Some aspects of the Sato-Tate conjecture

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The Sato-Tate conjecture is concerned with the distribution of the number of solutions mod p of a given system of equations, when p varies. The lecture was meant as an introduction to the next two, by K. Kedlaya and D. Kohel. We shall concentrate on the general aspects of the conjecture ; for more details, the reader is referred to [3, Chap.8].

Using Hironaka's resolution of singularities, together with Grothendieck and Deligne's results on ℓ-adic cohomology, the question can be reduced to the following.

Let $X^i, i \in I$, be a finite family of smooth projective varieties over $\mathbf{Q}$ [1]. For each $i \in I$, let n_i be a positive integer. Choose a finite set S of primes such that the X^i have good reduction outside S ; for every $p \notin S$, let us denote by $t_i(p)$ the trace of the geometric Frobenius at p, acting on the n_i-th cohomology of $X^i_{\overline{\mathbf{Q}}}$. If B^i denotes the Betti number dim $H^{n_i}(X^i_{\overline{\mathbf{Q}}})$, Deligne's theorem shows that $f^i(p) = t_i(p)/p^{i/2}$ belongs to the interval $T^i = [-B^i, +B^i]$. If we denote by T the box $\prod_i T^i$, we then have a map

$$f : P - S \to T,$$

where P is the set of all prime numbers.

The general Sato-Tate conjecture (see e.g. [2, §13]) predicts that the $f(p)$ are *equidistributed* with respect to a positive Radon measure μ on the space T. This means that, for every continuous function φ on T, we have

$$(*) \qquad \int_T \varphi(t)\mu(t) = \lim_{x\to\infty} \frac{1}{\pi_S(x)} \sum_{p\leqslant x} \varphi(f(p)),$$

where $\pi_S(x) \sim x/\log x$ is the number of $p \in P - S$ with $p \leqslant x$. Equivalently, μ is the limit, for the weak topology, of the mean values of the Dirac measures

1. Instead of $\mathbf{Q}$ one could take any finitely generated extension of $\mathbf{Q}$, for instance any number field.

at the points $f(p)$, for $p \leqslant x$ and $p \in P - S$. This can also be translated in terms of subsets of T, as follows : if $A \subset T$, define $N_A(x)$ as the number of $p \in P - S$ such that $f(p) \in A$ and $p \leqslant x$; assume that A is μ-quarrable, i.e. that its boundary has μ-measure 0 ; then

$$\lim_{x\to\infty} N_A(x)/\pi_S(x) \;=\; \mu(A).$$

The measure μ has some remarkable properties, for instance :

– It is invariant by the automorphism $(t_i) \mapsto ((-1)^{n_i} t_i)$ of T.

– If $\varphi : T \to \mathbf{R}$ is a polynomial in the t_i with coefficients in $\mathbf{Z}$, then $\int_T \varphi(t)\mu(t)$ belongs to $\mathbf{Z}$.

These properties are direct consequences of a (conjectural - but well motivated, see [2]) construction of μ in terms of compact Lie groups and characters. More precisely, there should exist a compact real Lie group K (the *Sato-Tate group*), together with :

(i) for every $p \in P - S$, an element s_p of the set Cl K of conjugacy classes of K ;

(ii) for every $i \in I$ a continuous linear representation $\rho^i : K \to \mathbf{GL}_{B^i}(\mathbf{C})$.

These data should fulfill several conditions, the most important ones being :

(a) The s_p are equidistributed in Cl K with respect to the image of the normalized Haar measure of K.

(b) Tr $\rho^i(s_p) = t_i(p)$ for every $i \in I$ and every $p \in P - S$.

Conditions (a) and (b) imply the equidistribution stated at the beginning; more precisely, they show that the measure μ on T is the image of the Haar measure of K by the map $K \to T$ defined by the characters of the ρ^i.

The original Sato-Tate case ([4], [1]) is the one where I has one element, the corresponding X^i being an elliptic curve without complex multiplication and the integer n_i being chosen equal to 1, so that $B^i = 2$. In that case, the group K is $\mathbf{SU}_2(\mathbf{C})$, the space T is the interval $[-2, +2]$ and the data (i) and (ii) are defined in an obvious way. Property (b) is true by construction. The real problem is the equidistribution property (a), which was proved only recently (over $\mathbf{Q}$, and more generally over every totally real number field) ; see the references in [3, §8.1.5].

References

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[4] J. Tate, *Algebraic cycles and poles of zeta functions*, in *Arithmetic Algebraic Geometry* (*Proc. Conf. Purdue Univ., 1963*), Harper & Row, New York (1965), 93-110.

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