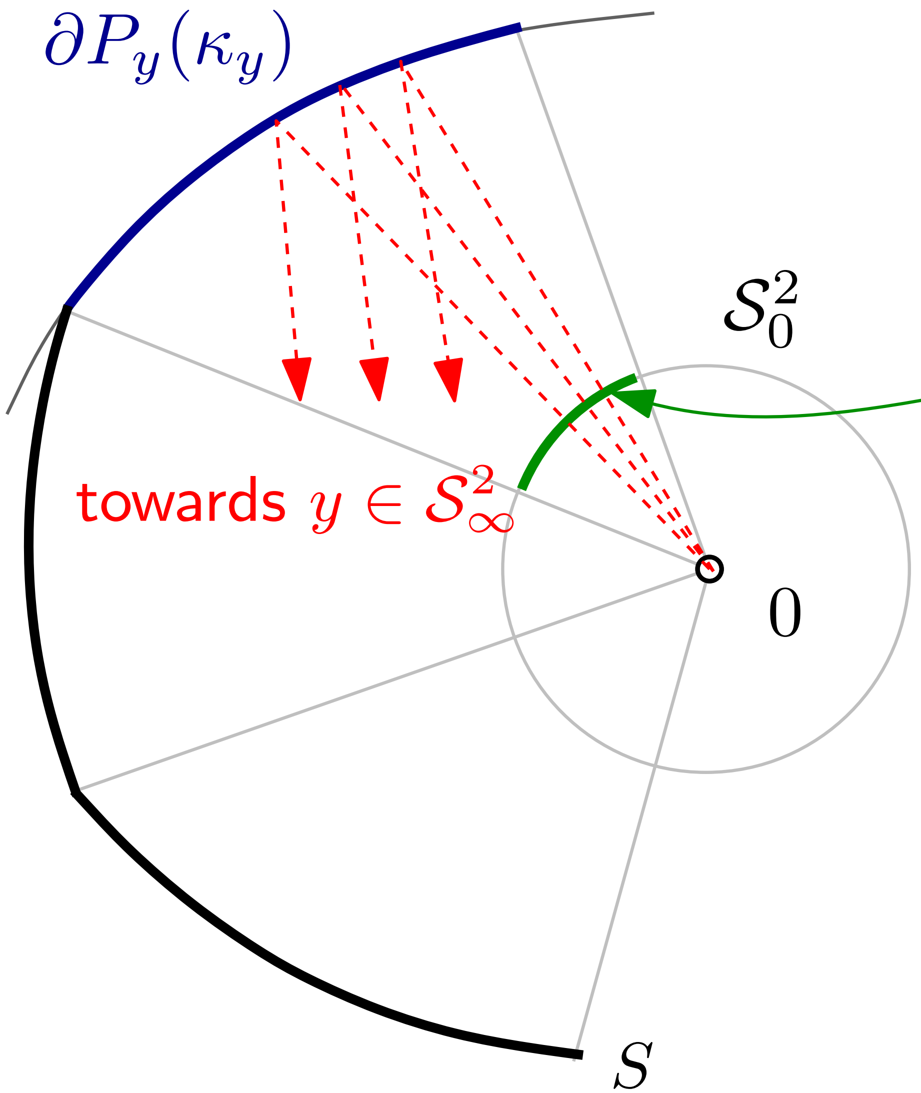

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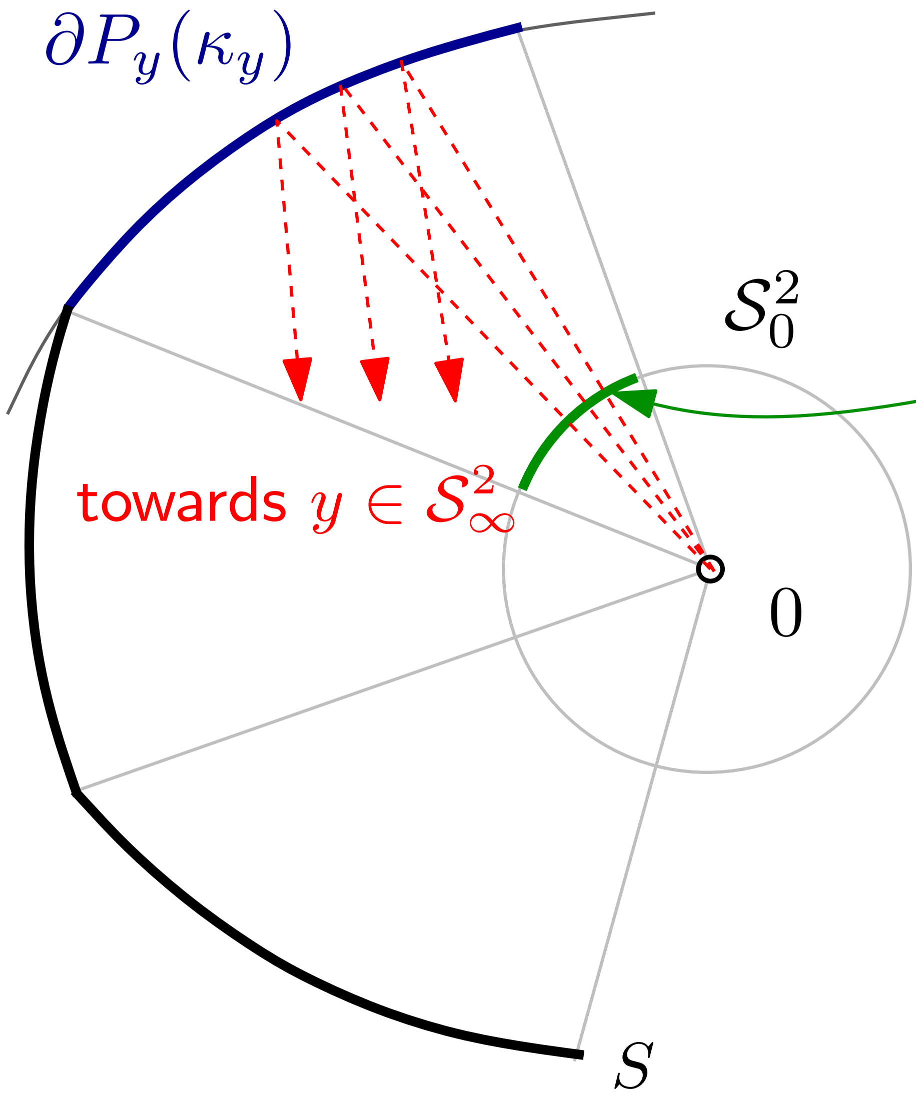

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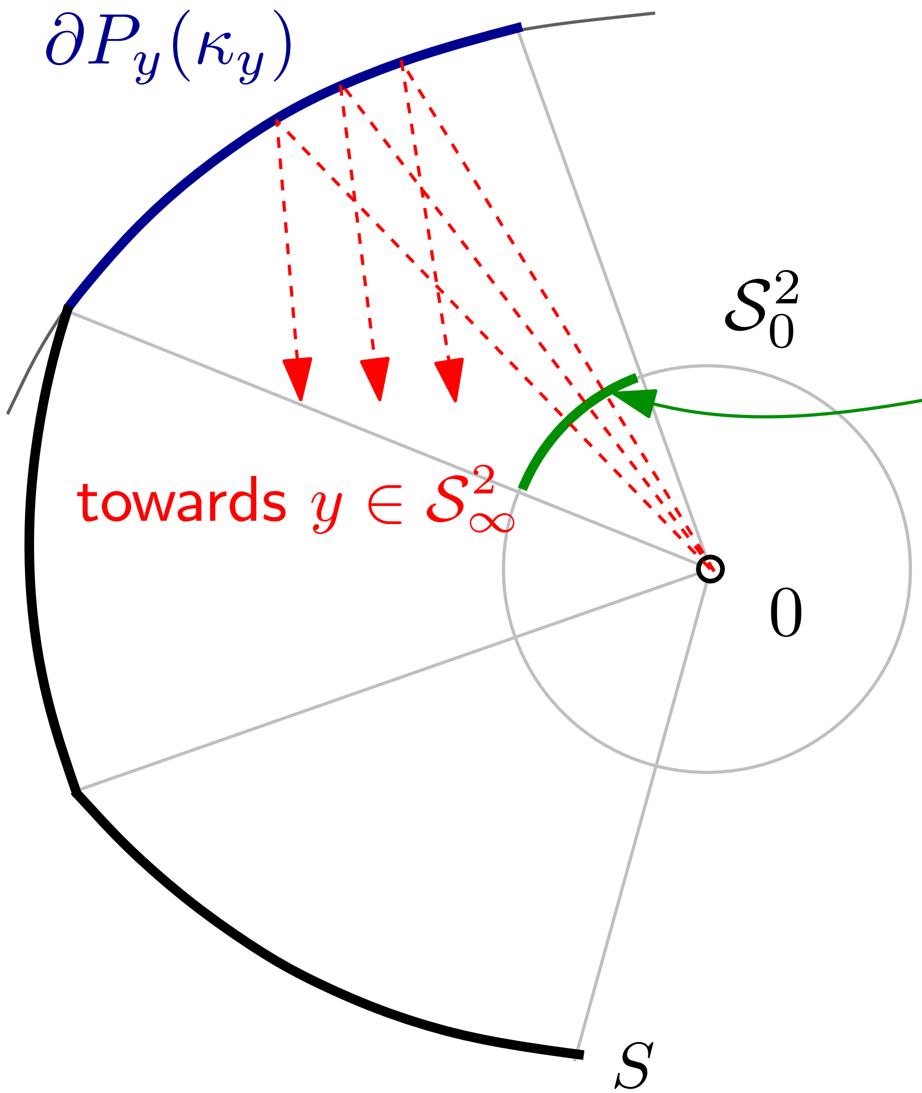

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$\longrightarrow$ Can be adjusted by playing with focal distance κ_y

$\longrightarrow$ Focal distance $\simeq$ prices in the economic example and, indeed, $V_y(\kappa)$ is a **Laguerre cell** !

$V_y(\kappa) = \mathrm{Lag}_y(\psi)$ for $\psi(y) = \log(\kappa_y)$ and $c(x,y) = -\log(1 - \langle x|y \rangle)$

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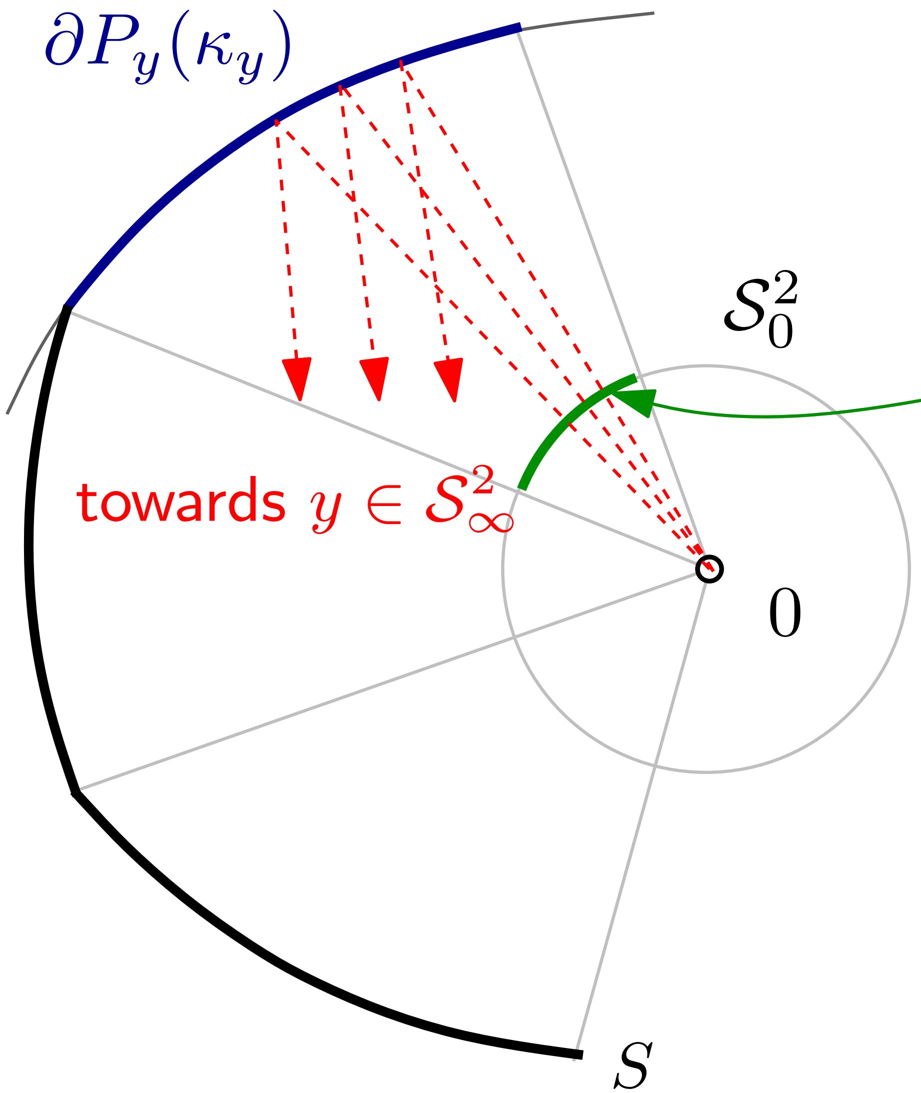

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Theorem: Semidiscrete Inverse Reflector Problem

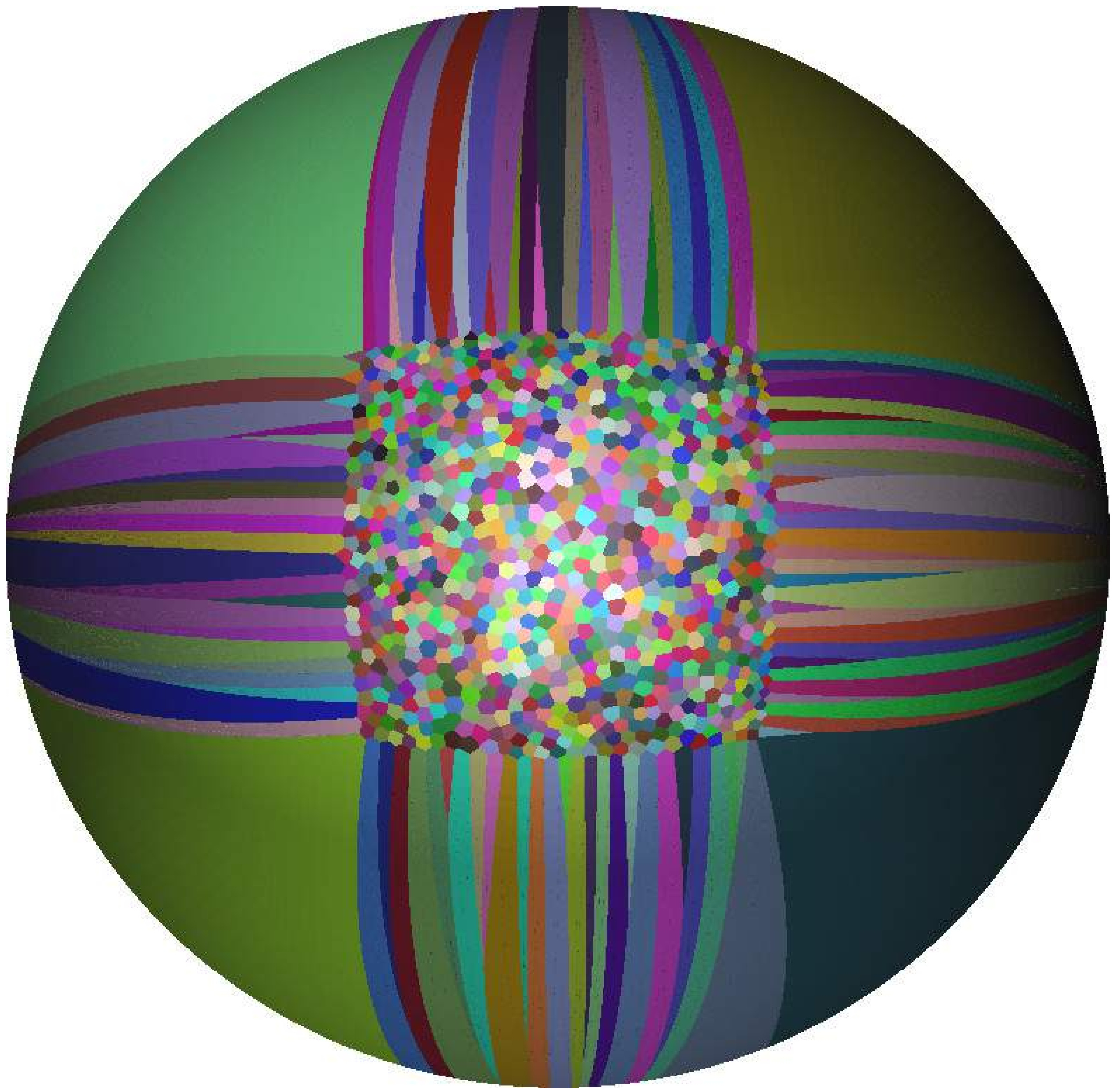
$\Longleftrightarrow$ semidiscrete OT problem on $\mathcal{S}^2$ for $c(x,y) = -\log(1 - \langle x|y \rangle)$

$\simeq$ [Glimm-Oliker '03] [Wang '04]

Numerics 1

$\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ = discretization of a picture of G. Monge. [Machado, M., Thibert '14]

ρ = uniform measure on half-sphere $X := \mathcal{S}^2_+$ $N = 1000$



drawing of $(\mathrm{Lag}_\psi(y_i))$ on $\mathcal{S}^2_+$ for $\psi = 0$

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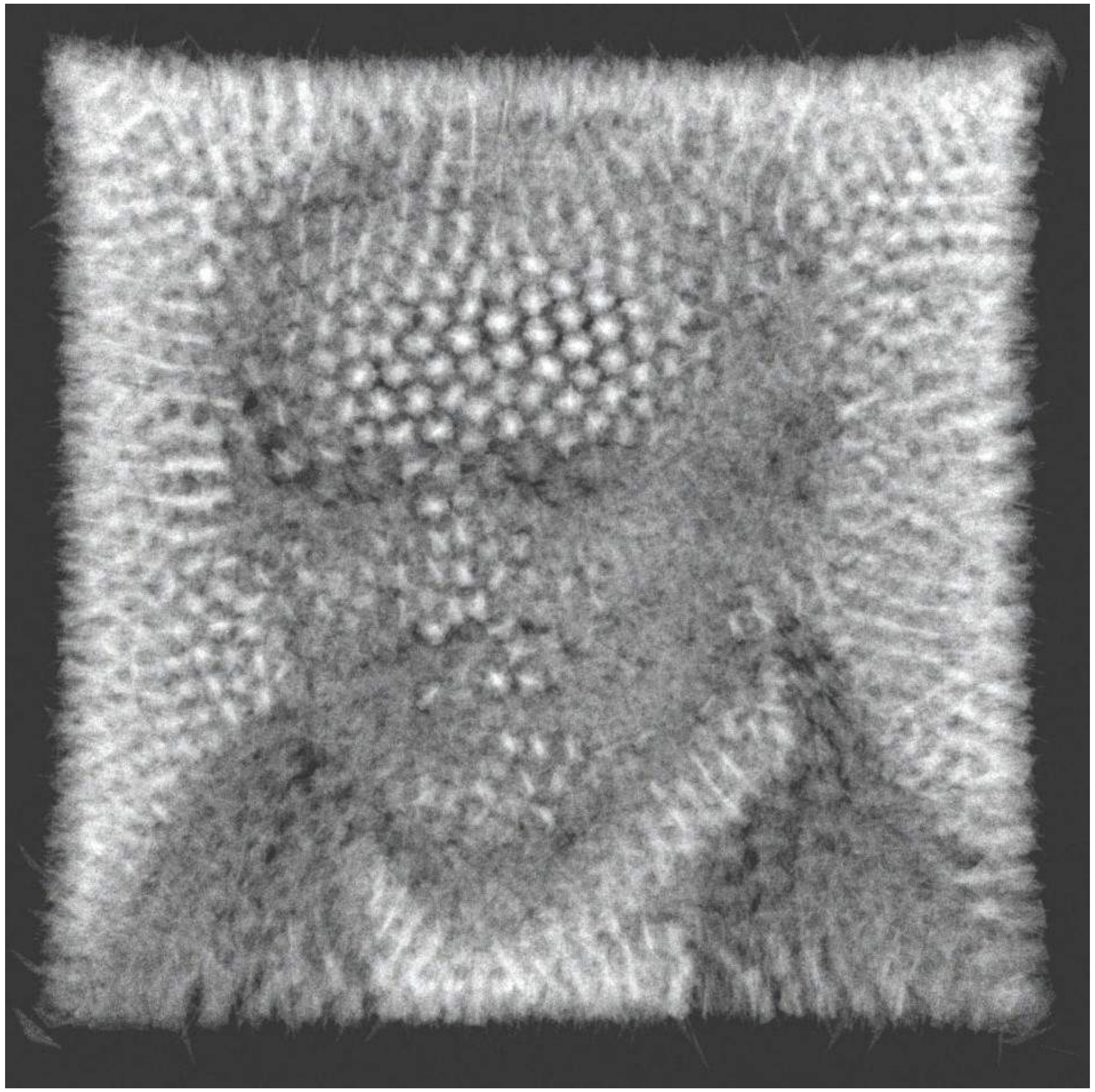

reflected image (using LuxRender)

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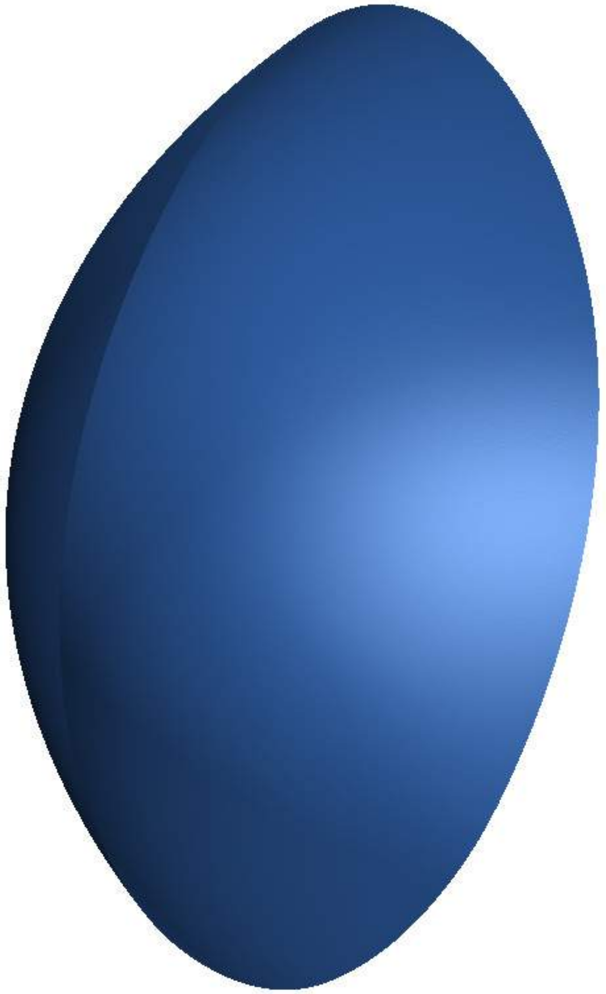

constructed reflector

Numerics 2

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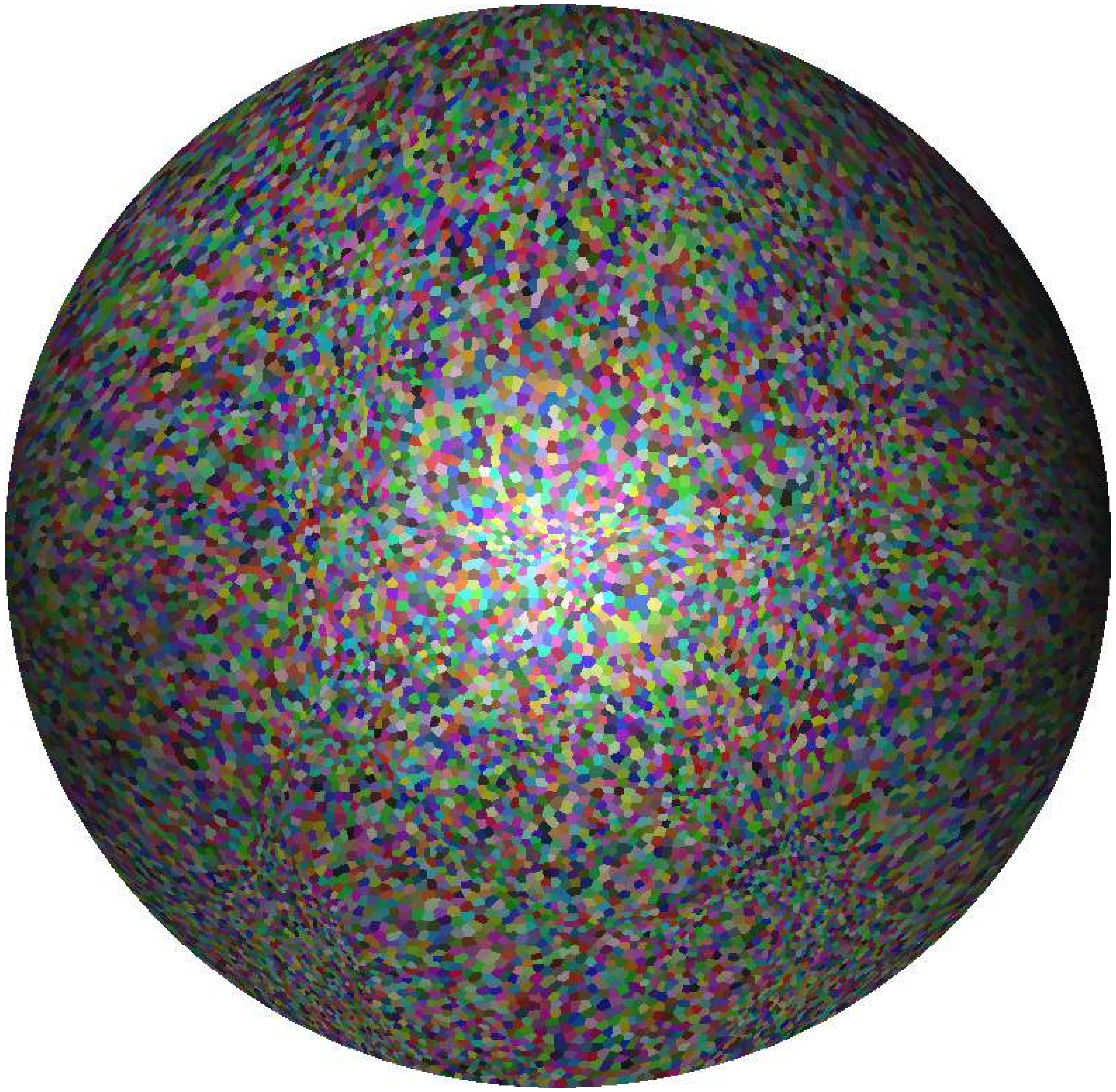

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Numerics 3

$\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ = discretization of the "Cameraman" picture [Meyron, M., Thibert '17]

ρ = non-uniform measure on half-sphere $X := \mathcal{S}^2_+$ $N = 250k$

Desired target ν

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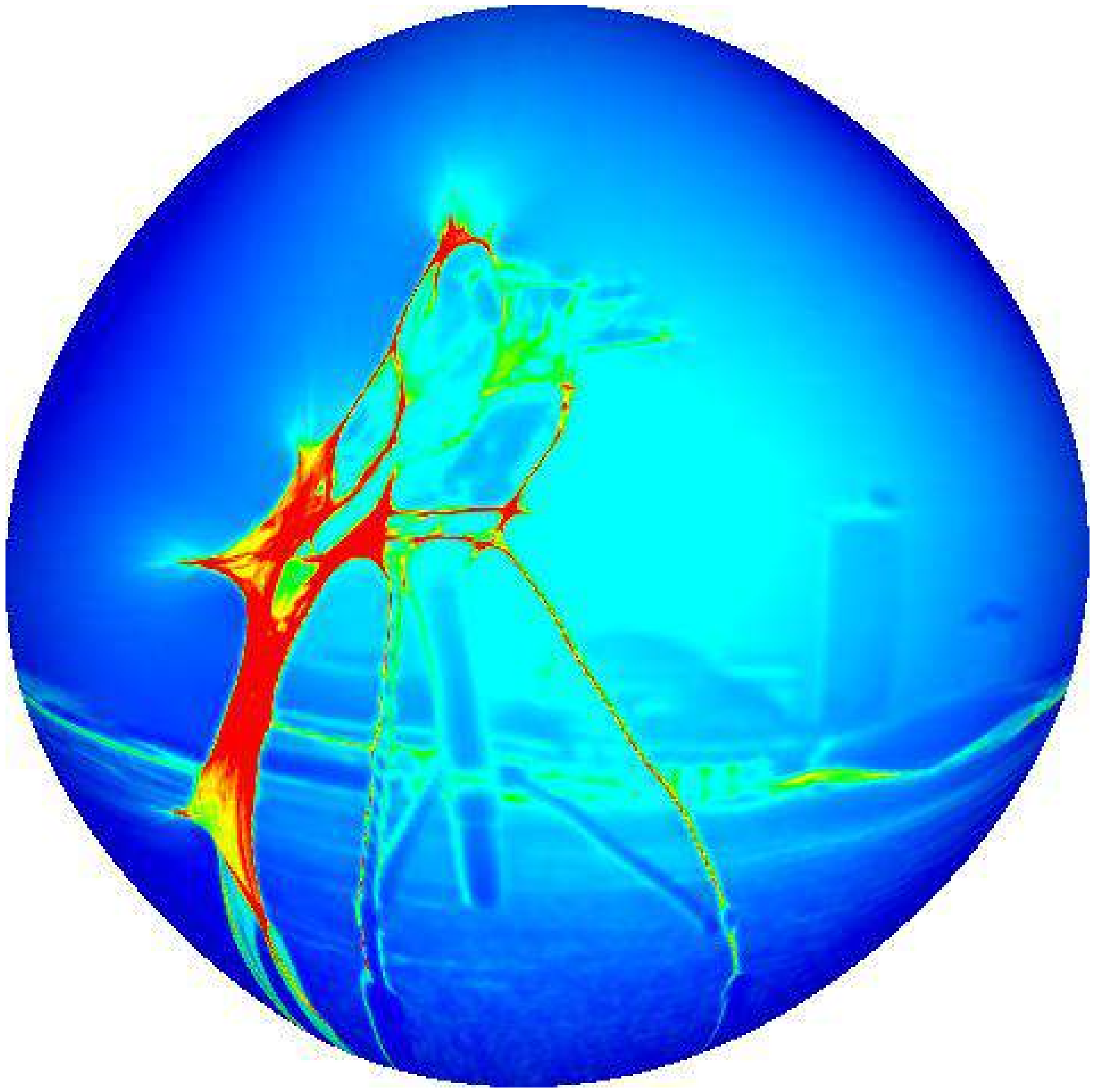

Constructed reflector
color = mean curvature

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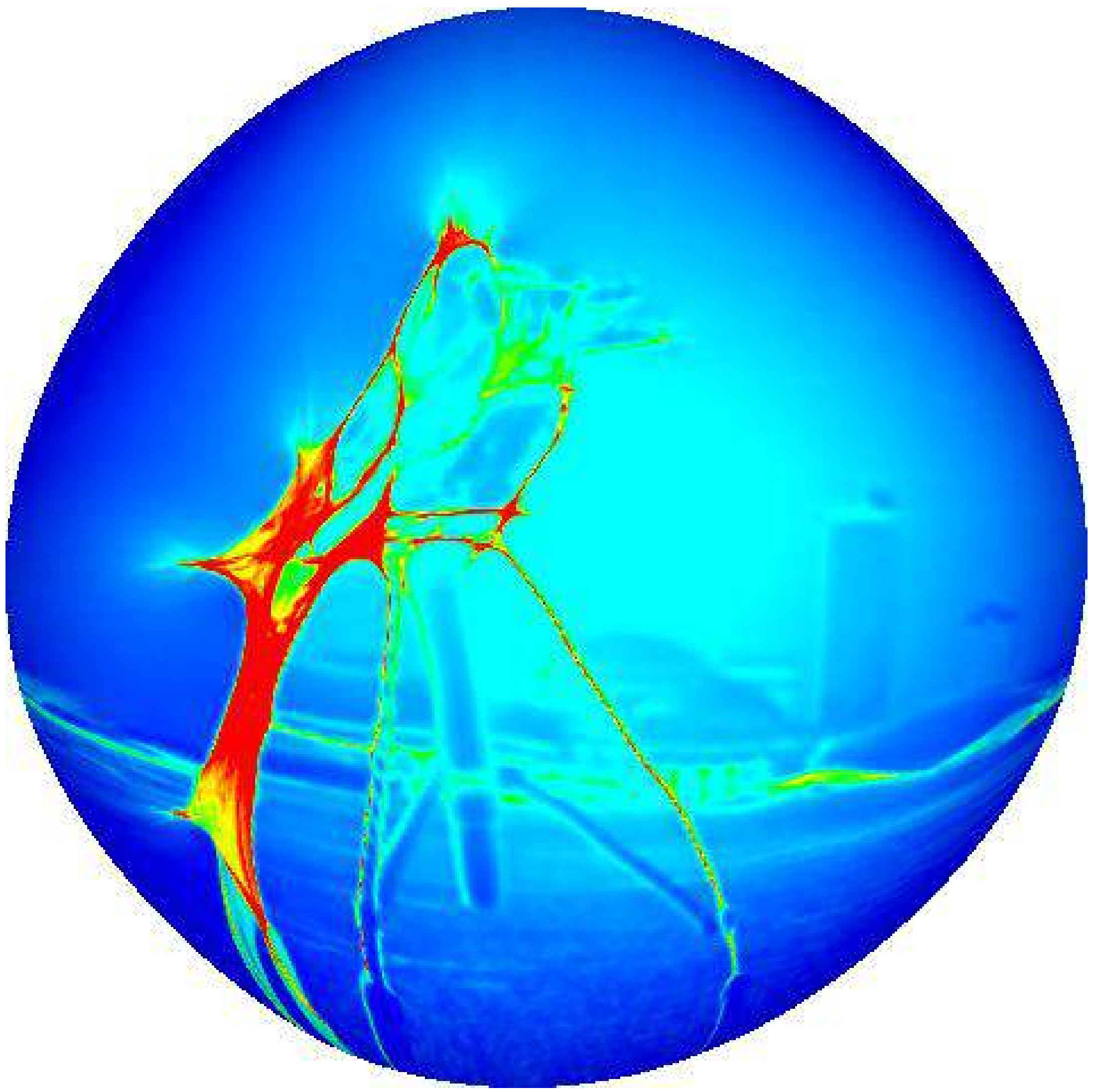

Constructed reflector
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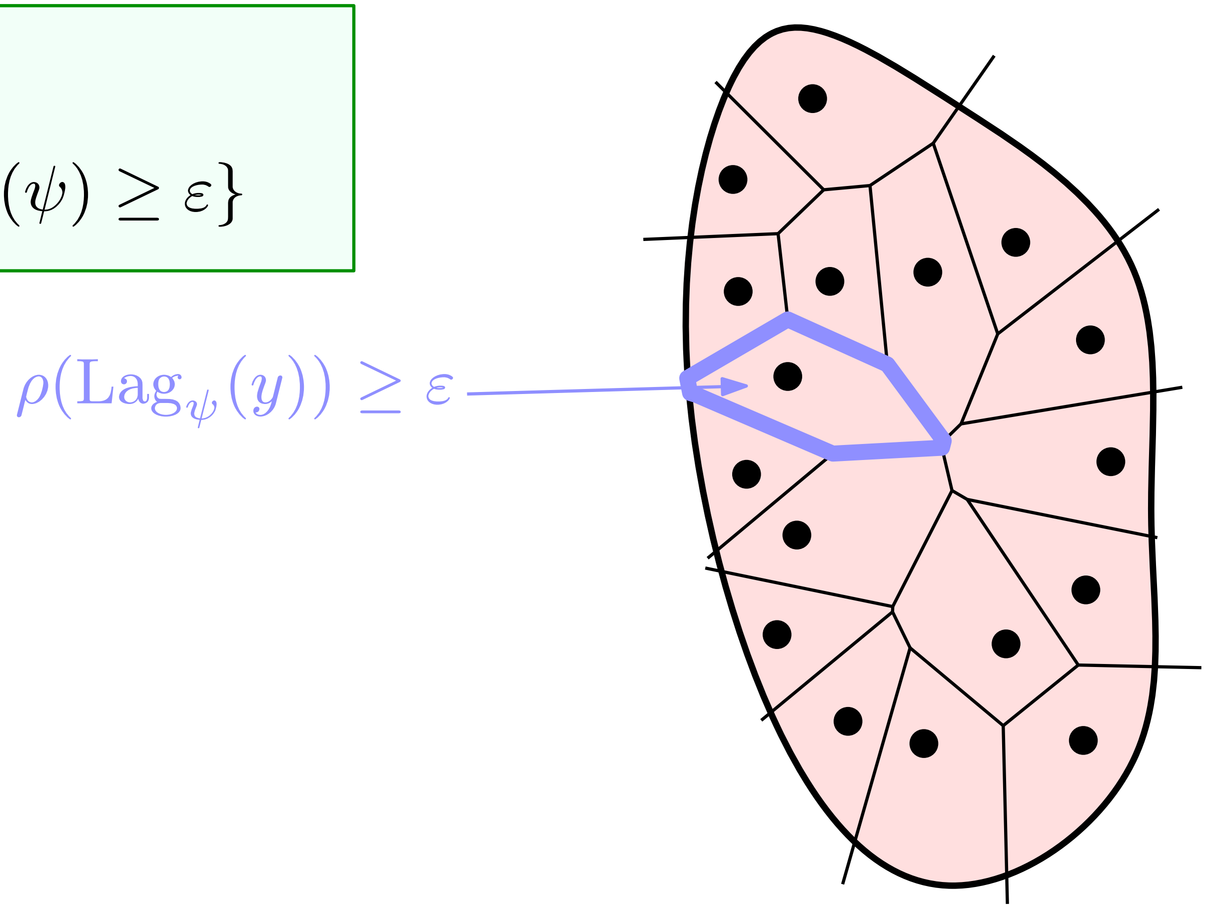
Resimulated image

Damped Newton's Algorithm

Recall: $G : \psi \in \mathbb{R}^Y \mapsto (\rho(\mathrm{Lag}_y(\psi)))_{y \in Y} \mathbb{R}^Y$.

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^Y \mid \forall y \in Y, G_y(\psi) \geq \varepsilon\}$



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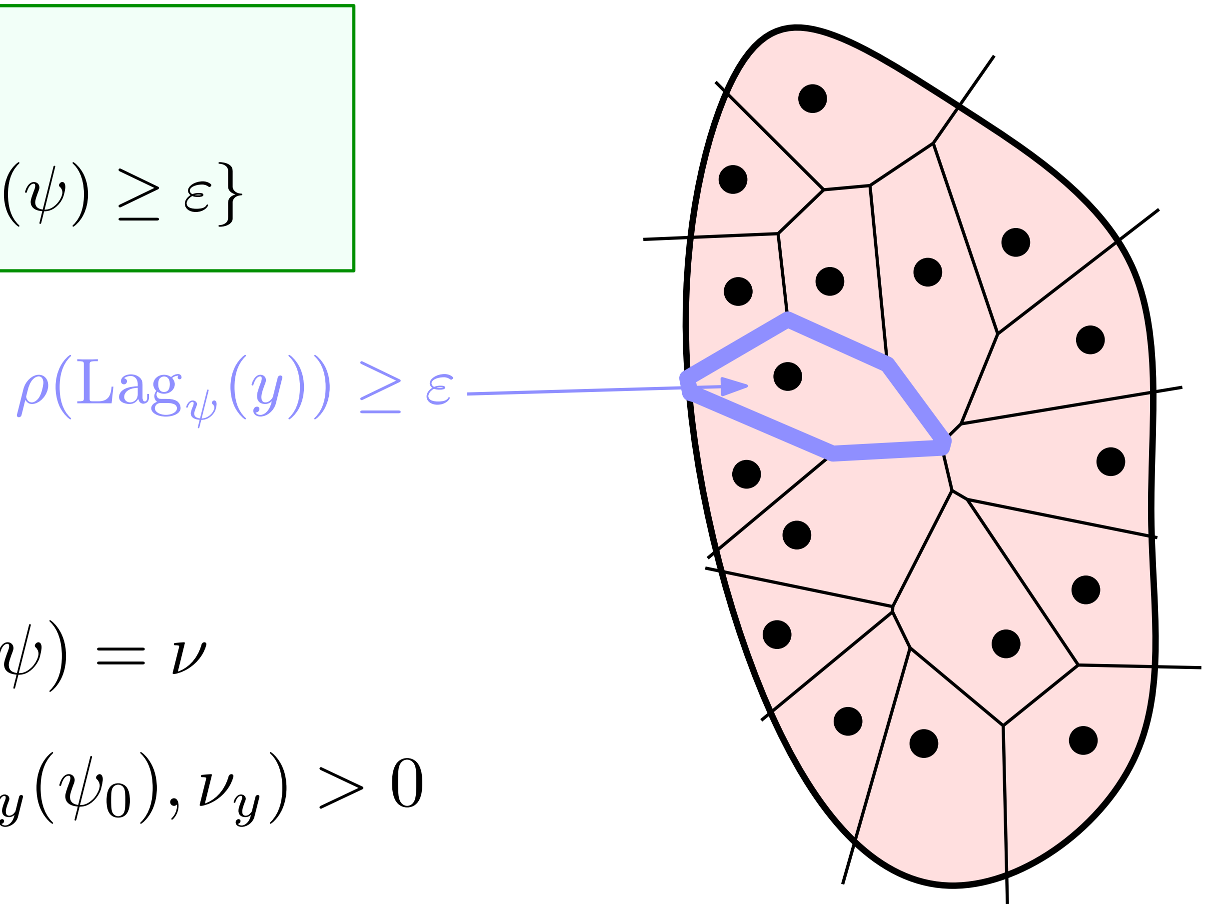

Damped Newton algorithm: for solving $G(\psi) = \nu$

Input: $\psi_0 \in Y^{\mathbb{R}}$ s.t. $\varepsilon := \frac{1}{2} \min_{y \in Y} \min(G_y(\psi_0), \nu_y) > 0$

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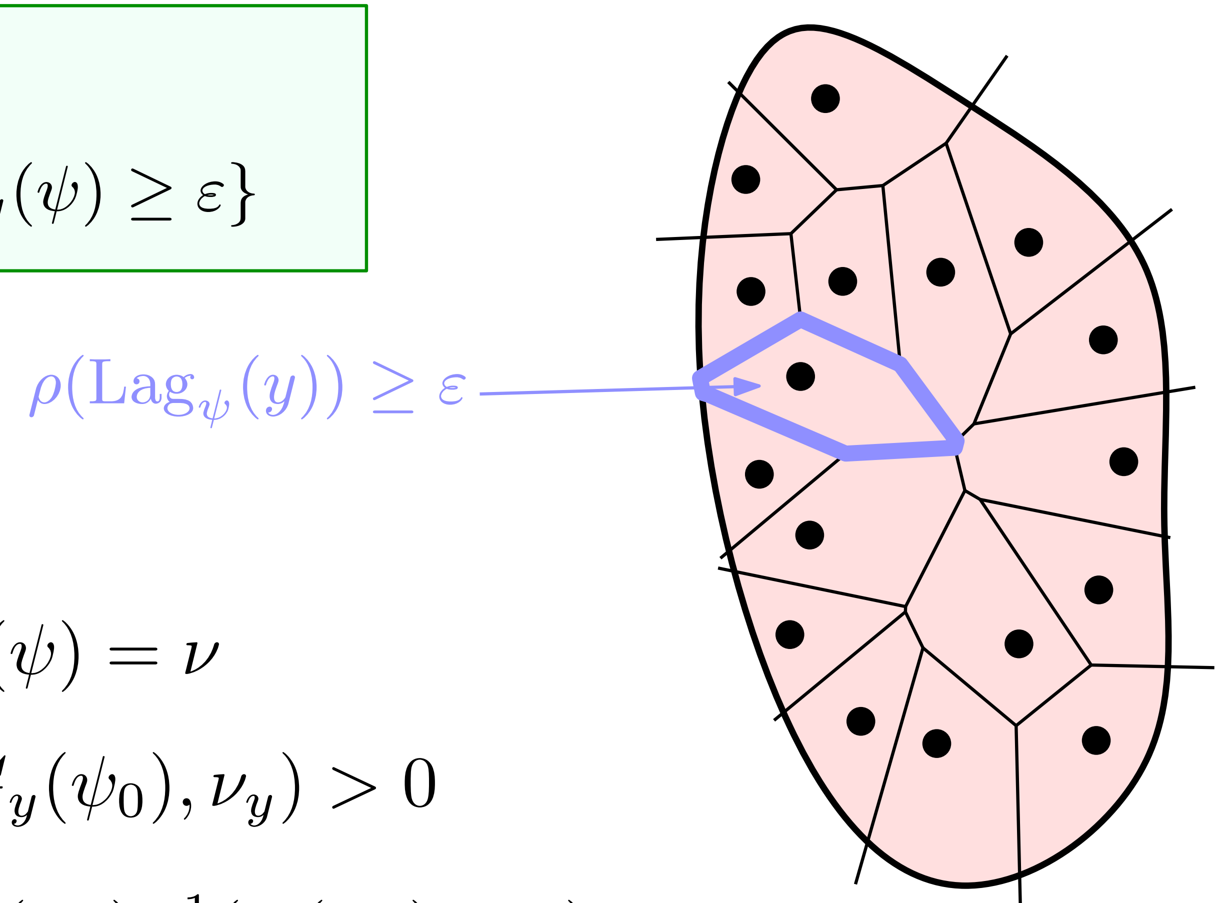

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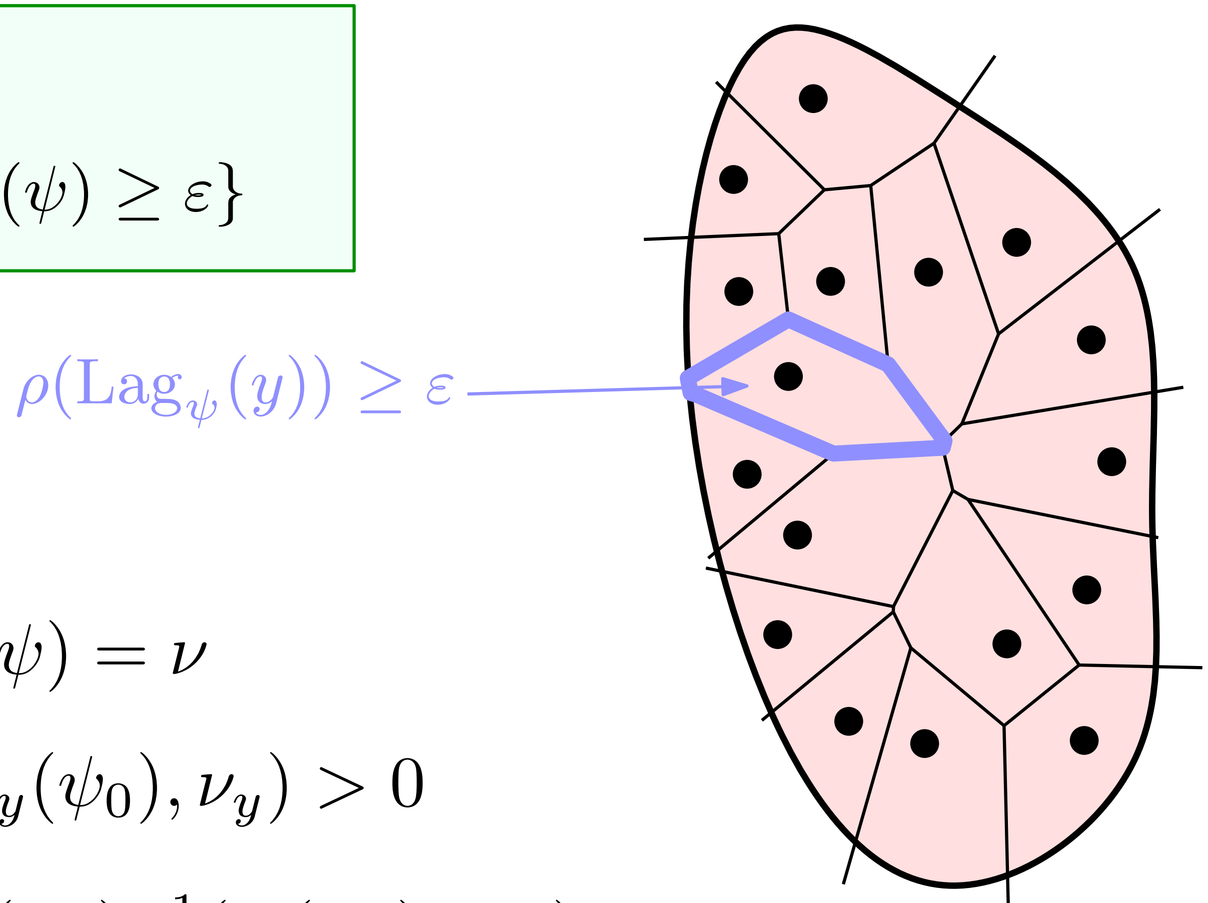

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(Strong monotonicity): for all $\psi \in E_\varepsilon$, $\mathrm{D}G$ is negative definite on $\{cst\}^\perp$

Damped Newton's Algorithm

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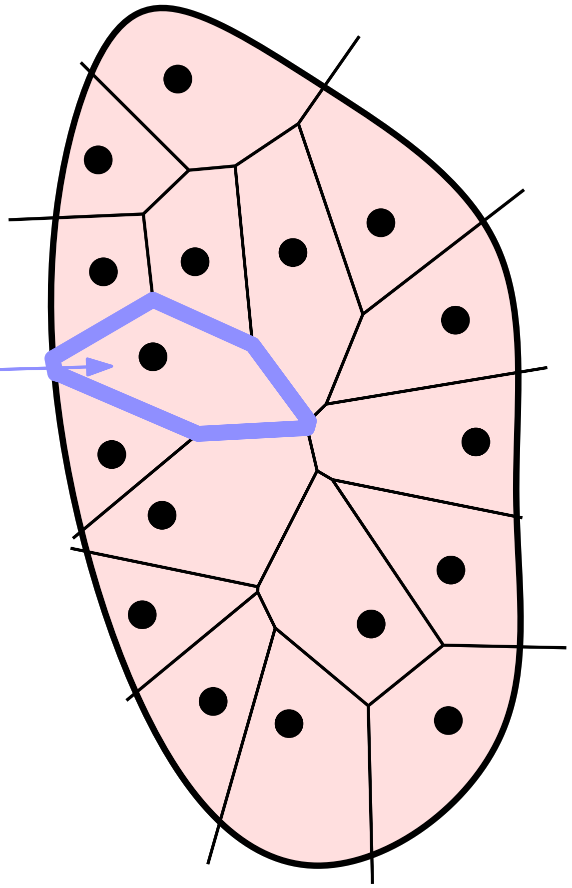
$\rho(\mathrm{Lag}_\psi(y)) \geq \varepsilon$



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(Smoothness): G is $\mathcal{C}^1$ on E_ε

Convergence of Damped Newton

Theorem: Let X be an hemisphere of $\mathcal{S}^2$. Assume that $Y \subset \mathcal{S}^2 \setminus X$ and that

$$\rho \in \mathcal{C}^\alpha(X) \text{ and } \{\rho > 0\} \text{ is connected}$$

Then, the damped Newton algorithm for SD-OT converges **globally** with linear rate and locally with rate $(1+\alpha)$. special case of [Kitagawa, M., Thibert '15]

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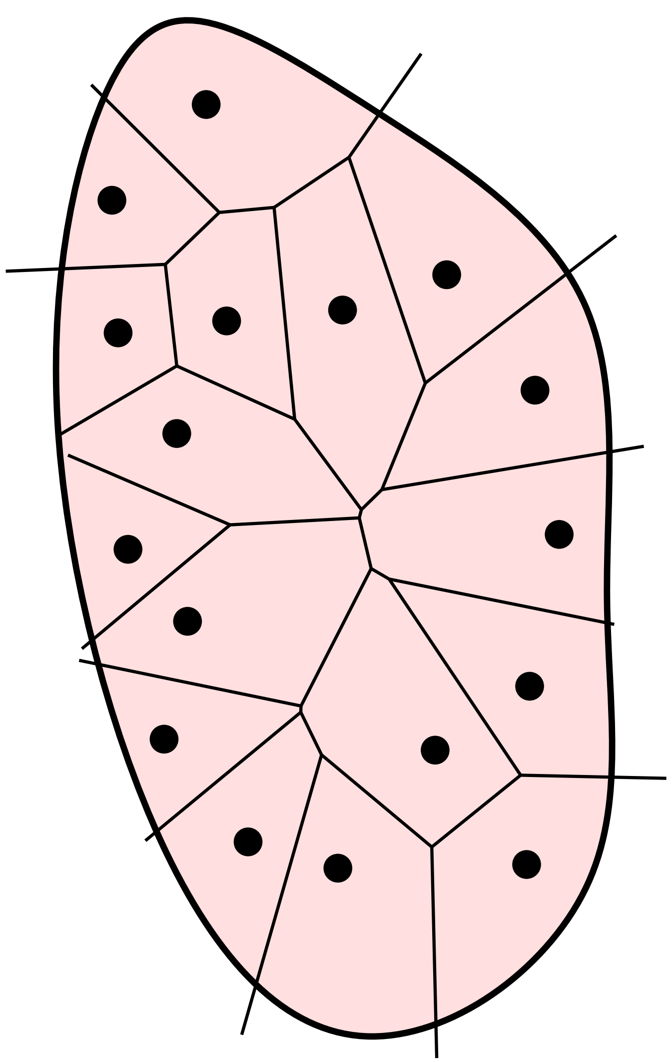

- Consider the matrix $(L_{yz}) := \left(\frac{\partial G_y}{\partial \mathbf{1}_z}(\psi)\right)$ and the graph H:

$(y,z) \in H \iff L_{zy} > 0$

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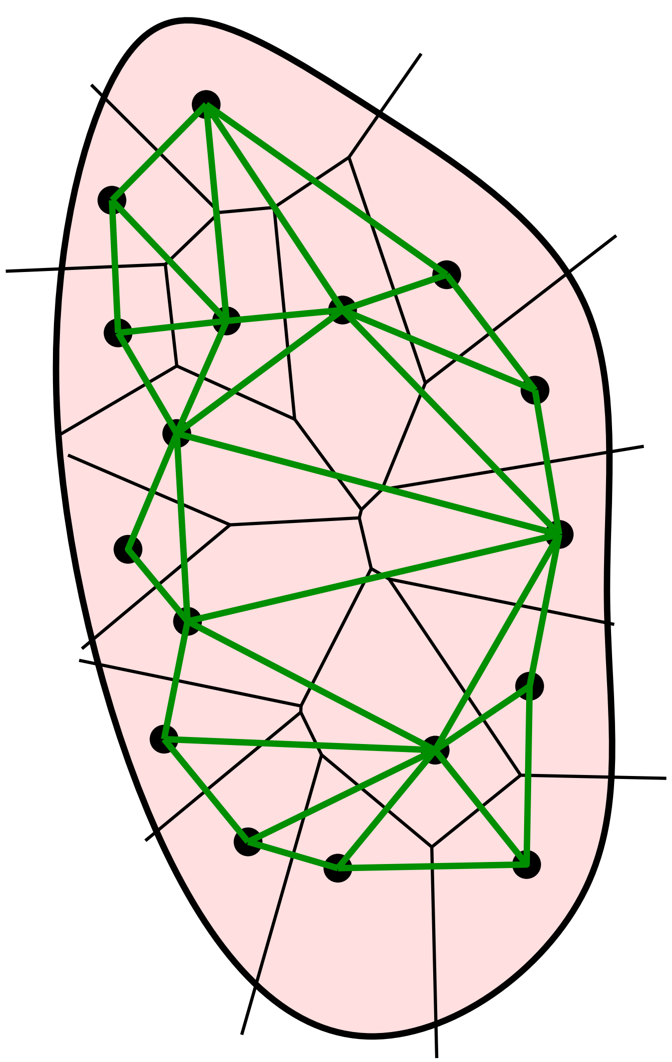

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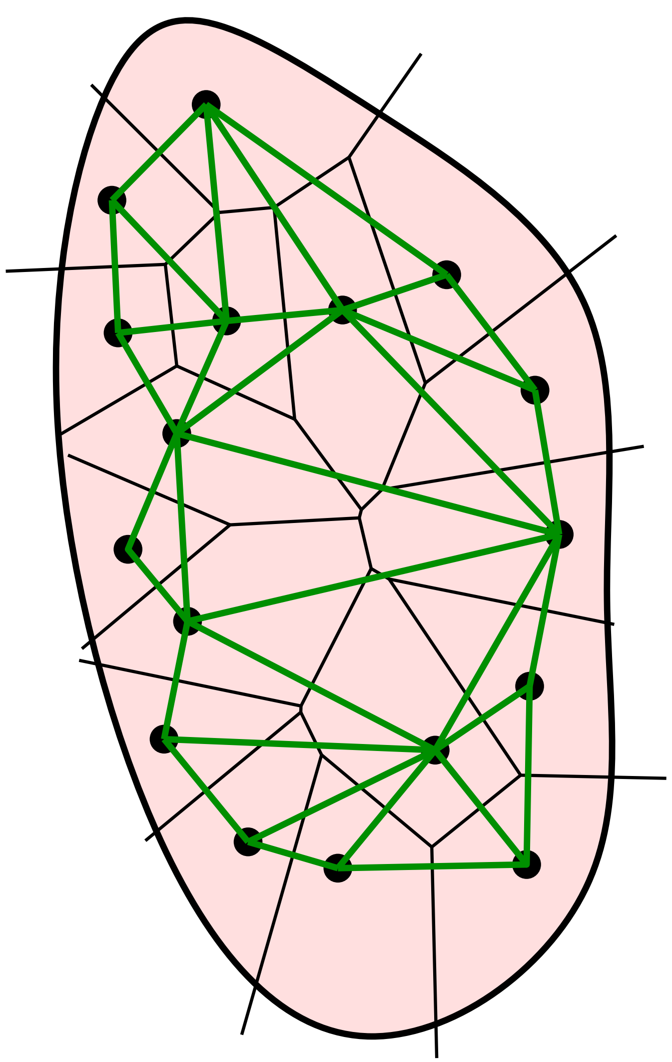

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- If $\{\rho > 0\}$ is connected and $\psi \in E_\varepsilon$, then H is connected

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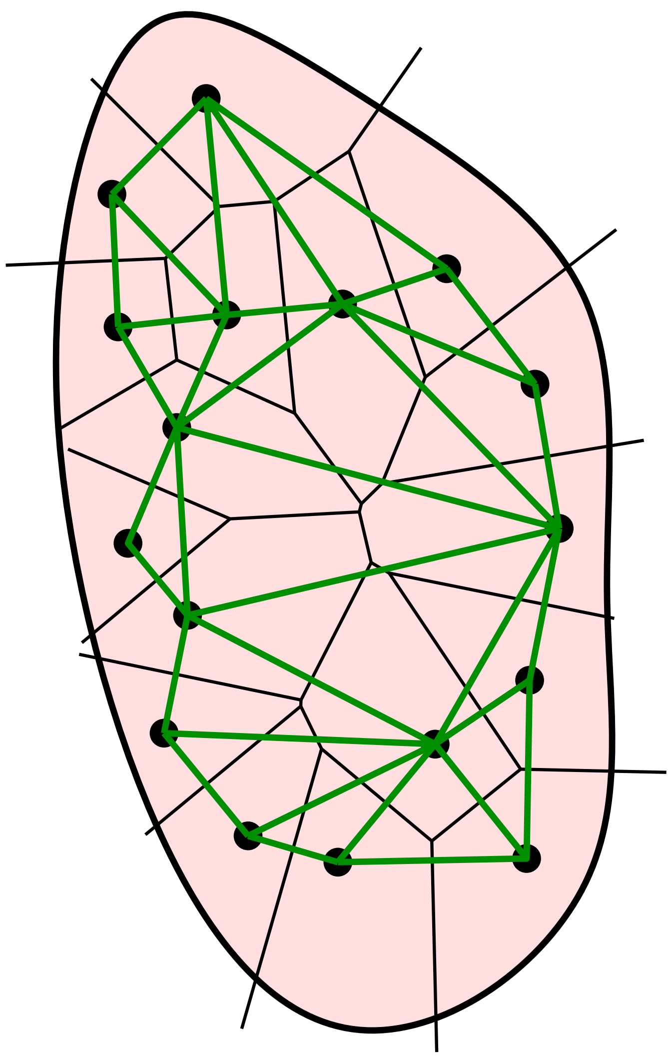

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  $(y,z) \in H \iff L_{zy} > 0$
- H = 1-skeleton of $(\mathrm{Lag}_y(\psi) \cap \{\rho > 0\})_{y \in Y}$
- If $\{\rho > 0\}$ is connected and $\psi \in E_\varepsilon$, then H is connected
- L is the Laplacian of a connected graph $\Longrightarrow \mathrm{Ker} L = \mathbb{R} \cdot \mathrm{cst}$

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Theorem: Let X be an hemisphere of $\mathcal{S}^2$. Assume that $Y \subset \mathcal{S}^2 \setminus X$ and that

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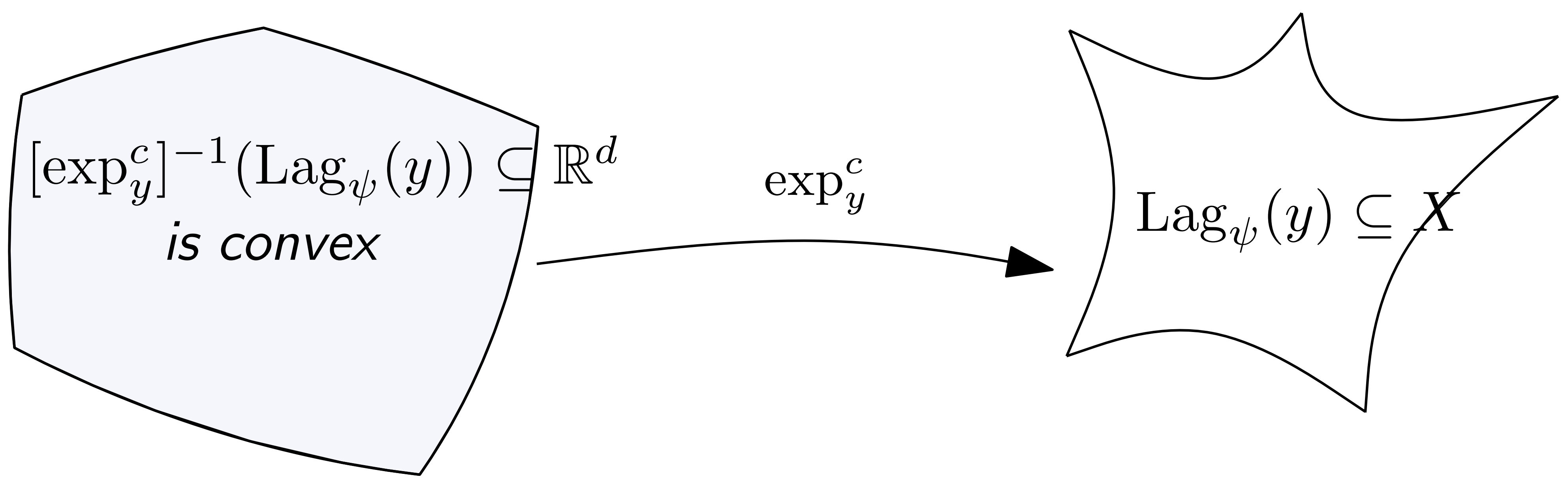

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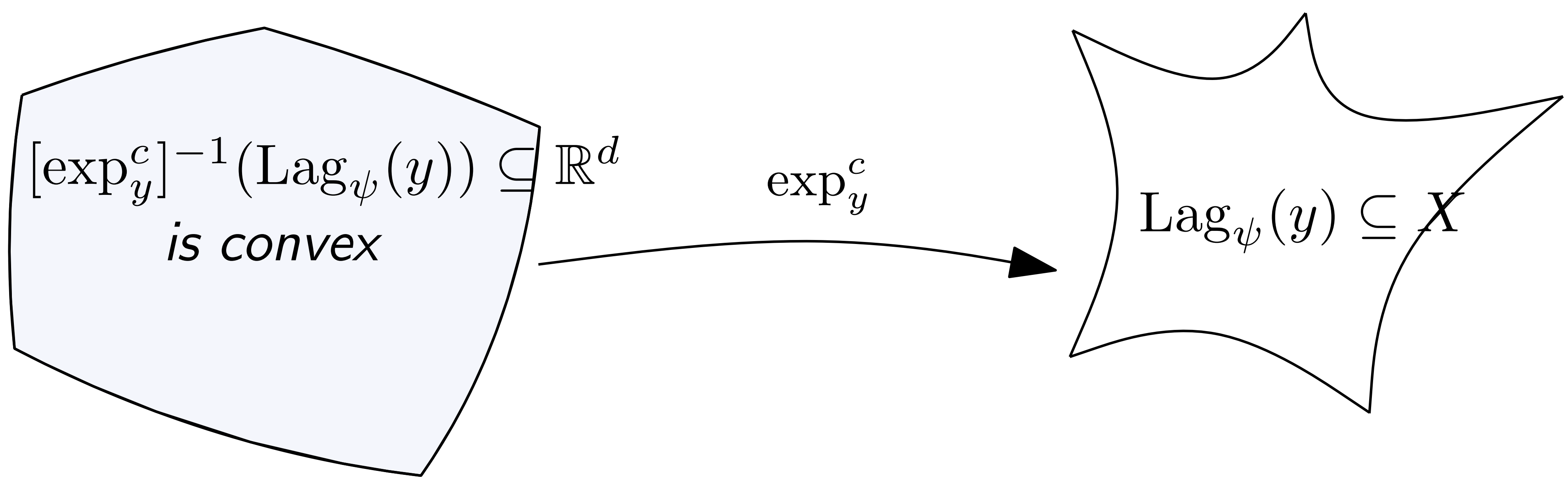

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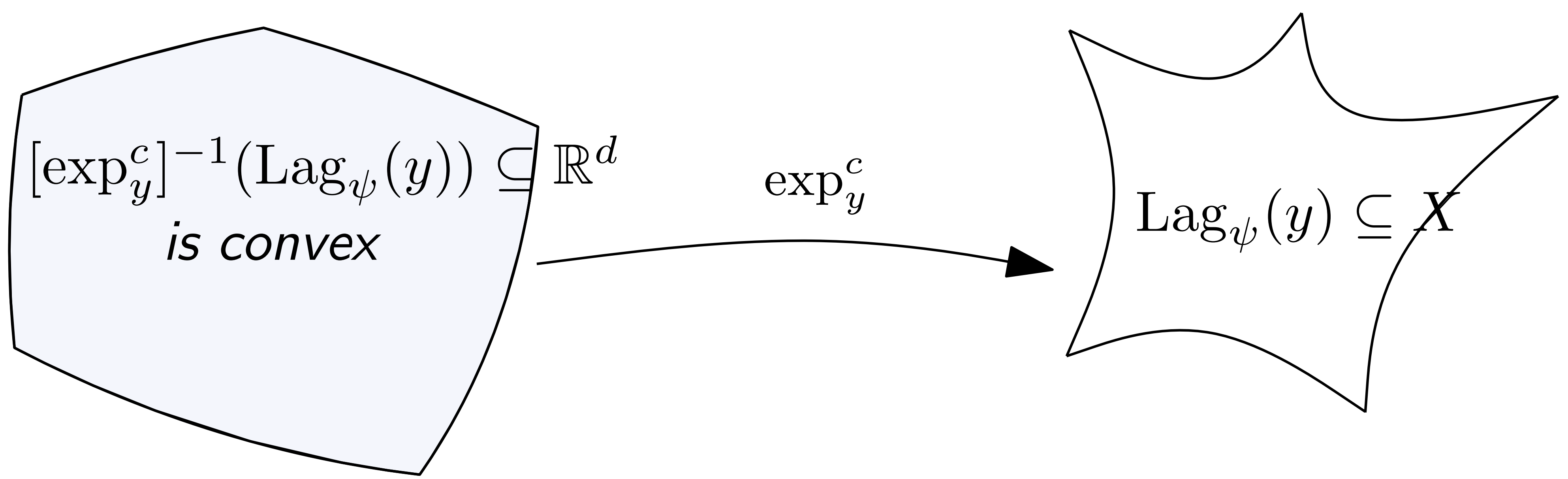

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$\longrightarrow$ Loeper's condition originates from regularity theory for OT...

3. Second application: enforcing incompressibility

Joint work with J.M. Mirebeau

Geodesics between incompressible maps

Thm: Smooth solutions to Euler equations for incompressible fluids are geodesics in

$$\mathbb{S}\mathrm{Diff} = \{\textbf{volume-preserving diffeo. from } X \text{ to } X\} \subseteq E := \mathrm{L}^2(X, \mathbb{R}^d)$$

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A point cloud cannot be exactly incompressible $\Longrightarrow$ penalization using optimal transport.

Distance to incompressible maps

Definition: Given $m = (M^1, \ldots, M^N) \in \mathbb{R}^{Nd}$, we define

$$\mathrm{d}^2_{\mathbb{S}}(m) = \text{ min. transport cost between } \rho \text{ and } \nu = \frac{1}{N}\sum_{k=1}^{N} \delta_{M^k}$$

where ρ is uniform on X and $c(x,y) = \|x-y\|^2$.

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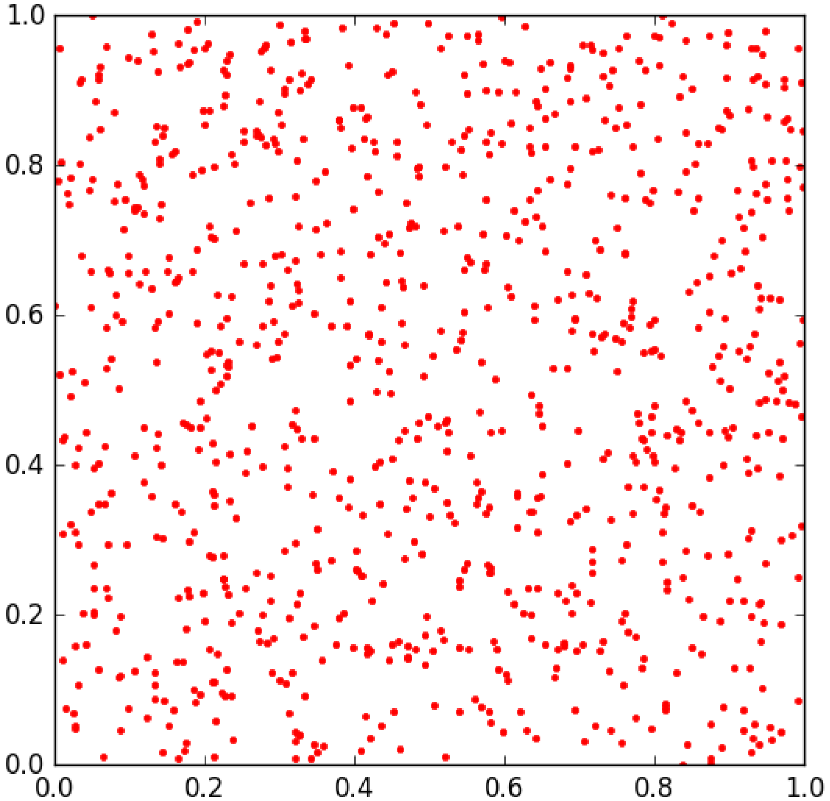

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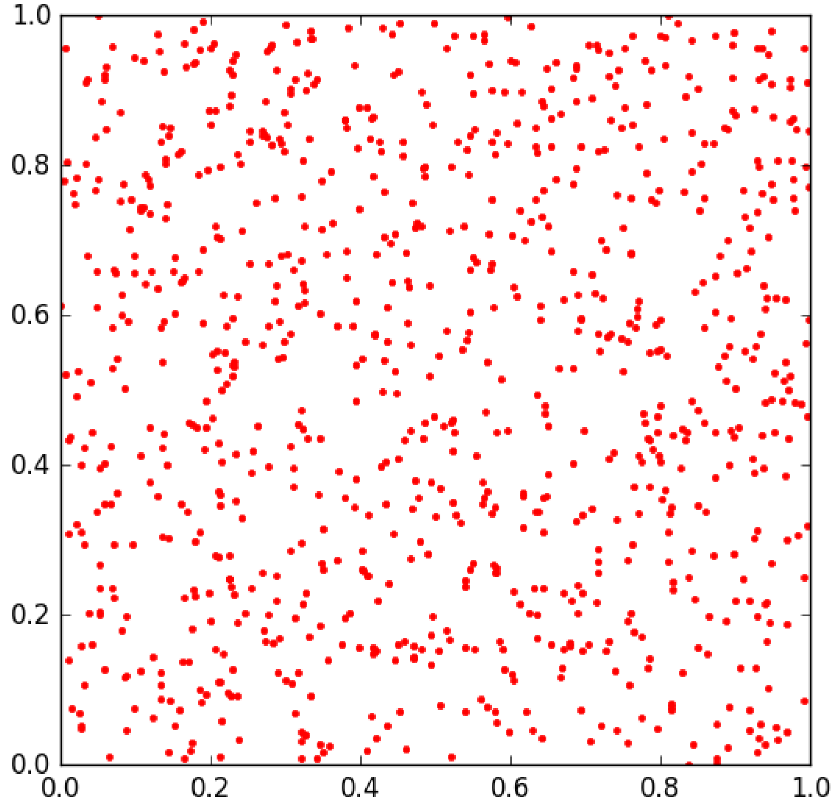

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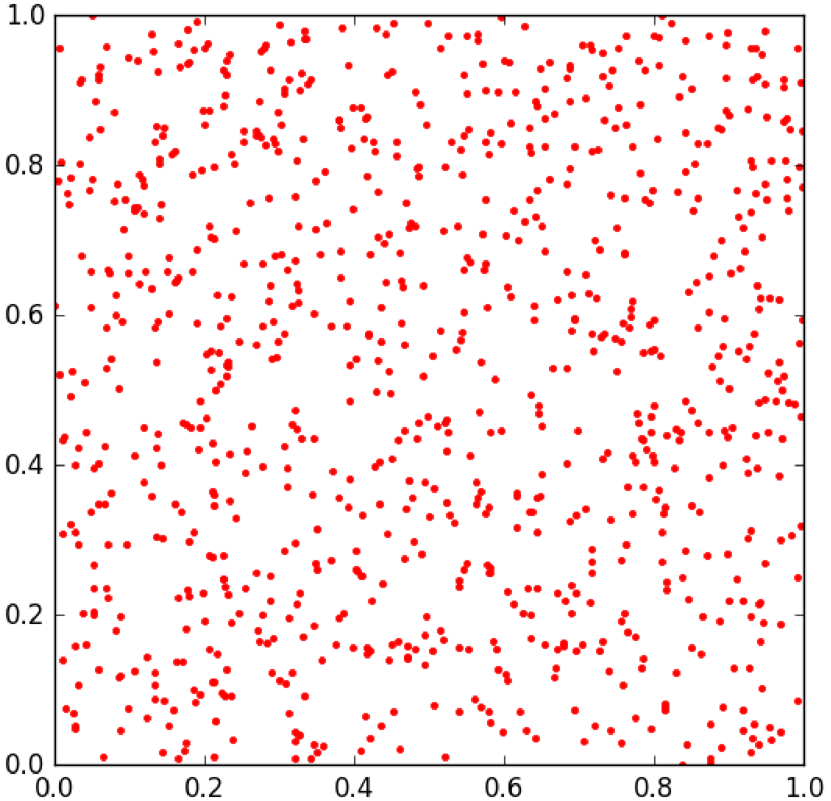

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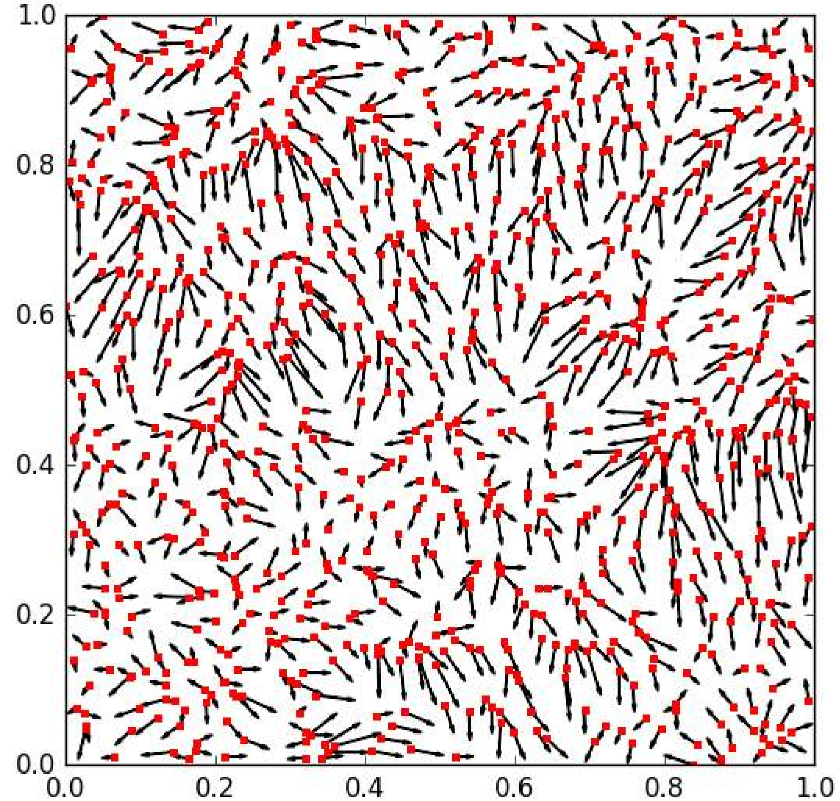

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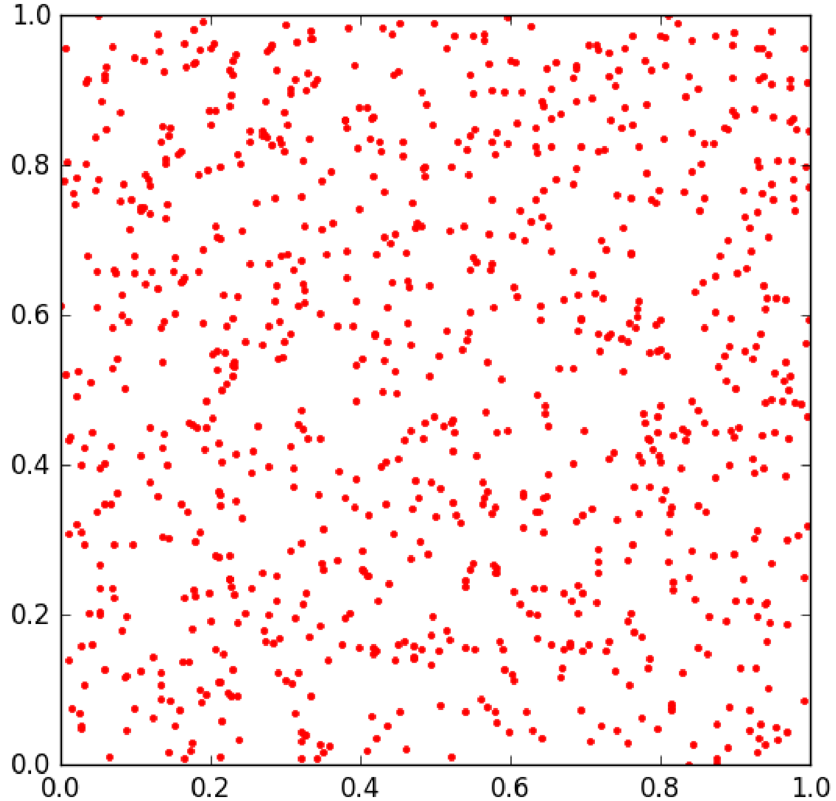

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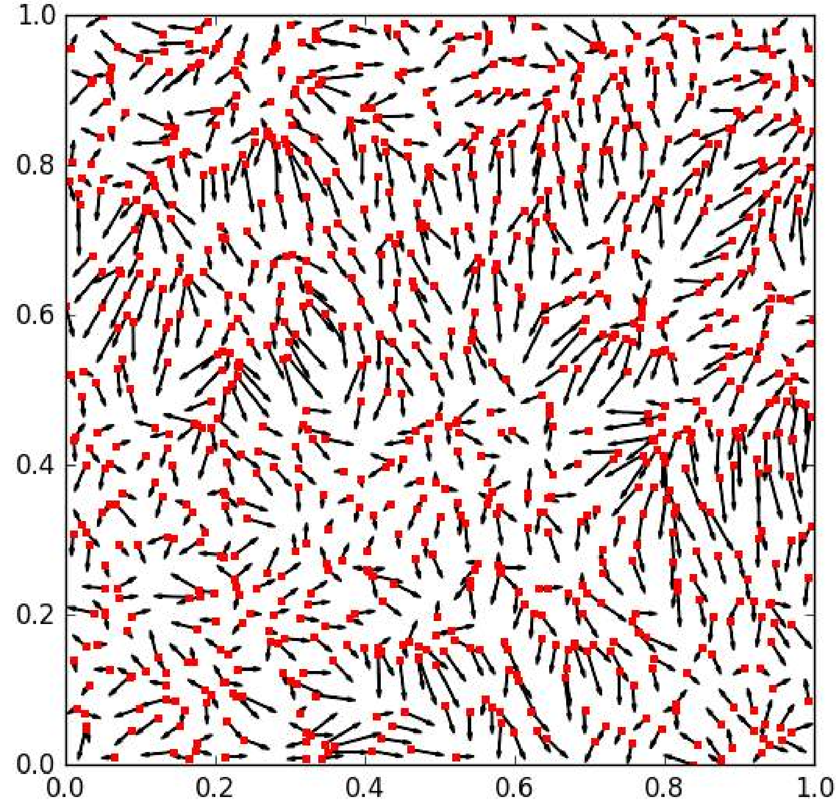

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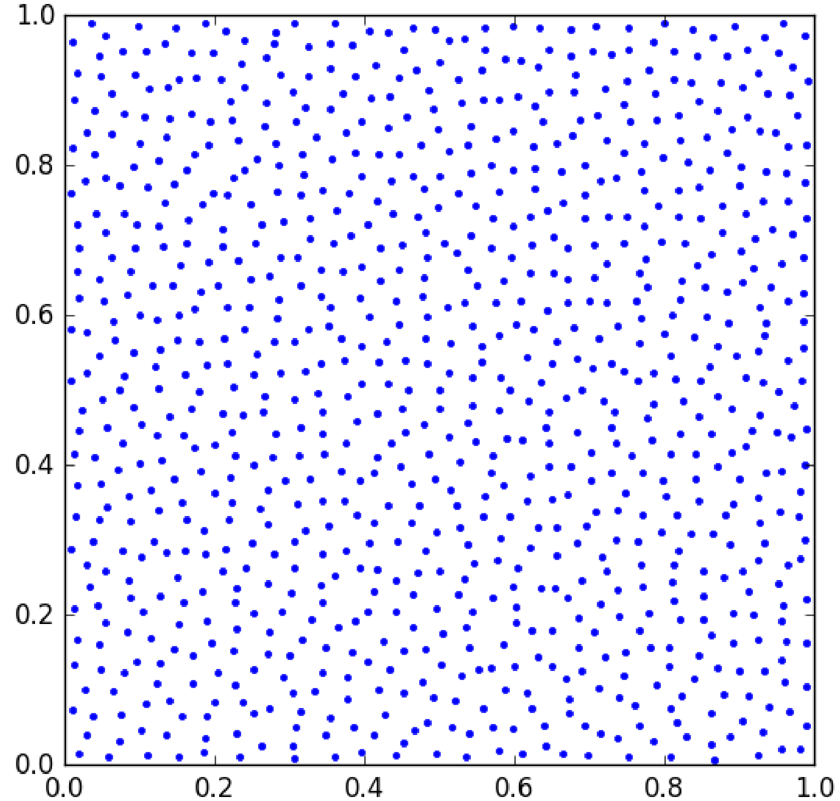

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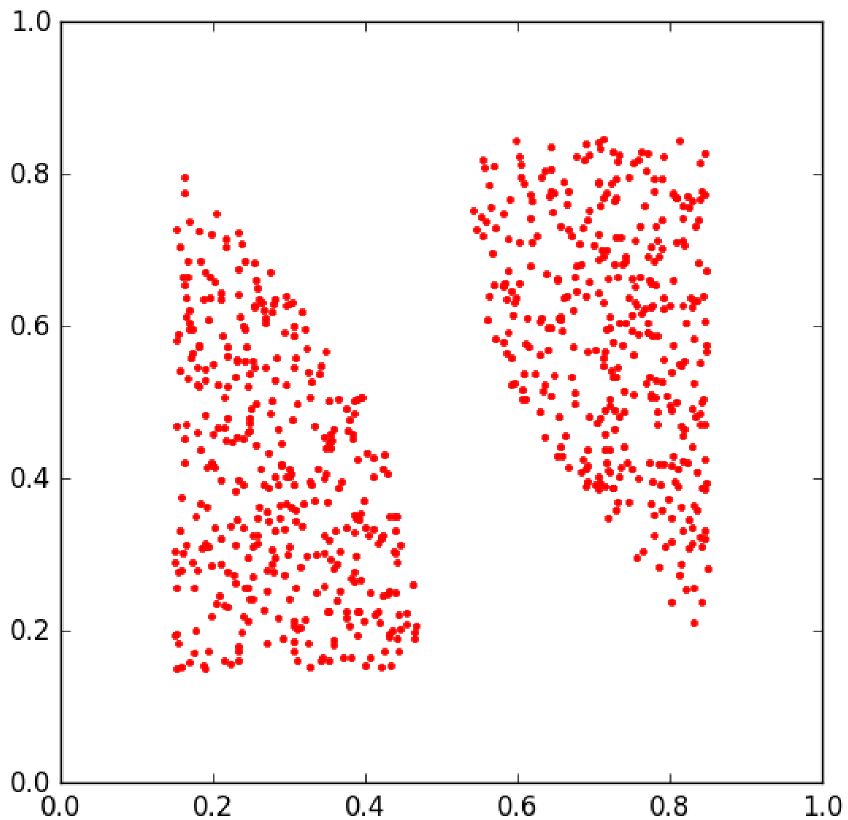

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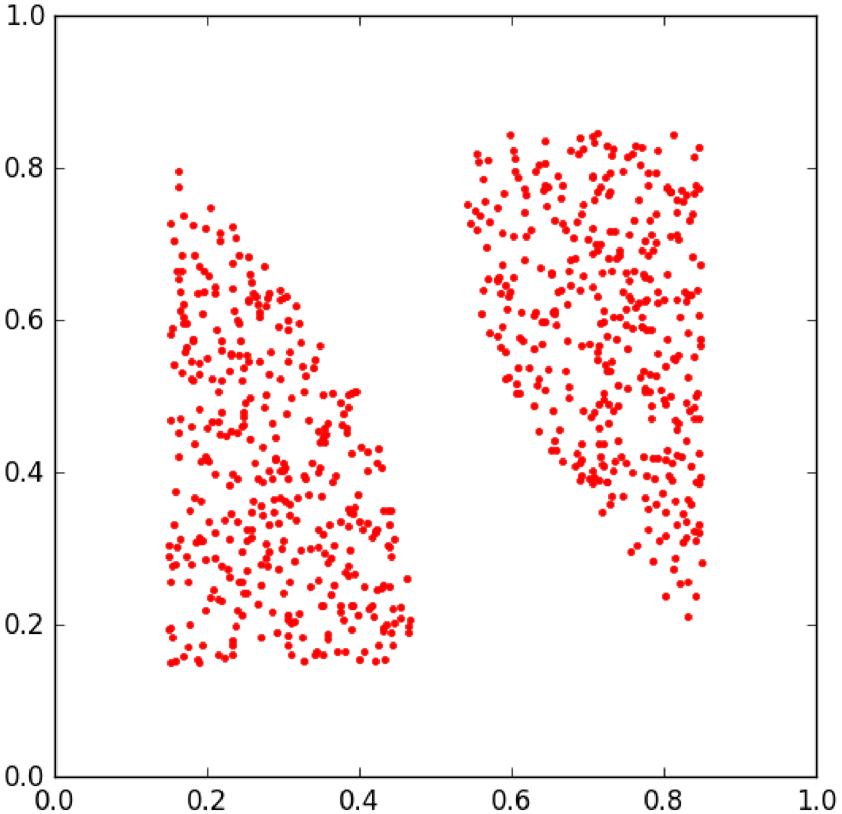

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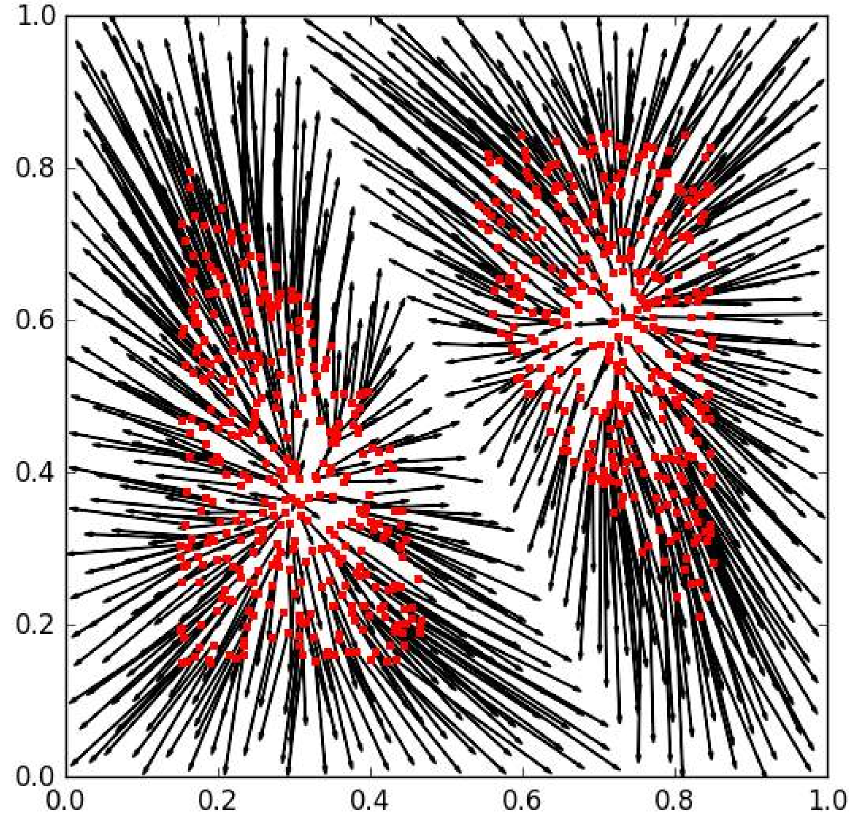

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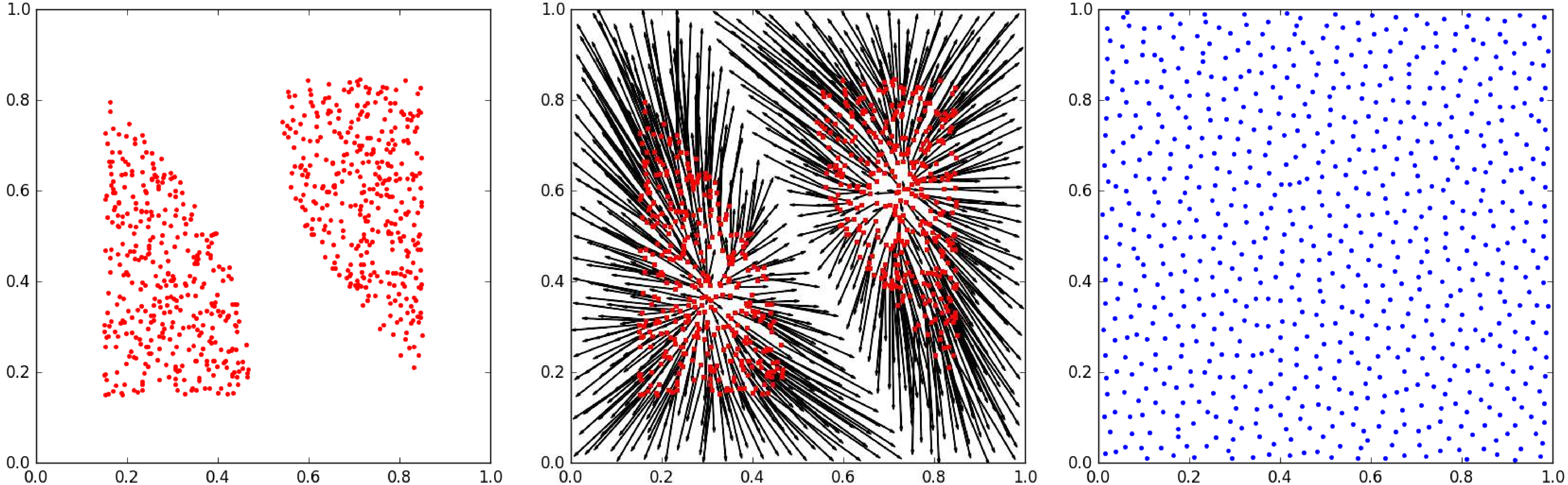

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From particles to paths

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action · boundary conditions · incompressibility

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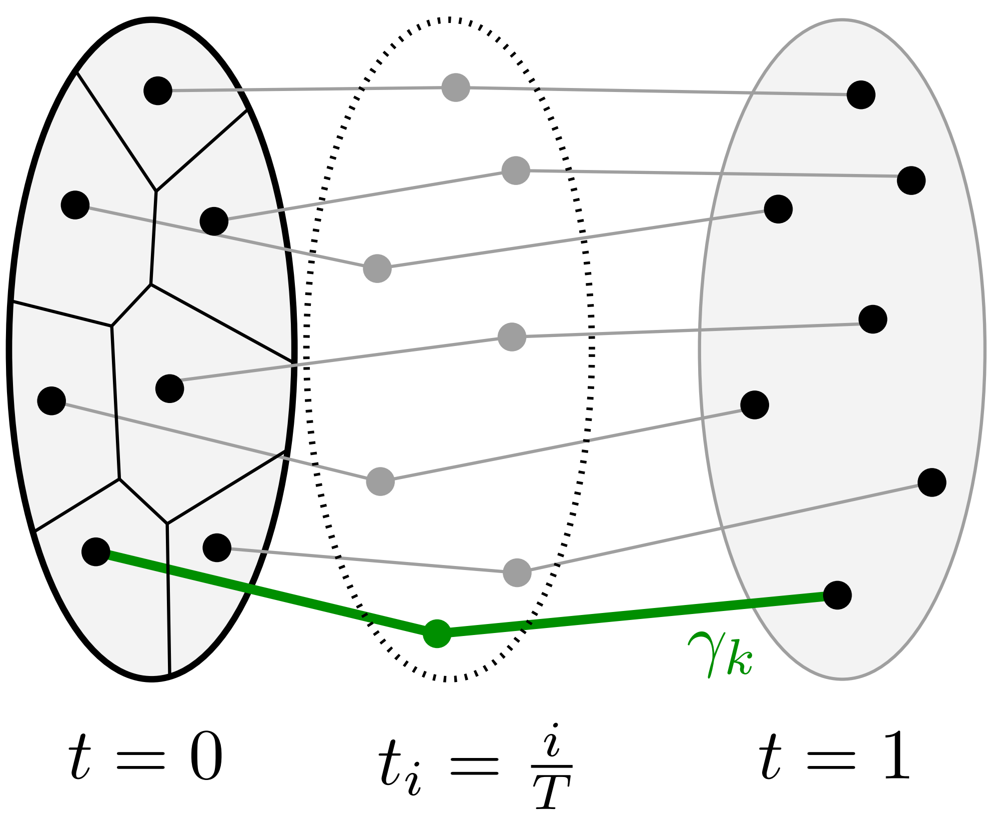

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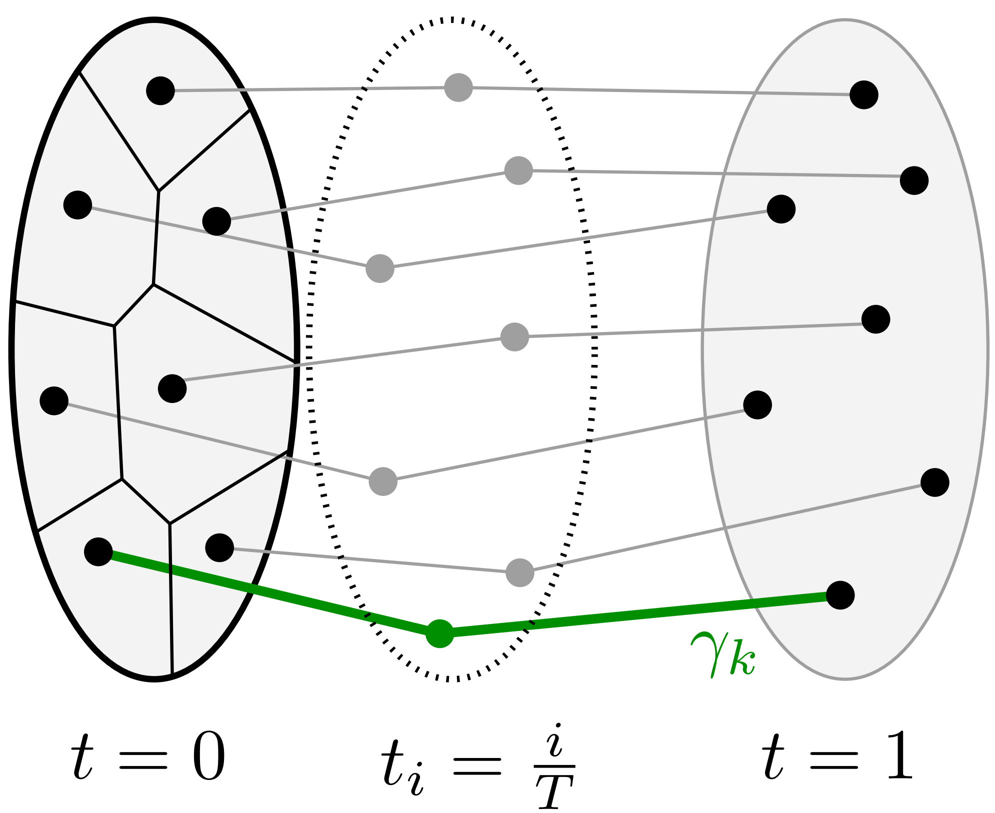

$\longrightarrow$ One can associate to m a probability measure over the set of $\mathcal{C}^0$ paths:

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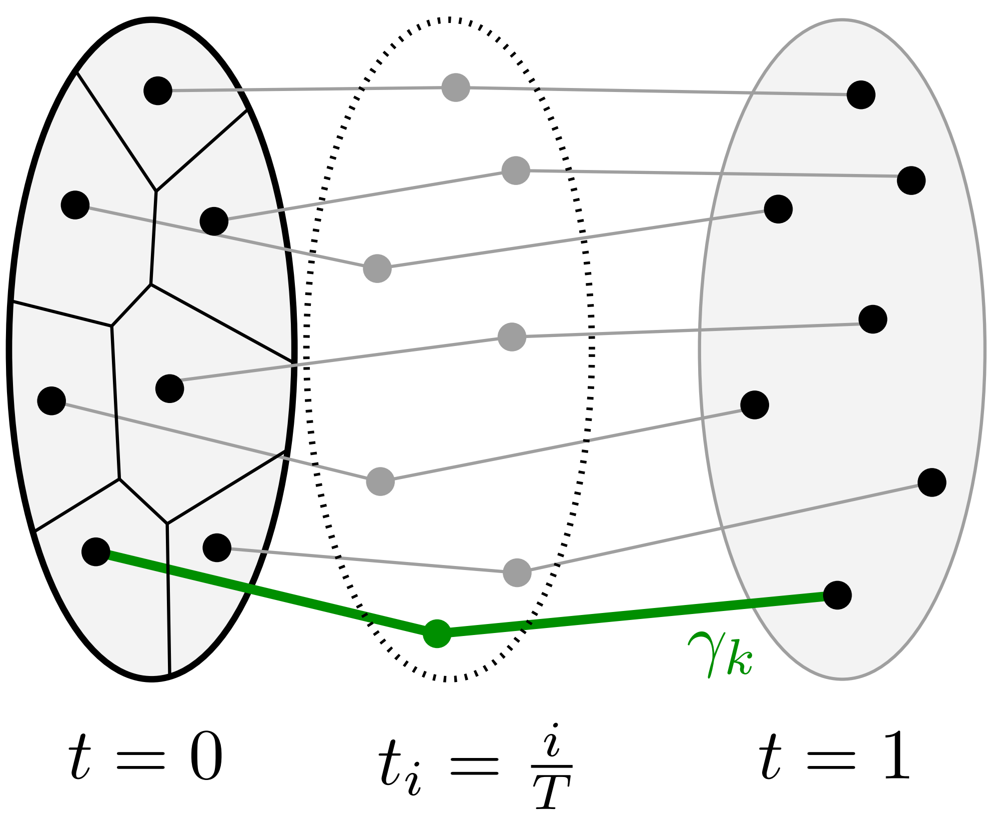

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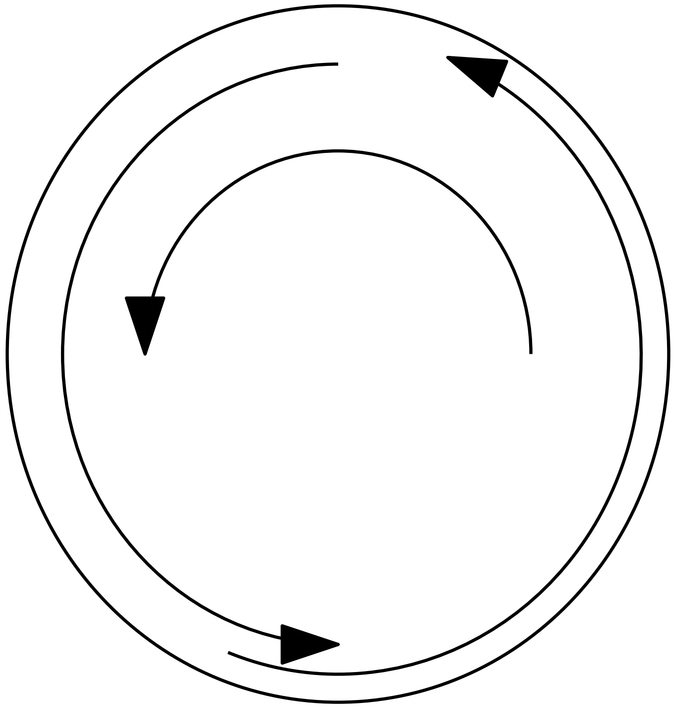
$\longrightarrow$ Under suitable hypotheses, minimizers of the discrete problem converge to a so-called **generalized minimizing geodesic,**

$$\mu \in \mathrm{Prob}(\mathcal{C}^0([0,1], \mathbb{R}^d)).$$

Numerical result: Inversion of the Disk



$X = \mathrm{B}(0,1) \subseteq \mathbb{R}^2 \qquad (s_*, s^*) = (\mathrm{id}, -\mathrm{id})$

Classical solutions: clockwise/counterclockwise rotations $\mu_\pm$

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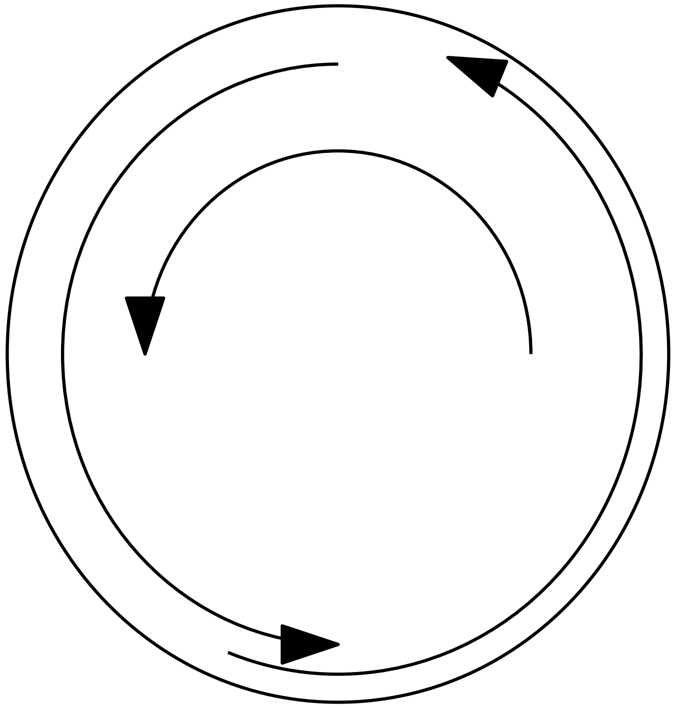

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Examples of generalized solutions:

linear combination $\mu_{\frac{1}{2}}$ of $\mu_\pm$ constructed from rotations

NB: $\dim(\mathrm{spt}(\mu_{\frac{1}{2}})) = 2$

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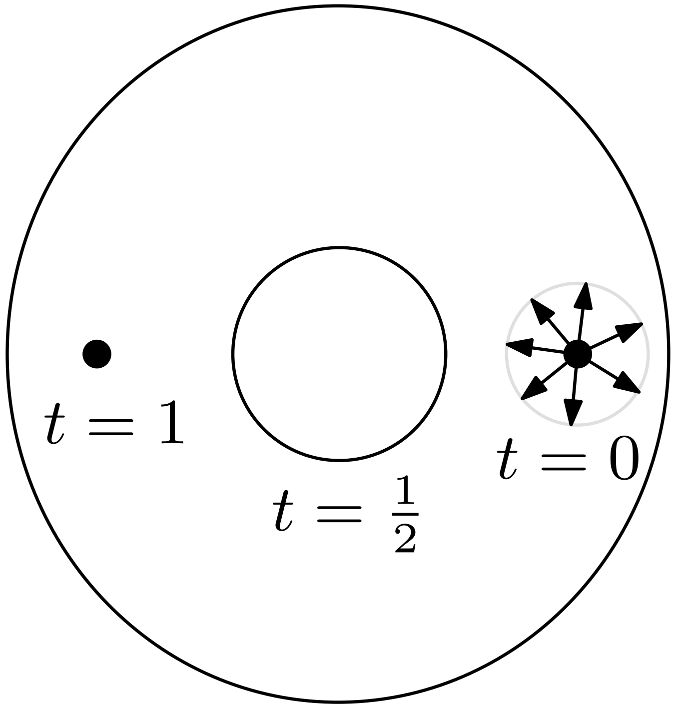

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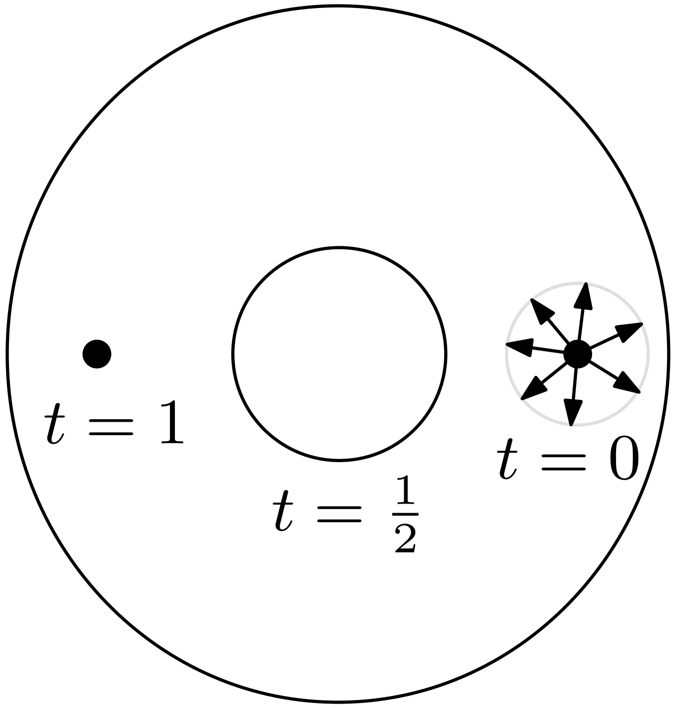

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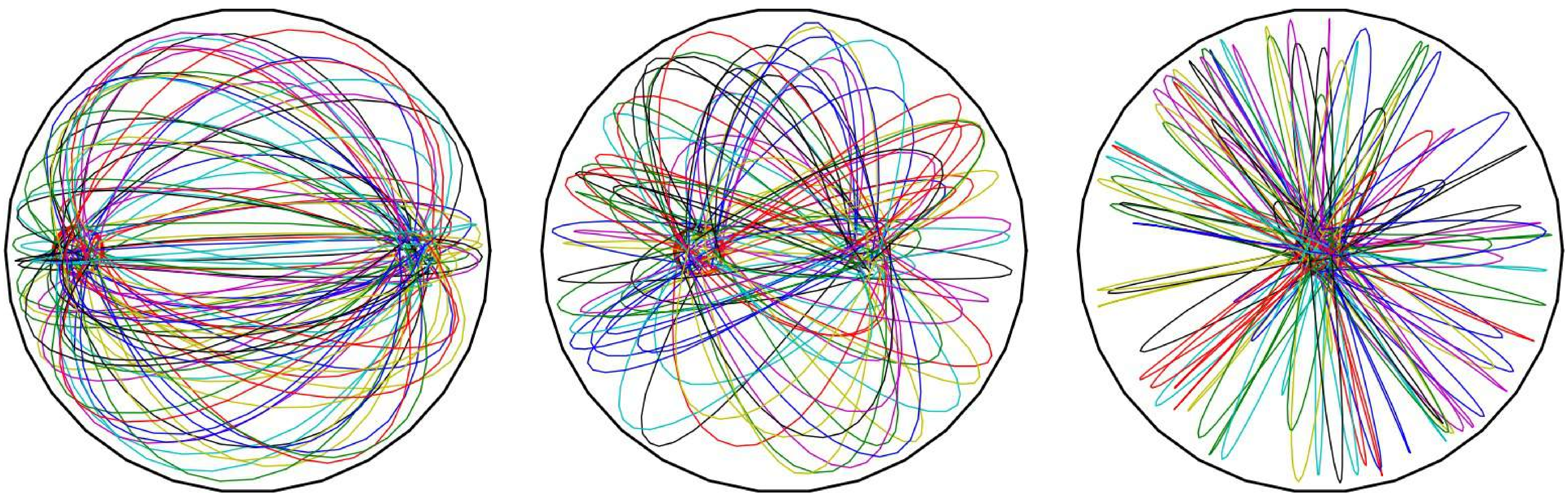
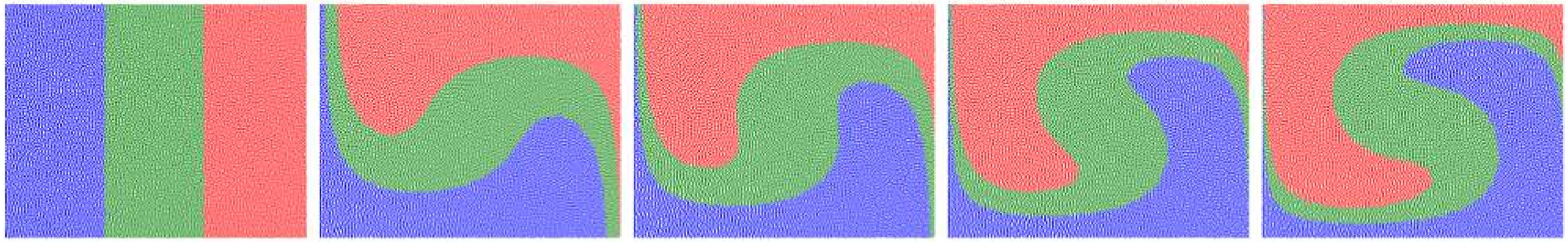
Computed trajectories for $N = 10^5,\ T = 17$



Numerical result: Beltrami Flow in Square

forward
simulation



(a) $t = 0.0$ (b) $t = 0.95$ (c) $t = 1.1$ (d) $t = 1.3$ (e) $t = 1.5$

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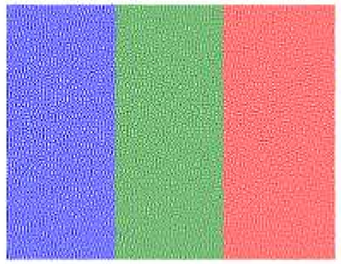
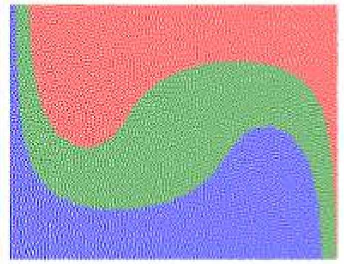
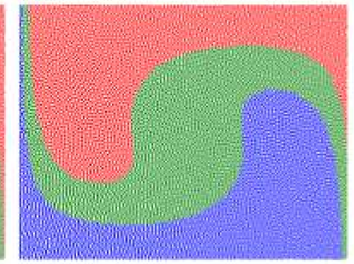


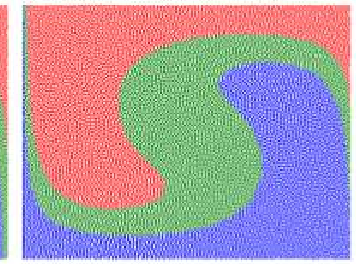
(a) $t = 0.0$



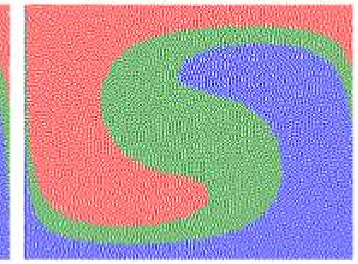
(b) $t = 0.95$



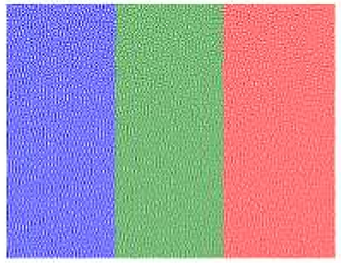
(c) $t = 1.1$



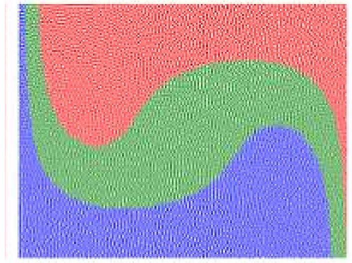
(d) $t = 1.3$



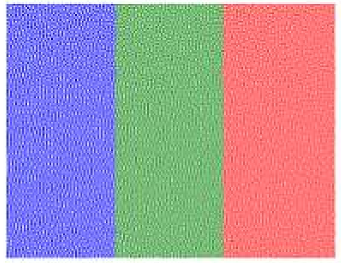
(e) $t = 1.5$



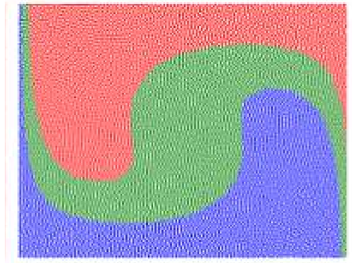
(f) $t = 0.0$



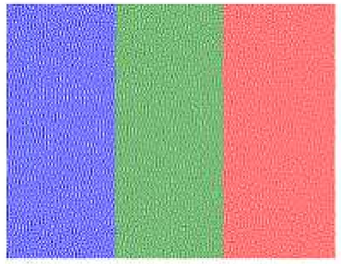
(j) $t = t_{\max} = 0.9$



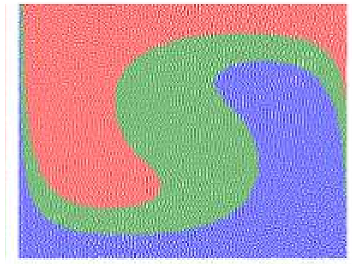
(k) $t = 0.0$



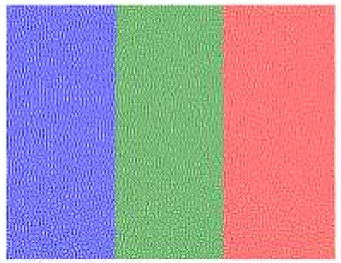
(o) $t = t_{\max} = 1.1$



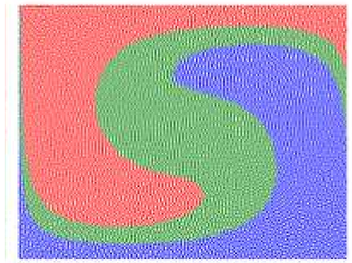
(p) $t = 0.0$



(t) $t = t_{\max} = 1.3$



(u) $t = 0.0$



(y) $t = t_{\max} = 1.5$

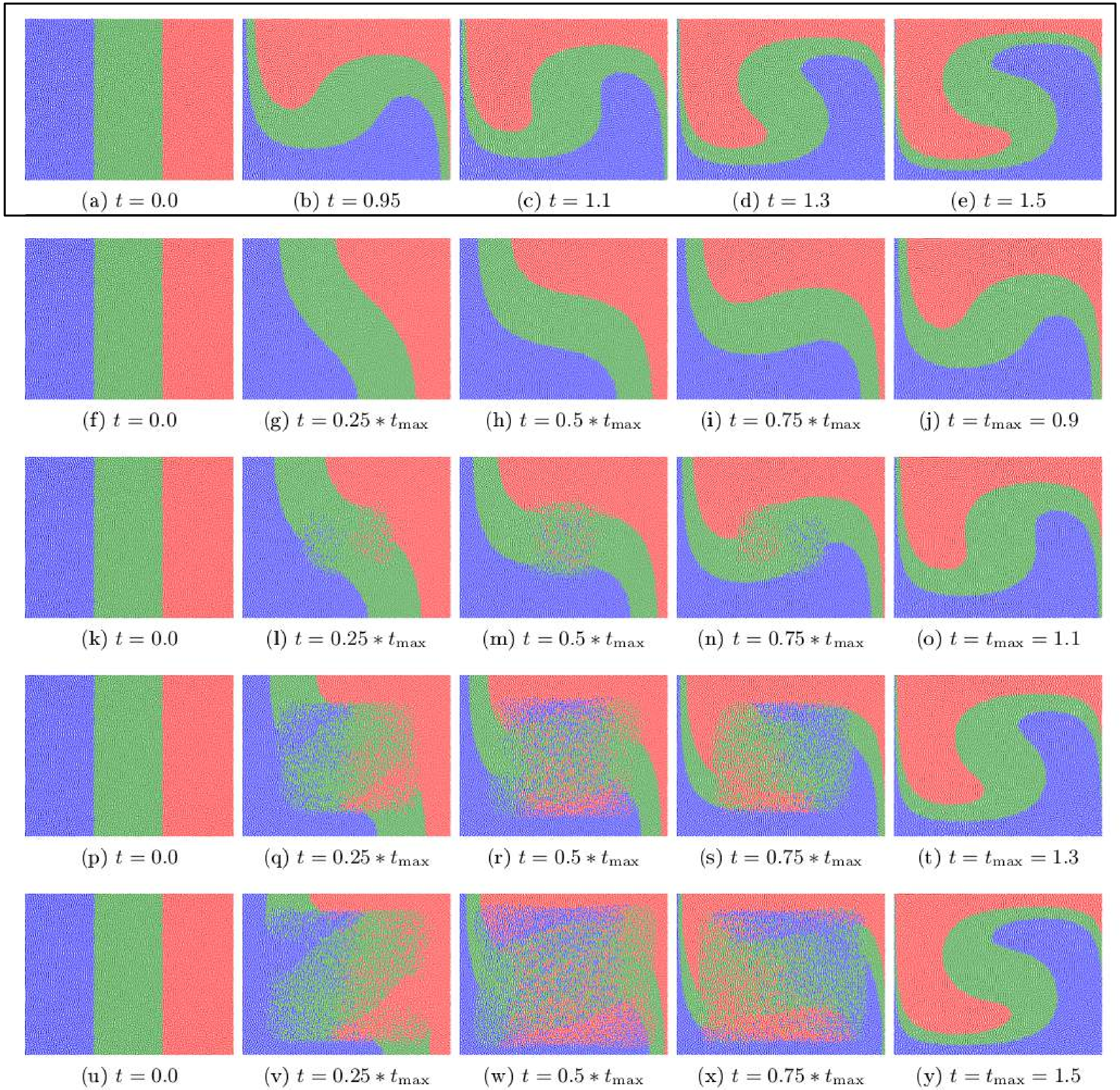
s_*

s^*

Numerical result: Beltrami Flow in Square

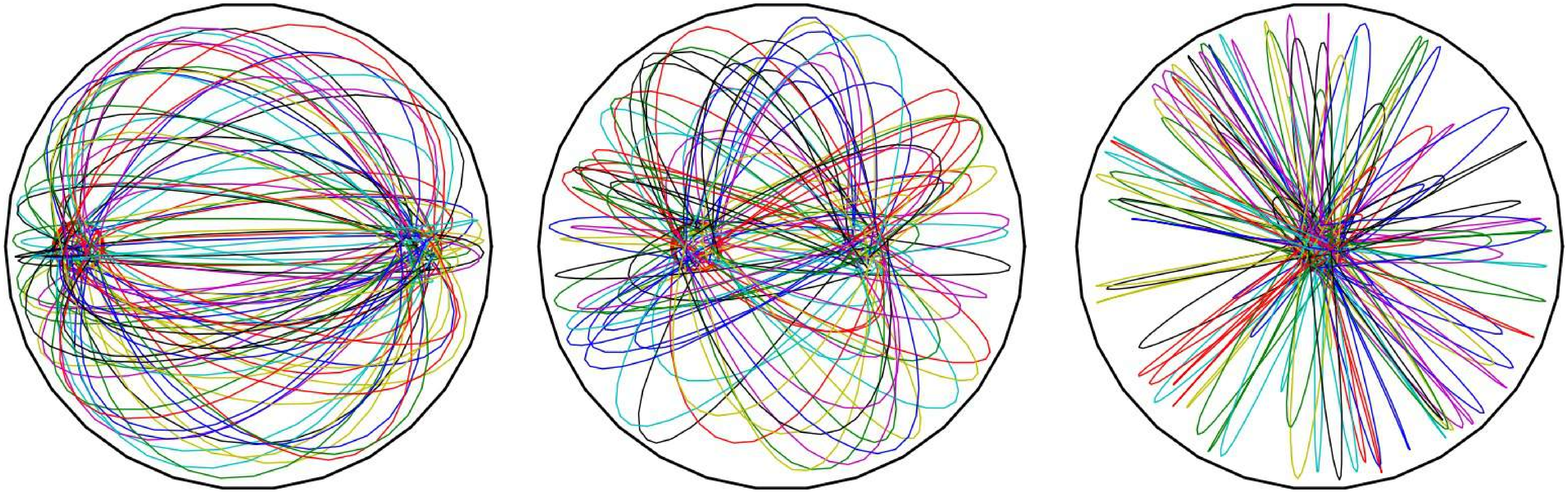
forward simulation

reconstructed generalized geodesics

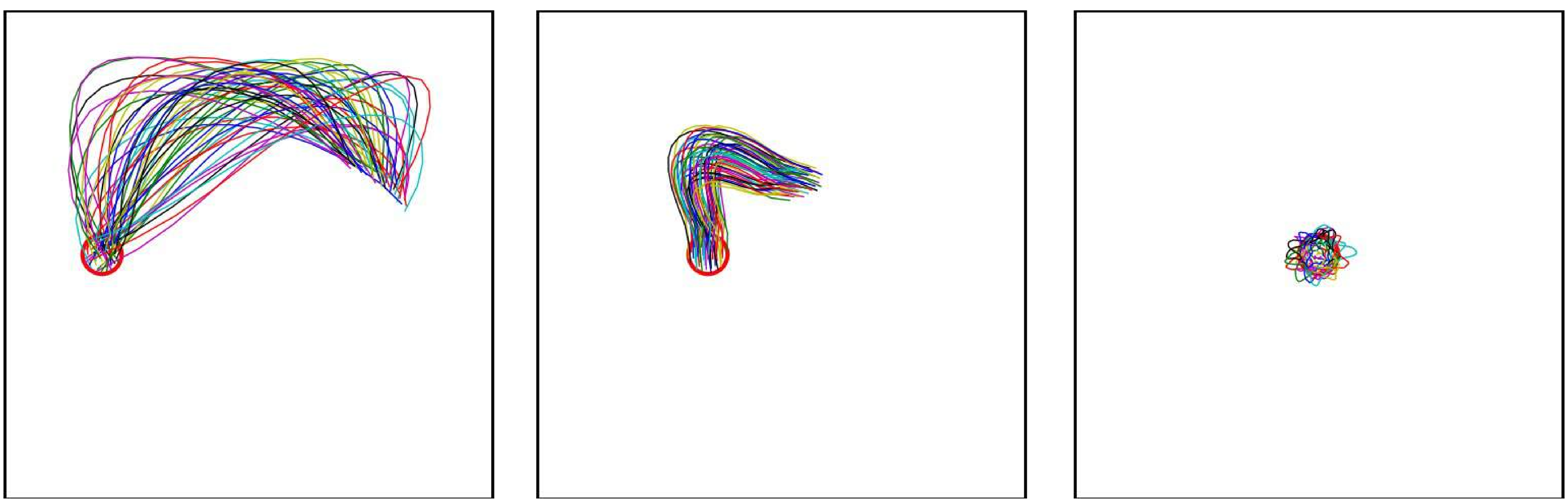


s_* $\qquad$ s^*

Numerical result: Comparison of Trajectories



Disk inversion

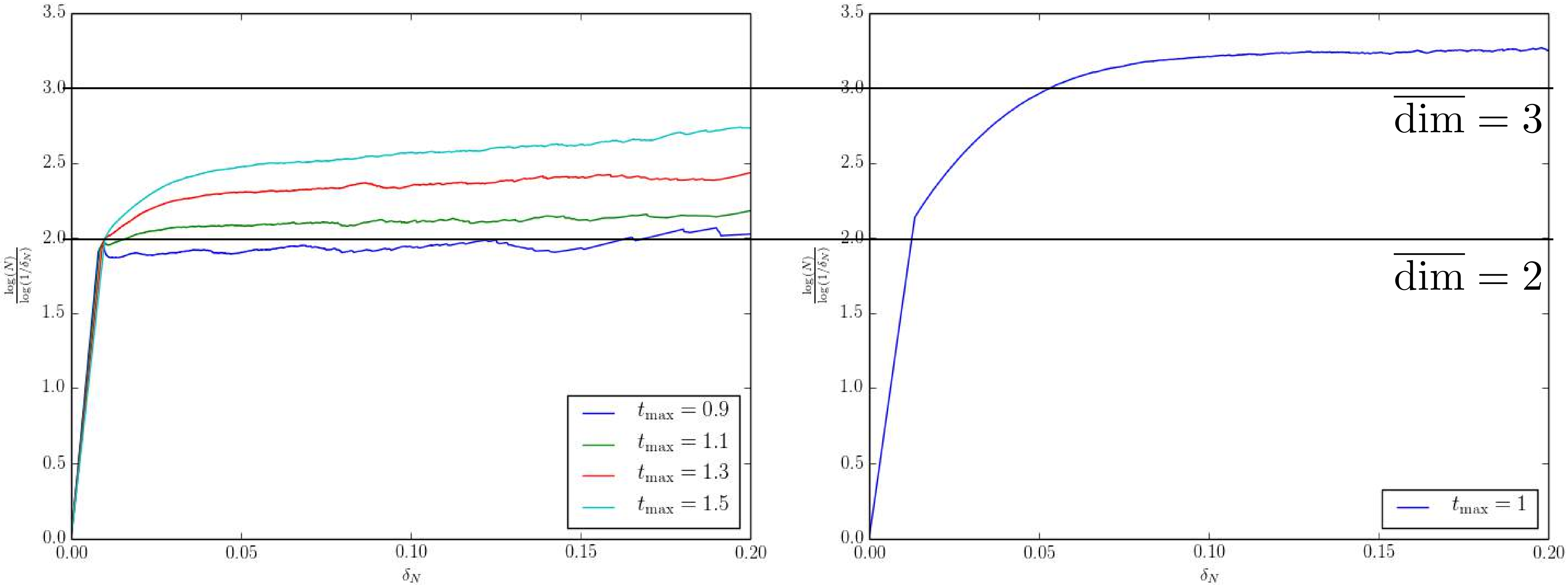


Square, $t_{\max} = 1.5$

Comparison of Minkowski dimensions

Estimation of $\dim(\mathrm{spt}(\mu))$ via $\log(N)/\log(1/\delta_N)$

where $\delta_N =$ minimum radius required to cover $\mathrm{spt}(\mu)$ with N balls.



Square rotation, $t_{\max} \in \{0.9, 1.1, 1.3, 1.5\}$ Disk inversion

Summary

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Optimal transport $\longmapsto$ physics

Monge-Ampère type PDEs
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Thank you for your attention!