Cohomological invariants mod 2 of Weyl groups

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Let G be the Weyl group of a root system, i.e. a crystallographic finite Coxeter group, cf. [LIE], chap.VI, §4.1. Let k_0 be a field of characteristic $\neq 2$, let $H^\bullet = \oplus_{n\geqslant 0} H^n(k_0, \mathbf{Z}/2\mathbf{Z})$ and let $I_G = \mathrm{Inv}(G)_{k_0}$ be the ring of cohomological invariants mod 2 of G, as defined in [Se], §1 ; it is a graded $H^\bullet$-algebra. When G is of type A, G is a symmetric group Sym_N, and I_G is $H^\bullet$-free of rank $n = [N/2]$, with an explicit basis $w_0 = 1, w_1, ..., w_n$, cf. [Se], chap.VII.

In order to extend this description of I_G to the general case, define S_G to be the set of elements $g \in G$ with $g^2 = 1$; an element of S_G shall be called an *involution* of G. Let Σ be the set of conjugation classes of elements of S_G.

Theorem A. *There is a canonical map $e : \Sigma \to I_G$ whose image is an $H^\bullet$-basis of I_G.*

[Equivalently : the module I_G is canonically isomorphic to the set of all maps $\Sigma \to H^\bullet$.]

The map e is compatible with the grading of I_G : if $g \in G$ is an involution, let $\deg(g)$ be the multiplicity of -1 as an eigenvalue of g in the canonical linear representation of G as a Coxeter group ; let Σ_n be the set of involution classes of degree n. *If $\sigma \in \Sigma_n$, then $e(\sigma)$ belongs to the n-th component $H^n(k_0)$ of $H^\bullet$.*

Examples. 1. When $G = \mathrm{Sym}_N$, the elements of Σ are the conjugation classes of the product of i disjoint transpositions, with $2i \leqslant N$, and we recover the fact that $H^\bullet$-free of rank $n = [N/2]$, with a basis made up of elements of degree $0, 1, ..., [N/2]$, as above.
2. When $G = \mathrm{Weyl}(\mathsf{E}_8)$, we have $|\Sigma_n| = 1$ for $n \leqslant 8$, with the only exception of $n = 4$ where $\Sigma_n = 2$; and, of course, $|\Sigma_n| = 0$ for $n > 8$. Hence I_G is a free $H^\bullet$-module of rank 10, with a basis made up of elements of degree $0, 1, 2, 3, 4, 4, 5, 6, 7, 8$.
3. For E_7 and E_6, the degrees are $0, 1, 2, 3, 3, 4, 4, 5, 6, 7$ and $0, 1, 2, 3, 4$.

Definition of the map $e : \Sigma \to I_G$.

Let a be an element of I_G and let g be an involution of G of degree n. We first define a «scalar product» $\langle a, g\rangle$, which is an element of $H^\bullet$. To do so, choose a splitting $g = s_1 \cdots s_n$, where the s_i are commuting reflections (recall that a reflection is an involution of degree 1) ; such a splitting always exists. Let $C = \langle s_1, ..., s_n\rangle$ be the group generated by the s_i, and let $a_c \in I_C$ be the restriction of a to C. The algebra I_C has a natural basis (α_I) indexed by the subsets I of $[1, n]$, cf. [Se], §16.4. Let $a_C \in H^\bullet$ be the coefficient of the «top» element $\alpha_{[1,n]}$ in a, c. It is possible to show that a_C *is independent of the chosen splitting of g, i.e., that it only depends on* (a, g). We then define the scalar product $\langle a, g\rangle$ as a_C ; we have $\langle a, g\rangle = \langle a, g'\rangle$ if g and g' are conjugate in G ; this allows us to define $\langle a, \sigma\rangle$ for every $\sigma \in \Sigma_G$. For a given σ, the map $a \mapsto \langle a, \sigma\rangle$ is $H^\bullet$-linear ; if a has degree m, then $\langle a, \sigma\rangle$ has degree $m - n$ (one may view $a \mapsto \langle a, \sigma\rangle$ has an m-th fold residue).

Theorem B.

(i) *If* $\langle a,\sigma\rangle = 0$ *for every* σ, *then* $a = 0$.

(ii) *For every* $\sigma \in \Sigma_G$, *of degree* $n \geqslant 0$, *there exists an invariant* $a_\sigma \in I_G^n$ *such that* $\langle a_\sigma,\sigma\rangle = 1$ *and* $\langle a_\sigma,\sigma'\rangle = 0$ *for every* $\sigma' \neq \sigma$.

[Note that, by (i), such an a_σ is unique.]

It is clear that Theorem B implies Theorem A, by defining $e(\sigma)$ as a_σ.

Indications on the proof of part (i) of Theorem B.

An induction argument shows that, if $\langle a,\sigma\rangle = 0$ for every σ, then the restriction of a to every subgroup generated by commuting reflections is 0. In that case, if the characteristic of k_0 is ≪good≫ for G, the arguments of [Se], §25, apply without change. This already covers the case where the irreducible components of G are of classical type. The exceptional types can be reduced to that case, thanks to the fact that, if G is such a Weyl group, there exists a subgroup G' of G, generated by some reflections of G, which is a product of groups of classical type, and has *odd index* in G ; for instance Weyl(E_8) contains Weyl(D_8) with index 135, Weyl(E_6) contains Weyl(D_5) with index 27. One then uses the fact that the restriction map $I_G \to I_{G'}$ is injective, cf. [Se], prop.14.4.

Indications on the proof of part (ii) of Theorem B.

We need to construct enough cohomological invariants. For most Weyl groups, this is done by using Stiefel-Whitney classes. There are however three cases where we have to do otherwise . For each one, there are two distinct classes of involutions of the same degree n for which it is hard to find $a \in I_G^n$ with $\langle a,\sigma\rangle = 0,\ \langle a,\sigma'\rangle = 1$. These cases are : D_{2n}, $n \geqslant 3$; E_7, $n = 3$ and 4 ; E_8, $n = 4$.

For these cases, one uses the relation given by Milnor's conjecture (now Voevodsky's theorem). The method applies to every linear group $\mathcal{G}$. The ring $\mathrm{Inv}(\mathcal{G}, W)$ of Witt invariants of $\mathcal{G}$ (as defined in [Se], §27.3) has a natural filtration : an invariant h has filtration $\geqslant n$ is, for every $\mathcal{G}$-torsor t of $\mathcal{G}$ over any extension k/k_0, the element $h(t)$ of the Witt ring $W(k)$ belongs to the n-th power of the canonical ideal of $W(k)$; in that case, h defines (by the Milnor construction) an element a_h of $\mathrm{Inv}^n(\mathcal{G}, \mathbf{Z}/2\mathbf{Z})$, and $a_h = 0$ if and only if the filtration of h is $> n$. In other words, we have *an injective map :* $\mathrm{gr}\,\mathrm{Inv}(\mathcal{G}, W) \ \to\ \mathrm{Inv}(\mathcal{G}, \mathbf{Z}/2\mathbf{Z})$.

We apply this with $\mathcal{G} = G$, where G is as in the three cases above. One finds a linear orthogonal representation of G whose Brauer character χ is such that $\chi(\sigma) - \chi(\sigma') = 2^n$. This gives a G-quadratic form, hence an element of $\mathrm{Inv}(\mathcal{G}, W)$; one modifies slightly that element to make it of filtration $\geqslant n$, so that it gives a cohomological invariant a of G of degree n, and one checks that $\langle a,\sigma\rangle - \langle a,\sigma'\rangle = 1$; that information is enough to conclude the proof.

References

[LIE] N. Bourbaki, *Groupes et Algèbres de Lie*, chap.IV-VI, Hermann, Paris, 1968.

[Se] J-P. Serre, *Cohomological invariants, Witt invariants, and trace forms*, ULS 28, 1-100, AMS, 2003.

Note. After my lecture, Stefan Gille has pointed out to me that, using a different method (based on a theorem of Totaro, but not involving involutions), C. Hirsch had already computed in 2009 the structure of the cohomological invariants of all the finite Coxeter groups, under mild hypotheses on the ground field ; his method also applies to other types of invariants. Reference :

Christian Hirsch, **Cohomological invariants of reflection groups**, Diplomarbeit (Betreuer : Prof. Dr. Fabien Morel), Univ. München, 2009.

This text has not been published yet. I hope it will soon be.

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