On the mod p reduction of orthogonal representations

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to the memory of Bertram Kostant

Introduction

The following fact is probably known :

(F)   *Let G be a finite group and let ρ be a linear representation of G in characteristic* 0 *which is orthogonal* (resp. symplectic). *Let p be a prime number $\neq 2$. Then the reduction mod p of ρ is orthogonal* (resp. symplectic).

[Recall that a linear representation is said to be orthogonal (resp. symplectic) if it fixes a nondegenerate symmetric (resp. alternating) bilinear form. As for the precise meaning of "reduction mod p", see below.]

In what follows, we give a detailed proof of (F), and we extend it to linear representations of algebras with involution, including the group algebra of an infinite group.

Let us state the theorem explicitly, in the group algebra case :

Let R be a discrete valuation ring, with ring of fractions K and residue field $k = R/\pi R$, where π is a uniformizer. Let V be a finite dimensional K-vector space. Let G be a group and let $\rho : G \to \mathrm{GL}(V)$ be a homomorphism. Assume that there exists a lattice L of V which is G-stable (such a lattice always exists when G is finite).

[Recall that a *lattice* is a free R-submodule L of V such that $K.L = V$, cf. §1.1.]

Let L be a G-stable lattice of V. The k-vector space $V_L = L/\pi L$ is a $k[G]$-module. The structure of this module may depend on the choice of L. However, by a theorem of Brauer-Nesbitt (see §3), its *semisimplification* $V_L^{\text{ß}}$ is well-defined, up to isomorphism (cf. §2.2). It is $V_L^{\text{ß}}$ that we call "the reduction mod π" of V, and we denote it by V_k. The precise form of (F) is :

Theorem A. *Assume that there exists a G-invariant symmetric* (resp. alternating) *nondegenerate K-bilinear form on V. Then there exists a k-bilinear form on V_k with the same properties.*

Assume now that 2 is invertible in R, i.e. $\mathrm{char}(k) \neq 2$, and hence $\mathrm{char}(K) \neq 2$. We may identify symmetric bilinear forms with quadratic forms. The symmetric part of th.A can then be restated, and made more precise, as follows :

Theorem B. *Let q be a nondegenerate G-invariant quadratic form on V. There exists a nondegenerate G-invariant quadratic form on V_k, whose class in the Witt ring $W(k)$ of k is the sum of the two Springer residues $\partial_1(q), \partial_2(q)$ of q.*

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[For the definition of the Springer residues $\partial_1, \partial_2 : W(K) \to W(k)$, see §3.3.]

As mentioned above, we shall prove a generalized form of these two theorems, the generalization consisting in replacing the action of a group by the action of an R-algebra with involution (th.5.1.4 and th.5.1.7).

The first three sections are about lattices; we define and give the main properties of what we call the *lower middle* and the *upper middle* of two lattices. The last two sections give the proof of theorems A and B.

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§0. Notation

0.1. Lower middle and upper middle of a pair of integers

Let x, y be two integers.

The *lower middle* $m_-(x, y)$ of (x, y) is defined as the largest integer $\leqslant \frac{x+y}{2}$:

(0.1.1) $m_-(x, y) = \lfloor \frac{x+y}{2} \rfloor$.

The *upper middle* $m_+(x, y)$ of (x, y) is the smallest integer $\geqslant \frac{x+y}{2}$:

(0.1.2) $m_+(x, y) = \lceil \frac{x+y}{2} \rceil$.

[This is similar to calling Wednesday and Thursday the *middle days* of the week.]

We have :

(0.1.3) $m_-(x, y) = \frac{x+y}{2} = m_+(x, y)$ if $x \equiv y \pmod 2$

(0.1.4) $m_-(x, y) < \frac{x+y}{2} < m_+(x, y)$ and $m_+(x, y) = m_-(x, y) + 1$ if $x \equiv y + 1 \pmod 2$.

(0.1.5) $m_+(-x, -y) = -m_-(x, y)$.

(0.1.6) $m_-(x, y) = \sup_{n \in \mathbf{Z}} \ \inf(x - n, y + n)$.

(0.1.7) $m_+(x, y) = \inf_{n \in \mathbf{Z}} \ \sup(x - n, y + n)$.

(0.1.8) $m_-(x + 1, y) = m_+(x, y)$ and $m_+(x + 1, y) = m_-(x, y) + 1$.

0.2. Discrete valuations

We keep the same notation as in the introduction : K is a field with a discrete valuation $v : K^\times \to \mathbf{Z}$, which is extended to K by putting $v(0) = +\infty$. The valuation ring R is the set of all $x \in K$ with $v(x) \geqslant 0$; the maximal ideal $\mathfrak{m}$ of R is the set of all $x \in K$ with $v(x) \geqslant 1$; we choose a generator π of $\mathfrak{m}$; we have $v(\pi) = 1$. The residue field is $k = R/\mathfrak{m} = R/\pi R$.

The letter V denotes a finite dimensional K-vector space.

§1. The lower and upper middles of a pair of lattices

1.1. Definitions

Recall that a *lattice* of V is a free R-submodule L of V such that the natural map $K \otimes_R L \to V$ is an isomorphism.

Let L and M be two lattices. Then $L \cap M$ is a lattice, and so is :

$$L + M = \text{set of all } x + y, \text{ with } x \in L, y \in M.$$

We now define two lattices $m_-(L, M)$ and $m_+(L, M)$, which are sandwiched between $L \cap M$ and $L + M$; we call them the *lower middle* and *upper middle* of L and M. (We shall see in prop.1.1.5 below how they are related to the "middles" of §0.1.) They are defined as follows :

(1.1.1) $\quad m_-(L, M) = +_{n \in \mathbf{Z}} \; (\pi^n L \; \cap \; \pi^{-n} M)$

(1.1.2) $\quad m_+(L, M) = \bigcap_{n \in \mathbf{Z}} \; (\pi^n L \; + \; \pi^{-n} M).$

[In these formulas, the $+$ symbol means " submodule generated by ". For instance $m_-(L, M)$ is the R-submodule of V generated by the lattices $\pi^n L \; \cap \; \pi^{-n} M$.]

Note that L and M play a symmetric role : we have $m_\pm(L, M) = m_\pm(M, L)$.

Remark.
In (1.1.1) and (1.1.2), it is not necessary to run n through the full set $\mathbf{Z}$; a finite subset suffices. For instance, in the case of (1.1.1), if n is large enough, then $\pi^n L$ is contained in $L \cap M$, hence the nth term $\pi^n L \; \cap \; \pi^{-n} M$ is contained in the 0th-one; we may delete it from (1.1.1) without changing the sum. The same is true if $-n$ is large enough, by a similar argument. Same thing for (1.1.2). Hence the above infinite sums and intersections can be replaced by finite ones; this shows that $m_+(L, M)$ and $m_+(L, M)$ are lattices.

As an example, suppose that $\pi^2 L \subset M$ and $\pi^2 M \subset L$. Then the terms with $|n| > 1$ can be deleted, and the formulas reduce to :

$m_-(L, M) = \quad \pi L \cap \pi^{-1} M \quad + \quad L \cap M \quad + \quad \pi^{-1} L \cap \pi M,$

$m_+(L, M) = \quad (\pi L + \pi^{-1} M) \; \cap (L + M) \; \cap \; (\pi^{-1} L + \pi M).$

More generally, if $a \geqslant 0$ is such that $\pi^a L \subset M$ and $\pi^a M \subset L$, the terms with $|n| > a/2$ can be deleted.

Example : *middles of twisted lattices*

If L is a lattice, and if $a \in \mathbf{Z}$, the *a-twist* $L(a)$ of L is defined as :

(1.1.3) $\quad L(a) = \pi^{-a} L.$

We have $a \leqslant b \Rightarrow L(a) \subset L(b)$, and :

(1.1.4) $\quad L(a) + L(b) = L(\sup(a, b)) \quad \text{and} \quad L(a) \cap L(b) = L(\inf(a, b)).$

Proposition 1.1.5. *If $x, y \in \mathbf{Z}$, then* :

(1.1.6) $\quad m_-(L(x), L(y)) = L(m_-(x, y)),$

(1.1.7) $\quad m_+(L(x), L(y)) = L(m_+(x, y))$

[In other words : on the set of all twists of a given lattice, the "middle" operations coincide with those defined on $\mathbf{Z}$ in §0.1.]

Proof.
This follows from (1.1.4), combined with (0.1.6) and (0.1.7).

1.2. **Basic properties of the lower and upper middles of two lattices**

As above, let L and M be two lattices of V.

Proposition 1.2.1. *We have* :

(1.2.2) $L \cap M \ \subset \ m_-(L, M) \ \subset m_+(L, M) \ \subset \ L + M$.

Proof.

The inclusions $L \cap M \ \subset \ m_-(L, M)$ and $m_+(L, M) \ \subset \ L + M$ are clear.
Let us show that $m_-(L, M) \ \subset m_+(L, M)$. Note first that we have :

(1.2.3) $\pi^a L \cap \pi^{-a} M \ \subset \ \pi^b L + \pi^{-b} M$ for every $a, b \in \mathbf{Z}$.

Indeed, if $a \geqslant b$, this follows from :

$$\pi^a L \cap \pi^{-a} M \ \subset \pi^a L \ \subset \pi^b L \ \subset \ \pi^b L + \pi^{-b} M \, ;$$

similarly, if $b \geqslant a$, (1.2.3) follows from :

$$\pi^a L \cap \pi^{-a} M \ \subset \pi^{-a} M \ \subset \pi^{-b} M \ \subset \ \pi^b L + \pi^{-b} M.$$

Since $m_-(L, M)$ is generated by the $\pi^a L \cap \pi^{-a} M$, formula (1.2.3) shows that
it is contained in every $\pi^b L + \pi^{-b} M$, hence also in their intersection, which is
$m_+(L, M)$.

Remark.

The R-modules $(L + M)/m_+(L, M)$ and $m_-(L, M)/(L \cap M)$ are (non canoni-
cally) isomorphic ; this is proved by the method of reduction to dimension 1
used in the proof of th. 1.2.4 below.
Similarly, we have $(L + M)/m_-(L, M) \ \simeq \ m_+(L, M)/(L \cap M)$.

Theorem 1.2.4. *We have* $\pi.m_+(L, M) \ \subset \ m_-(L, M)$.

Proof.

Suppose first that $\dim V = 1$. In that case, all the lattices of V are twists of one
of them. By prop.1.1.5, the formula $\pi.m_+(L, M) \ \subset \ m_-(L, M)$ is equivalent to
the obvious formula $-1 + m_+(x, y) \leqslant m_-(x, y)$ for $x, y \in \mathbf{Z}$.

We now reduce the general case to that one. To do so, let $V = \oplus V_i$ be
a splitting of V as a direct sum of subspaces V_i. We say that this splitting is
compatible with a lattice P if $P = \oplus P_i$, where $P_i = P \cap V_i$. If $V = \oplus V_i$ is
compatible with two lattices L and M, then the same is true for $L \cap M$, and we
have $(L \cap M)_i = L_i \cap M_i$; same for $L + M$, $\pi^a L \cap \pi^b M$, $m_\pm(L, M)$. Hence, *if
th. 1.2.4 is true for the V_i, it is true for V*.

It only remains to prove :

Lemma 1.2.5. *There exists a splitting* $V = \oplus V_i$, *with* $\dim V_i = 1$ *for every i,
which is compatible with both L and M.*

Proof.

By replacing M by a suitable $\pi^n M$, we may assume that $M \subset L$. Since R is a
principal ideal domain, there exists an R-basis (x_i) of L, and nonzero elements
a_i of R such that the $(a_i x_i)$ make up a basis of M, cf. e.g. [A VII, §4, th.1].
The splitting $V = \oplus K e_i$ has the required properties : one has $L = \oplus R e_i$ and
$M = \oplus R a_i e_i$.

Remark.

Let us mention three formulas which can also be proved by reduction to dimen-
sion 1 :

(1.2.6) $m_-(\pi L, M) = \pi.m_+(L, M),$

(1.2.7) $m_+(\pi L, M) = m_-(L, M),$

(1.2.8) $m_\pm(L, M) = m_\pm(L + M, L \cap M).$

Building interpretation.

When R is complete for the π-adic topology, the lattices of V may be viewed as the vertices of the *affine building* X associated by Goldman-Iwahori to $\mathrm{GL}(V)$, cf. [GI 63] and [Ge 81]; the space X is isomorphic to the product of $\mathbf{R}$ by the Bruhat-Tits building of $\mathrm{SL}(V)$, cf. [BT 84]; the apartments of X correspond to the splittings of V as direct sums of 1-dimensional subspaces, and lemma 1.2.5 is a special case of the fact that any two points of X are contained in an apartment, cf. [Ge, 2.3.4]. If L, M are two lattices, and $[L], [M]$ are the corresponding vertices of X, any barycenter $x[L] + y[M]$, with x, y real $\geqslant 0$, $x + y = 1$, makes sense as a point of X (such barycenters make up the *geodesic segment* joining $[L]$ to $[M]$). In particular, $\frac{1}{2}[L] + \frac{1}{2}[M]$ makes sense; it is the middle (in the standard meaning of the word) of that geodesic segment. This middle point is not always a vertex, i.e., it does not always correspond to a lattice[1]. One can check that it is indeed a vertex if and only if $m_+(L, M)$ and $m_-(L, M)$ coincide, and in that case we have the good-looking formula :

$$\tfrac{1}{2}[L] + \tfrac{1}{2}[M] = [m_+(L, M)] = [m_-(L, M)].$$

1.3. The lower and upper middle submodules of a torsion R-module

[The content of this section will not be used in the rest of this paper.]

Let T be a torsion R-module of finite exponent, i.e., an R-module such that there exists $N \geqslant 1$ with $\pi^N T = 0$. Let us define its *lower middle* and *upper middle* submodules by the following formulas :

(1.3.1) $m_-(T) = +_{n \geqslant 0} (\mathrm{Im}\ \pi_T^n\ \cap\ \mathrm{Ker}\ \pi_T^n),$

(1.3.2) $m_+(T) = \bigcap_{n \geqslant 0} (\mathrm{Im}\ \pi_T^n\ +\ \mathrm{Ker}\ \pi_T^n),$

where π_T is the endomorphism of T defined by π. In these formulas, it is enough to let n run from 1 to $N - 1$: larger n's contribute nothing, since then $\pi_T^n = 0$, and $n = 0$ does not give anything either.

The properties of $m_-(T)$ and $m_+(T)$ are analogous to the ones of §1.1 and §1.2. One may sum them up by :

Proposition 1.3.3. *We have* :

(1.3.4) $\pi.m_+(T) \subset m_-(T) \subset m_+(T).$

(1.3.5) $m_+(T)\ \simeq\ T/m_-(T).$

(1.3.6) $m_-(T)\ \simeq\ T/m_+(T).$

The proof is analogous to that of th.1.2.4 : check first the case where T is cyclic, i.e., $T \simeq R/\pi^m R$ for some $m \leqslant N$, and then use the well known fact that T is a direct sum (finite or infinite) of cyclic submodules.

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1. But it does after extension of scalars to $K(\sqrt{\pi})$.

This construction is related to that of §1.1 and §1.2 : if L and M are two lattices of V, we have $m_\pm(T) = m_\pm(L, M)/(L \cap M)$, where $T = (L+M)/(L \cap M)$.

Application to isogenies between abelian varieties

Let A and B be abelian varieties over a field F, and let $\varphi : A \to B$ be an isogeny. Assume that $\deg(\varphi)$ is a power of a prime number ℓ distinct from $\operatorname{char}(F)$. Let F^s be a separable closure of F, and let $T = \operatorname{Ker} A(F^s) \to B(F^s)$. By assumption, T is a finite abelian ℓ-group of order $\deg(\varphi)$, hence is a torsion module over $\mathbf{Z}_\ell$. Let $m_-(T), m_+(T)$ be the middles of T. The Galois group $\operatorname{Gal}(F^s/F)$ acts on T, and stabilizes $m_-(T)$ and $m_+(T)$. Thus, there is a factorization of φ as $A \to A_- \to A_+ \to B$, where A_- and A_+ are abelian varieties over F, and the kernel of $A(F^s) \to A_\pm(F^s)$ is $m_\pm(T)$; by (1.3.4), the kernel of $A_- \to A_+$ is killed by ℓ.

An interesting special case is when B is the dual variety of A, and φ is a polarization of degree a power of ℓ. In that case, one can show that A_- and A_+ are dual of each other. Hence, *if an abelian variety over F has a polarization of ℓ-power degree, there is an F-isogenous one which has the same property, and for which the polarization kernel is killed by ℓ.*

§2. The Brauer-Nesbitt theorem

2.1. Notation

Let A be an R-algebra and let $(a, x) \mapsto ax$ be an R-bilinear map $A \times V \to V$ which makes V into an A-module.

[If A is the group algebra $R[G]$ of a group G, this means that G acts linearly on V.]

Lemma 2.1.1. *The following properties are equivalent* :

(2.1.2) *There exists a lattice L of V with is stable under the action of A, i.e., $A.L = L$.*

(2.1.3) *The image of A in $\operatorname{End}(V)$ is a finitely generated R-module .*

Proof.

Let A_V be the image of A in $\operatorname{End}(V)$. If L is as in (2.1.2), then we have $A_V \subset \operatorname{End}_R(L)$, and (2.1.3) follows. Conversely, if A_V is finitely generated, let M be a lattice, and let $L = A_V.M$; then L is a finitely generated R-module which contains M ; hence it is a lattice such that $A.L = L$.

If (2.1.2) and (2.1.3) hold, we shall say that the action of A on V is *bounded*. A lattice L with $A.L = L$ will be called an *A-lattice*.

2.2. Semisimplification

Let L be an A-lattice ; then $E_L = L/\pi L$ is a module over the k-algebra $A_k = A/\pi A$. Let $S_1, ..., S_m$ be the successive quotients of a Jordan-Hölder filtration of E_L ; the direct sum $E_L^{\mathrm{ss}} = S_1 \oplus ... \oplus S_m$ is a semisimple A_k-module which is independent (up to isomorphism) of the chosen Jordan-Hölder filtration ; that module is called the *semisimplification* of E_L.

Theorem 2.2.1 (Brauer-Nesbitt). *Let L and M be two A-lattices of V. Then the A_k-modules E_L^{ss} and E_M^{ss} are isomorphic.*

This was proved by Brauer and Nesbitt when A is the R-algebra of a finite group. The proof in the general case is similar. There are two steps :

(2.2.2) *The special case where $\pi L \subset M \subset L$.*

Let $T = L/M$. We have an exact sequence of A_k-modules :

(2.2.3) $\qquad\qquad 0 \to T \to E_M \to E_L \to T \to 0,$

where the map $T \to E_M$ is induced by $x \mapsto \pi x$. This implies an isomorphism :

(2.2.4) $\qquad\qquad T^{\mathrm{ss}} \oplus E_L^{\mathrm{ss}} \simeq E_M^{\mathrm{ss}} \oplus T^{\mathrm{ss}} \,;$

hence $E_L^{\mathrm{ss}} \simeq E_M^{\mathrm{ss}}$.

(2.2.5) *Reduction of the general case to the special case.*

Replacing M by a multiple $\pi^a M$ does not change E_M^{ss} ; hence we may assume that $M \subset L$. There exists $n \geqslant 0$ such that $\pi^n L \subset M$; choose such an n and use induction on n. The case $n = 0$ is trivial since $M = L$; if $n > 0$, define $N = \pi^{n-1}L + M$. Since $\pi^{n-1}L \subset N \subset L$, the induction hypothesis implies that $E_L^{\mathrm{ss}} \simeq E_N^{\mathrm{ss}}$. Since $\pi N \subset M \subset N$, we have $E_M^{\mathrm{ss}} \simeq E_N^{\mathrm{ss}}$ by (2.2.2). Hence $E_L^{\mathrm{ss}} \simeq E_M^{\mathrm{ss}}$.

Alternate proof.

Since E_L^{ss} and E_M^{ss} are semisimple, to prove that they are isomorphic it is enough to show that, for every $a \in A$, the characteristic polynomials of a, acting on these two modules, are the same (this criterion, for group algebras of finite groups, is due to Brauer - for the general case, see Bourbaki [A VIII.377, §20, th.2]) ; but this is clear, since these polynomials are the reduction mod π of the characteristic polynomial of a acting on V.

Notation. The A_k-module E_L^{ss} will be called the *reduction* mod π of V ; we shall denote it by V_k. It is defined up to a non canonical isomorphism.

§3. Bilinear forms and lattices

Let $B(x, y)$ be a nondegenerate K-bilinear form on V, which is ϵ-symmetric, with $\epsilon = \pm 1$, i.e., $B(x, y) = \epsilon B(y, x)$ if $x, y \in V$.

3.1. Dual lattices

Let L be a lattice of V, and let L' be the *dual lattice* of L, namely the set of all $x \in V$ such that $B(x, y) \in R$ for every $y \in L$. If $x \in L'$, the map $y \mapsto B(x, y)$ is R-linear ; this gives an isomorphism $L' \to \mathrm{Hom}_R(L, R)$; we may thus identify L' with the usual "R-dual" of L. We have $(L')' = L$.

A lattice L is *self-dual* (or *unimodular*) if $L' = L$; if $(e_1, ..., e_n)$ is an R-basis of L, this means that both the matrix $\mathsf{B} = (B(e_i, e_j))$ and its inverse have coefficients in R, i.e., we have $\mathsf{B} \in \mathrm{GL}_n(R)$.

We say that L is *almost self-dual* if $\pi L' \subset L \subset L'$, i.e., if the matrices B et $\pi \mathsf{B}^{-1}$ have coefficients in R.

Theorem 3.1.1 *Let L be a lattice and let L' be its dual. Then the lower middle $m_-(L, L')$ of L and L' is an almost self-dual lattice whose dual is $m_+(L, L')$.*

Proof.

The functor "lattice $\mapsto$ dual lattice" transforms finite intersections into finite

sums, and conversely. By (1.1.1) and (1.1.2), this shows that $m_-(L, L')$ and $m_+(L, L')$ are dual of each other. By prop.1.2.1 and th.1.2.4, we have

$$\pi.m_+(L, L') \subset m_-(L, L') \subset m_+(L, L'),$$

hence $m_-(L, L')$ is almost self-dual.

3.2. The bilinear forms b_1 and b_2 associated with an almost self-dual lattice

Let L be an almost self-dual lattice. The inclusions $\pi L' \subset L \subset L'$ give an exact sequence of k-vector spaces :

(3.2.1) $0 \to L'/L \to L/\pi L \to L/\pi L' \to 0,$

where the map $L'/L \to L/\pi L$ is given by $x \mapsto \pi x$.

By passage to quotients, the bilinear form B gives a k-bilinear forms b_1 on $L/\pi L'$; similarly, πB defines a k-bilinear form b_2 on L'/L. These forms are ϵ-symmetric, and nondegenerate.

3.3. The quadratic case, and the Springer residues

Let us assume now that $\epsilon = 1$, i.e., that B is symmetric. Assume also that $\mathrm{char}(k) \neq 2$, i.e., that 2 is invertible in R. We may thus identify symmetric bilinear forms and quadratic forms. Let $q(x) = B(x, x)$ be the quadratic form defined by b, and let $[q]$ be its image in the Witt ring $W(K)$ of the field K (for the definition and basic properties of the Witt ring, see [La 05, chap.I-II]). Similarly, let q_1, q_2 be the quadratic forms defined by the k-bilinear forms b_1, b_2 of §3.2, and let $[q_1], [q_2]$ be their images in the Witt ring $W(k)$.

Recall (cf. [Sp 55], [La 05, chap.VI]) that Springer has defined two "residue" maps

$$\partial_1, \partial_2 : W(K) \ \to \ W(k).$$

The map ∂_1 is a ring homomorphism; the map ∂_2 is additive (and depends on the choice of the uniformizing element π). They are characterized by these properties, together with their values for 1-dimensional quadratic forms, which are as follows :

(3.3.1) If $u \in R^\times$ has image $\overline{u}$ in k, then the images of the 1-dimensional form $\langle u \rangle$ by ∂_1, ∂_2 are $\langle \overline{u} \rangle, 0$, and those of $\langle u\pi \rangle$ are $0, \langle \overline{u} \rangle$.

The map $(\partial_1, \partial_2) : W(K) \to W(k) \times W(k)$ is surjective; it is bijective if K is complete.

Theorem 3.3.2. *Let L be an almost self-dual lattice, and let q_1, q_2 be the corresponding quadratic forms over k. Then $\partial_1([q]) = [q_1]$ and $\partial_2([q]) = [q_2]$.*

Proof.
We use the same method as for th.1.2.4, namely reduction to dimension 1 (in which case the formulas are obvious). What is needed is the following orthogonal analogue of lemma 1.2.5 :

Lemma 3.3.3. *If M is any lattice of V, there exists an orthogonal splitting $V = \oplus V_i$, with $\dim V_i = 1$ for every i, which is compatible with M.*

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[The orthogonality assumption means that V_i and V_j are orthogonal for the bilinear form b if $i \neq j$. It implies that the splitting is compatible with the dual M' of M.]

Proof of the lemma.

Use induction on $\dim V$. Let $m = \inf_{x \in M} v(q(x))$; we have $m > -\infty$. Choose $x \in M$ with $v(q(x)) = m$. For $y, z \in M$, we have $v(B(y,z)) \geqslant m$: this follows from the formula $2B(y,z) = q(y+z) - q(y) - q(z)$ since the valuations of $q(y+z), q(y), q(z)$ are $\geqslant m$. If $y \in V$, put $\ell(y) = B(x,y)/B(x,x)$. The linear form $\ell : V \to K$ is such that $\ell(x) = 1$, and it maps M onto R. It thus gives a splitting of V as $Kx \oplus \operatorname{Ker} \ell$, namely $y \mapsto (\ell(y)x, y - \ell(y)x)$; this splitting is compatible with M. It is an orthogonal splitting, since $b(y - \ell(y)x, x) = 0$ for every y. The lemma follows by applying the induction assumption to the vector space $\operatorname{Ker} \ell$ and its lattice $M \cap \operatorname{Ker} \ell$.

Corollary 3.3.4. *The classes in $W(k)$ of the two quadratic forms associated with an almost self-dual lattice do not depend on the choice of that lattice.*

Remark. The fact that the class of q_1 in $W(k)$ is the same for any two almost self-dual lattices L and M does not imply that these two quadratic forms have the same dimension : they may differ by hyperbolic factors. But, if they do have the same dimension, a theorem of Bayer-Fluckiger and First shows that L and M are *isomorphic as quadratic R-modules* ([BF 17], th.4.1) ; hence they are conjugate of each other by an element of the orthogonal group of (V, q).

§4. Semisimplification of symplectic and quadratic modules over an algebra with involution

This section is essentially independent of §§1,2,3 : we work over a field, and not over a discrete valuation ring.

4.1. Notation

Let k be a field, and let A_k be a k-algebra with a k-linear involution denoted by $a \mapsto a^*$; we have $a^{**} = a$ and $(ab)^* = b^* a^*$ for every $a, b \in A_k$.

Let E be an A_k-module which is finite dimensional over k (i.e., a "linear representation" of A_k), and let b be a nondegenerate bilinear form on E with the following property :

(4.1.1) $b(ax, y) = b(x, a^*y)$ for every $a \in A_k$, $x, y \in E$.

We then say that b is *compatible* with the A_k-module structure of E. When A_k is the group algebra $k[G]$ of a group G, with its canonical involution $g^* = g^{-1}$ for every $g \in G$, this means that b is invariant by G.

We assume one of the following :

(4.1.2) b is alternating.

(4.1.3) b is symmetric and $\operatorname{char}(k) \neq 2$.

In the first case, E is called a *symplectic A_k-module*, and in the second case, it is called an *orthogonal A_k-module*.

4.2. Semisimplification

Let A_k, E, b be as above, and let E^{ss} be the semisimplification of the A_k-module E.

Theorem 4.2.1. *In case* (4.1.3) (*resp. in case* (4.1.2)), *there exists a symmetric* (*resp. alternating*) *bilinear form on* E^{ss} *with the following two properties* :

    (4.2.2) *It is compatible with the* A_k-*module structure of* E^{ss}.

    (4.2.3) *It is isomorphic to* b.

One may sum up (4.2.2) by saying that the semisimplification of a quadratic (resp. alternating) module is quadratic (resp. alternating). As for (4.2.3), it is obvious in the symplectic case, since all nondegenerate alternating forms of a given rank are isomorphic; in the orthogonal case, it means that the corresponding quadratic forms have the same class in the Witt ring $W(k)$.

Proof.
We use the method of [Th 84, §2]. Let S be an A_k-submodule of E, which is totally isotropic for b (i.e. $b(x, y) = 0$ for all $x, y \in E$), and is maximal for that property. Let $S_\perp$ be its orthogonal relative to b; because of (4.1.2), it is a submodule of E, and we have $0 \subset S \subset S_\perp \subset E$. The form b defines a nondegenerate form b_1 on $S_\perp/S$.

Lemma 4.2.4. (i) *The* A_k-*module* $X = S_\perp/S$ *is semisimple, and the only totally isotropic submodule of* X *is* 0.

    (ii) *In the orthogonal case* (4.1.3), *the form* b *is isomorphic to the direct sum of* b_1 *and an hyperbolic form of rank* $2 \dim S$.

Proof of (i).
If Y is a totally isotropic submodule of X, its inverse image in $S_\perp$ is totally isotropic, hence equal to S, i.e., $Y = 0$. If Z is a submodule of X, then $Y = Z \cap Z_\perp$ is totally isotropic, hence 0; we have $X = Z \oplus Z_\perp$; this shows that every submodule of X is a direct summand, i.e., X is semisimple.

Proof of (ii).
Let M be a subvector space of $S_\perp$ such that $S_\perp = S \oplus M$. The restriction of b to M is nondegenerate, and the projection $M \to S_\perp/S$ is an isomorphism of quadratic spaces; we have $E = M \oplus M_\perp$, and $\dim M_\perp = 2 \dim S$. The form b splits as $b_1 \oplus h$, where h is the quadratic form of $M_\perp$; the form h is hyperbolic, since that space contains the totally isotropic subspace S, of dimension $\frac{1}{2} \dim M$. This proves (ii).

End of the proof of th.4.2.1.
The bilinear form b defines a duality between S and $E/S_\perp$; hence we may identify $E/S_\perp$ with the k-dual S' of S (with its natural A_k-structure). We have

$$E^{\mathrm{ss}} = X \oplus (S^{\mathrm{ss}} \oplus S'^{\mathrm{ss}}),$$

since X is semisimple, cf. 4.2.4 (i). It is easy to see that S'^{ss} is isomorphic to the dual of S^{ss}. Hence, $E^{\mathrm{ss}} \simeq X \oplus (Y \oplus Y')$, where $Y = S^{\mathrm{ss}}$. We then put on E^{ss} the bilinear form which is the direct sum of the form b_1 on X and the natural bilinear form (symmetric or alternating, as needed) on $Y \oplus Y'$. By lemma 4.2.4, that form has properties (4.2.2) and (4.2.3).

§5. Proof of theorem B

We now prove the theorems stated in the introduction; the proofs will merely consist in putting together the results of §§2,3,4.

5.1. The setting

We go back to the standard notation (K, R, V) and we assume that V is a module over an R-algebra A with involution. We also assume that this action is *bounded* (cf. §2.1), i.e., that there exists a lattice of V which is stable under A.

Let B be a nondegenerate K-bilinear form on V, which is compatible with the action of A; this means (as in (4.1.1)) :

(5.1.1) $B(ax, y) = B(x, a^*y)$ for every $a \in A$, $x, y \in V$.

As in §4, we assume one of the following :

(5.1.2) B is alternating.

(5.1.3) B is symmetric, and $\operatorname{char}(k) \neq 2$.

By the Brauer-Nesbitt theorem (see §2.2), the semisimplification V_k of V is well-defined; it is a module over the k-algebra $A_k = A/\pi A$.

Theorem 5.1.4. *There exists a nondegenerate k-bilinear form b on V_k with the following properties* :

(5.1.5) *It is compatible with the action of A_k, i.e., it satisfies condition* (4.1.1).

(5.1.6) *It is alternating (resp. symmetric) if B is.*

[More shortly : if V is a symplectic (resp. orthogonal) module, so is V_k].

Note that condition (5.1.5) alone would be easy to satisfy : since V is isomorphic to its dual, the same is true for V_k, and that is equivalent to the existence of a bilinear form b compatible with the action of A_k.

In the orthogonal case (5.1.3), one may ask more of the form b. To state it, let us denote by q (instead of $[q]$) the Witt class in $W(K)$ of the quadratic form $q(x) = B(x, x)$, and let $q_1, q_2 \in W(k)$ be its images by the Springer residue maps, cf. §3.3. Then :

Theorem 5.1.7. *The form b of th.5.1.4 can be chosen to have the following property* :

(5.1.8) *There exists an A_k-orthogonal splitting $V_k = E_1 \oplus E_2$ such that the Witt class of the restriction to E_i $(i = 1, 2)$ of $x \mapsto b(x, x)$ is q_i.*

Note that (5.1.8) implies that the Witt class of the quadratic form $b(x, x)$ on V_k is $q_1 + q_2$. When A is a group algebra $R[G]$, we recover th.B of the Introduction. Similarly, th.5.1.4 applied to $R[G]$ gives th.A.

Theorems 5.1.4 and 5.1.7 will be proved in §5.3.

5.2. Existence of almost self-dual A-lattices

Theorem 5.2.1. *There exists an A-lattice of V which is almost self-dual* (cf. §3.1) *relative to the bilinear form B.*

Proof.

Let L be an A-lattice of V. Condition (5.1.1) shows that the dual L' of L is also an A-lattice, and so are the lattices $\pi^n L, \pi^n L'$, and hence also the lower middle $m_-(L, L')$, which is almost self-dual by th.3.1.1.

Alternate proof.

The proof above uses the "middle" notions of §§1,2. Here is a direct proof, taken from [Th 84] :

Choose an A-lattice L, with $L \subset L'$, and which is maximal for that property. Let m be the smallest integer such that $\pi^m L' \subset L$. If $m = 0$, or 1, then L is self-dual, or almost self-dual. Let us show that $m \geqslant 2$ is impossible. Define $M = \pi^{m-1} L' + L$; we have $M' = \pi^{1-m} L \cap L'$. The inequality $m \geqslant 2$ implies $\pi^{m-1} L' \subset \pi^{1-m} L$, hence $\pi^{m-1} L' \subset M'$; since $L \subset M'$, this shows that M is contained in M'. The maximality of L then implies $M = L$, hence $\pi^{m-1} L' \subset L$, which contradicts the minimality of m.

Remark. This short proof is not in fact very different from the first one. Indeed, it amounts to construct an almost self-dual A-lattice, starting with any lattice L contained in its dual, by choosing m with $\pi^m L' \subset L$, then replacing L by $\pi^{m-1} L' + L$, and iterating until one gets $m = 0$ or $m = 1$. But, if one writes down the end result of that process, a simple computation shows that one finds the lower middle lattice $m_-(L, L')$, exactly as in the first proof.

There is however a definite advantage in using a maximal lattice : in that case, L'/L does not contain any nonzero totally isotropic submodule, hence it is a semi-simple A_k-module, cf. lemma 4.2.4. This simplifies the proofs of the next section.

5.3. **Proof of th.5.1.4 and th.5.1.7**

Let L be an almost self-dual A-lattice, cf. th.5.2.1. Let $F_1 = L/\pi L'$, $F_2 = L'/L$. By (3.2.1), we have an exact sequence of A_k-modules :

(5.3.1) $0 \to F_2 \to L/\pi L \to F_1 \to 0$.

Let $E_1 = F_1^{\mathrm{ss}}, E_2 = F_2^{\mathrm{ss}}$,, and let $V_k = (L/\pi L)^{\mathrm{ss}}$. The exact sequence above gives a splitting :

(5.3.2) $V_k = E_1 \oplus E_2$.

As explained in §3.2, the bilinear form B defines k-bilinear forms b_1 and b_2 on F_1 and F_2 ; these forms are compatible with the action of A_k and they are alternating (resp. symmetric) if B is alternating (resp. symmetric). By th.4.2.1, applied to F_1 and F_2, there exist A_k-compatible forms b'_1, b'_2 on E_1, E_2 which are isomorphic (as bilinear forms) to b_1, b_2 ; in the orthogonal case, th.3.3.1 shows that the Witt classes of these forms are the Springer residues of q. Using (5.3.2), we define a bilinear form on V_k as the orthogonal sum of b'_1 on E_1 and b'_2 on E_2. All the properties of th.5.1.4 and th.5.1.7. hold.

This concludes the proof.

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