Circuits & 2-adic Integers

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Digital Synchronous Circuit

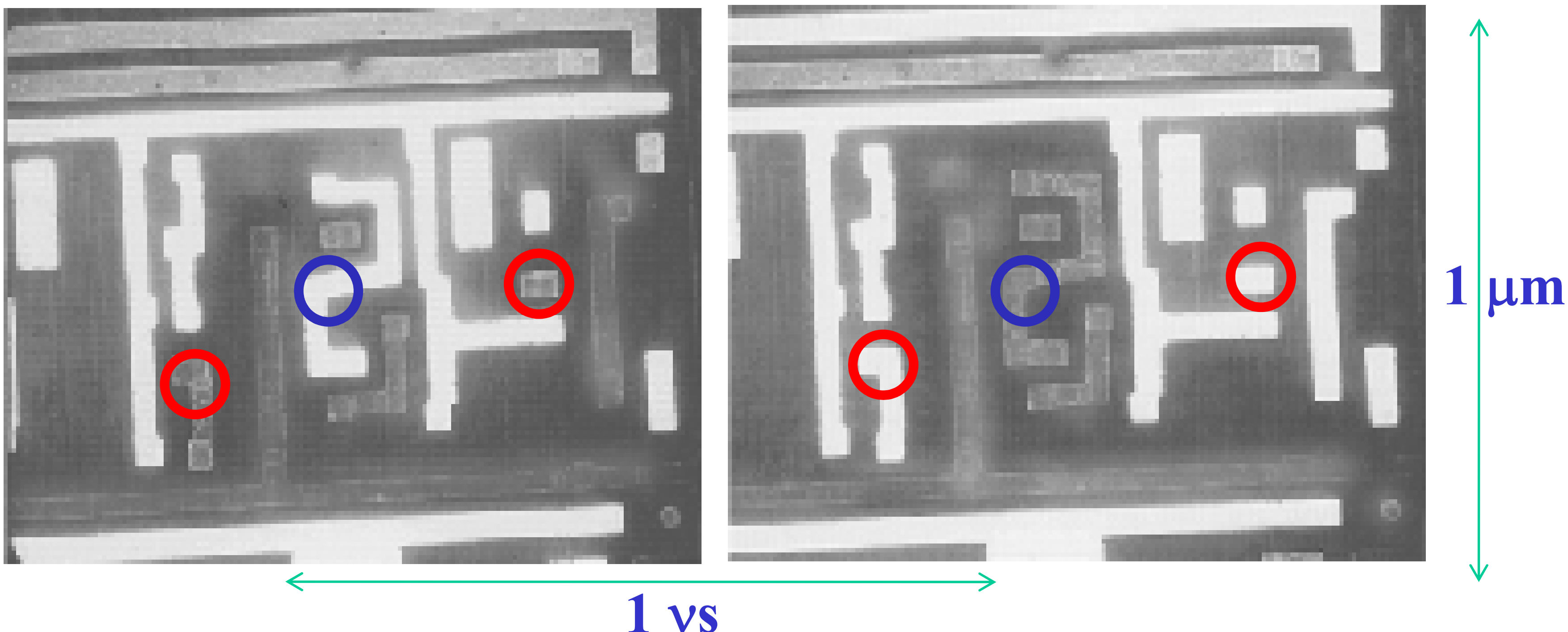
2-adic Integers

Arithmetic Circuits

LHC Application

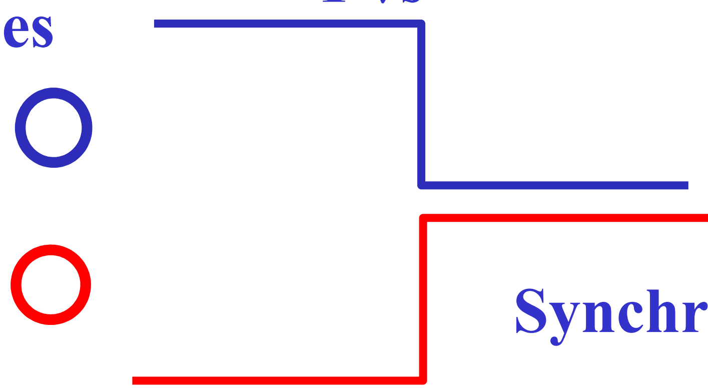
Synchronous Binary Signals

EBM Images @1GHz



Binary Values

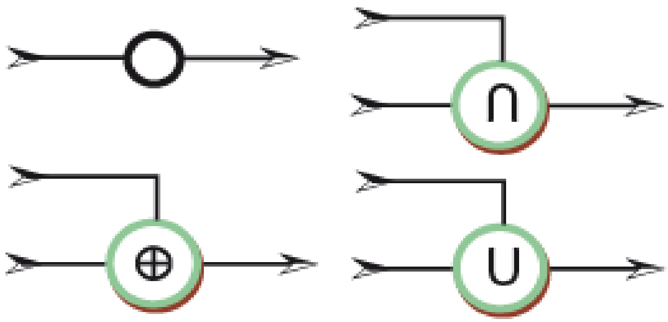
Discrete Time



Synchronous Transitions

Synchronous *Gates*

Boolean Gates



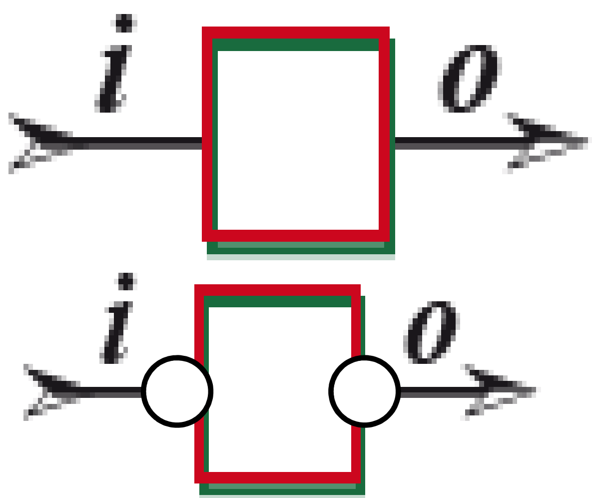
$$\neg a_t = 1 - a_t$$

$$a_t \oplus b_t = a_t + b_t - 2a_t b_t$$

$$a_t \cup b_t = a_t + b_t - a_t b_t$$

$$a_t \cap b_t = a_t b_t$$

Register



$$o_0 = 0,\ \ o_{t+1} = i_t$$

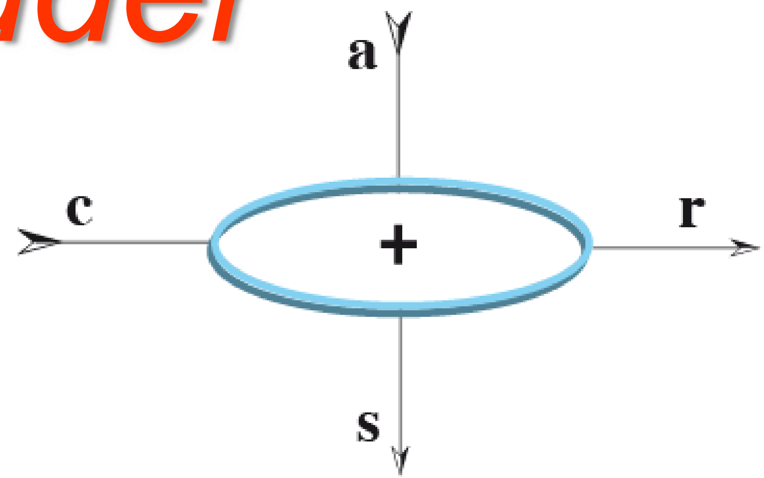
$$o_0 = 1,\ \ o_{t+1} = i_t$$

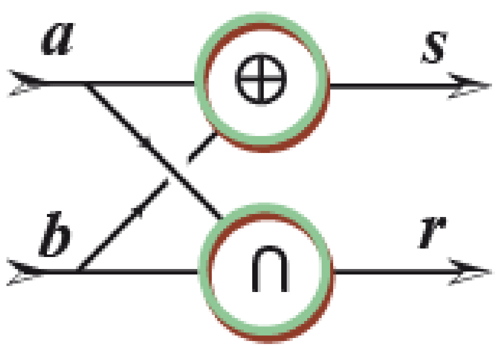
Half Adder

$$halfAdd(a,b,c)=(s,r)$$

$$\{\quad s=a\oplus b$$

$$r=a\cap b\quad\}$$





$$s_t=a_t\oplus b_t=a_t+b_t-2a_tb_t$$

$$r_t=a_t\cap b_t=a_tb_t$$

$$\forall t:\ a_t+b_t=s_t+2r_t$$

Digital Synchronous Circuit

- Syntax
  - Compose *In, And,Xor, Reg*
  - Feed-forward Boolean Gates & Clocked feed-back
  - Hierarchical Netlist
- Semantics
  - All variables have binary values at all times
  - Zero delay: *Not, And,Xor* Unit delay: *Reg*
  - All transitions happen at clock signal
- Global clock
  - Isochronous distribution through entire chip
  - Clock period can be constant or variable
  - Clock period exceeds all transition delays

Binary Integers

n	B_0^n	B_1^n	B_2^n	B_3^n	$B_{k>3}^n$	$\{k \in n\}$
0	0	0	0	0	0	$\{\}$
1	1	0	0	0	0	$\{0\}$
2	0	1	0	0	0	$\{1\}$
3	1	1	0	0	0	$\{0\ 1\}$
4	0	0	1	0	0	$\{2\}$
5	1	0	1	0	0	$\{0\ 2\}$
6	0	1	1	0	0	$\{1\ 2\}$
7	1	1	1	0	0	$\{0\ 1\ 2\}$
8	0	0	0	1	0	$\{3\}$

n	B_0^n	B_1^n	B_2^n	B_3^n	$B_{k>3}^n$	$\{k \in n\}$
–1	1	1	1	1	1	$\neg\{\}$
–2	0	1	1	1	1	$\neg\{0\}$
–3	1	0	1	1	1	$\neg\{1\}$
–4	0	0	1	1	1	$\neg\{0\ 1\}$
–5	1	1	0	1	1	$\neg\{2\}$
–6	0	1	0	1	1	$\neg\{0\ 2\}$
–7	1	0	0	1	1	$\neg\{1\ 2\}$
–8	0	0	0	1	1	$\neg\{0\ 1\ 2\}$
–9	1	1	1	0	1	$\neg\{3\}$

$$n = \sum_k B_k^n 2^k = \sum_{k \in n} 2^k$$

$$\neg 0 = 1 + 2 + 4 + 8 + 16 + \cdots = -1$$

$$k \in n \Leftrightarrow 1 = B_k^n$$

$$\neg n = -(n+1)$$

2-adic Addition

	a	1	1	1	0	1	0 $\cdots$
+	b	1	0	1	1	1	0 $\cdots$
	c	0	1	1	1	1	1
	s	0	0	1	0	1	1

$\cdots$

$$\boxed{a_t + b_t + c_t = s_t + 2c_{t+1}}$$

Magic Masks

k	B_k^{-1}	B_k^{-2}	B_k^{-3}	B_k^{-4}	B_k^{-5}		μ_k
0	1	0	1	0	1	$\cdots$	$-1/3$
1	1	1	0	0	1	$\cdots$	$-1/5$
2	1	1	1	1	0	$\cdots$	$-1/17$

$$\mu_k = \sum_n B_k^{\neg n} 2^n = \sum_{n \notin k} 2^n$$

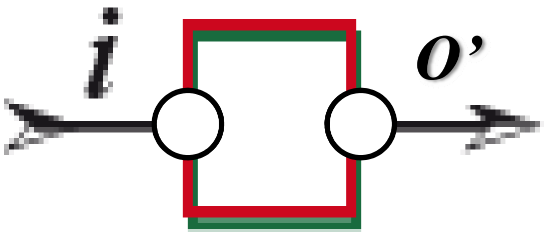
$$\mu_k = (1^{2^k} 0^{2^k})^\infty = \frac{-1}{1+2^{2^k}}$$

$$\begin{array}{rcl} \mu_0 & : & 10101010\cdots \\ +2\mu_0 & : & 01010101\cdots \\ =3\mu_0 & : & 11111111\cdots \end{array}$$

$$3\mu_0 = -1$$

$$\mu_0 = 1 + 4 + 16 + \cdots = -\frac{1}{3}$$

2-adic Gates

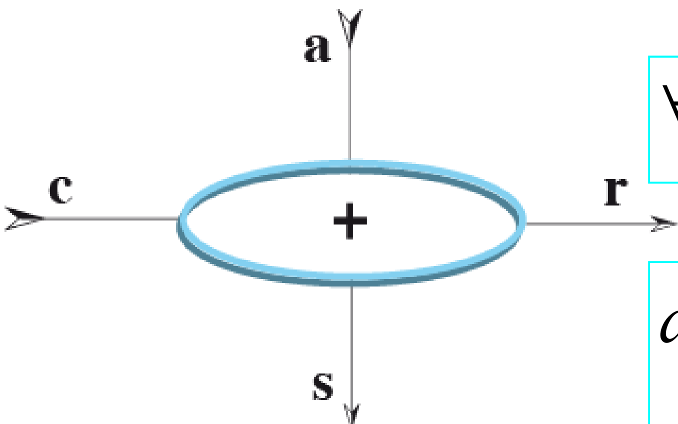


$$o_0 = 0, \; o_{t+1} = i_t$$

$$o = \sum o_t 2^t \quad i = \sum i_t 2^t$$

$$\boxed{o = 2i}$$

$$\boxed{o' = 1 + 2i}$$



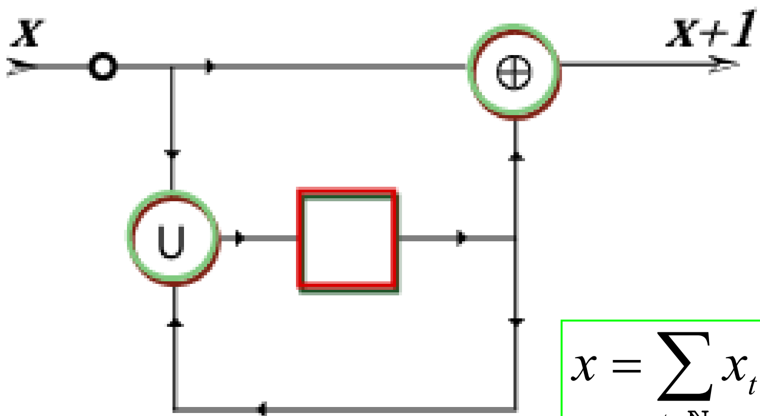
$$\forall t: \; a_t + b_t = s_t + 2r_t$$

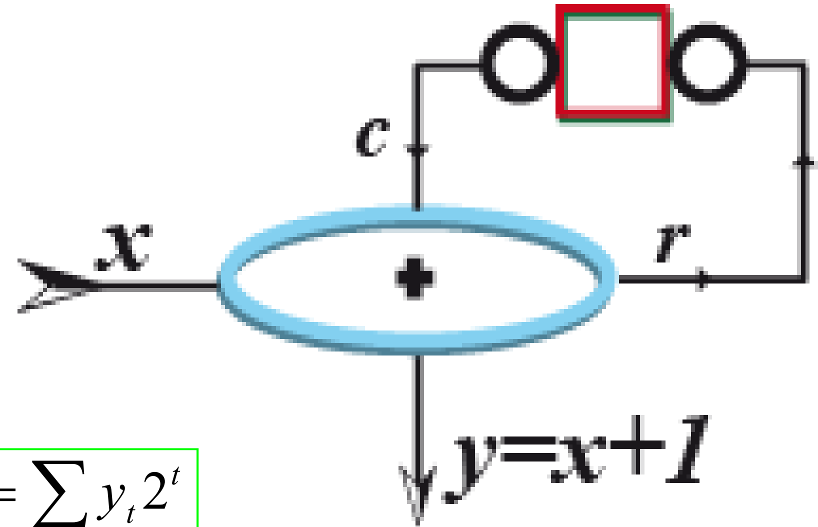
$$a = \sum a_t 2^t \quad b = \sum b_t 2^t$$

$$s = \sum s_t 2^t \quad r = \sum r_t 2^t$$

$$\boxed{a + b = s + 2r}$$

Serial Increment





$$x = \sum_{t \in \mathbb{N}} x_t 2^t \qquad y = \sum_{t \in \mathbb{N}} y_t 2^t$$

$$c = \sum_{t \in \mathbb{N}} c_t 2^t \qquad r = \sum_{t \in \mathbb{N}} r_t 2^t$$

$$\forall t \in \mathbb{N}:$$

$$c_0 = 1 \qquad c_{t+1} = r_t$$

$$x_t + c_t = y_t + 2r_t$$

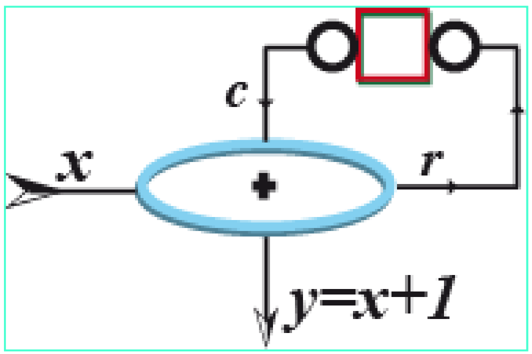
$$c = 1 + 2r$$

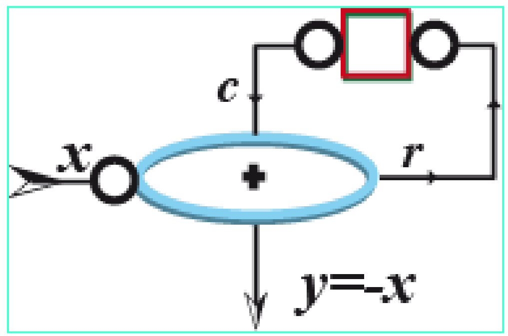
$$x + c = y + 2r$$

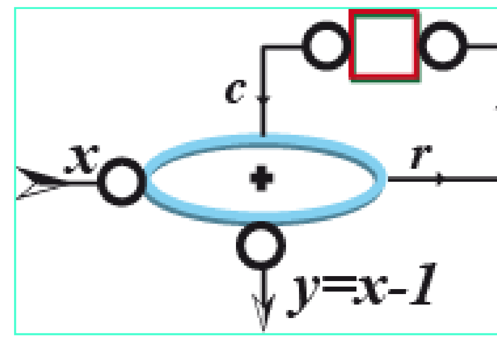
$$\Rightarrow \quad \boxed{y = x + 1}$$

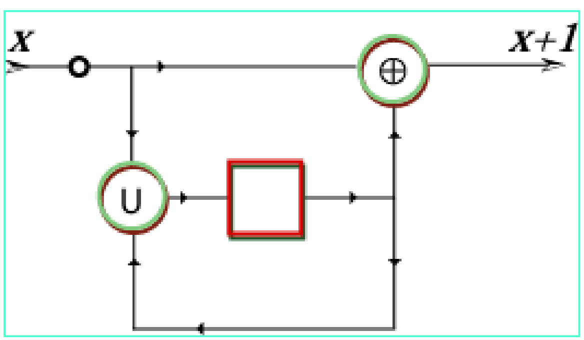
One & Two's Complement

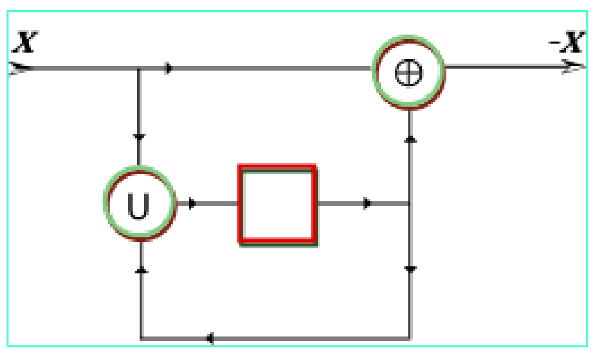
$$
\begin{aligned}
-1 &= x + \neg x \\
-x &= 1 + \neg x \\
x+1 &= -\neg x \\
x-1 &= \neg - x
\end{aligned}
$$

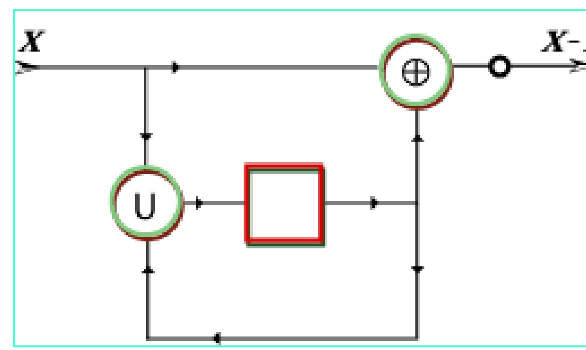




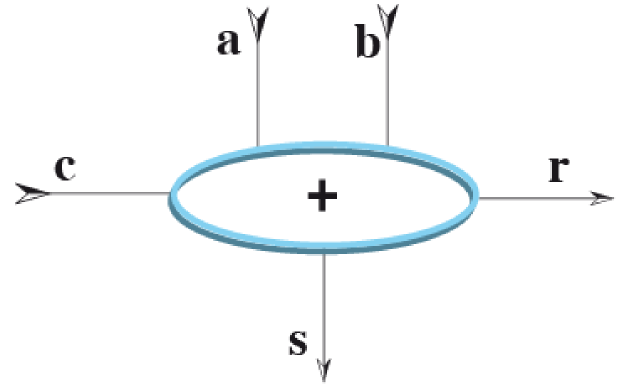








Full Adder

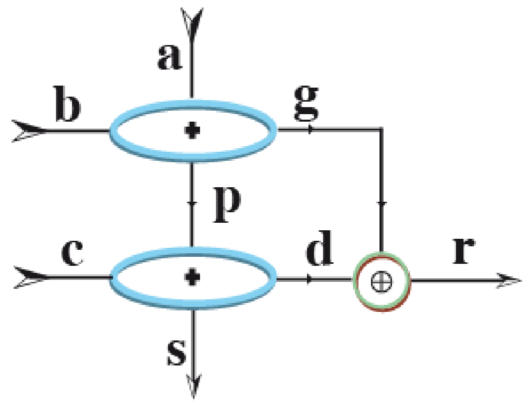


$$fullAdd(a,b,c)=(s,r)$$

$$\{ \quad (p,g)=halfAdd(a,b)$$

$$(s,d)=halfAdd(c,g)$$

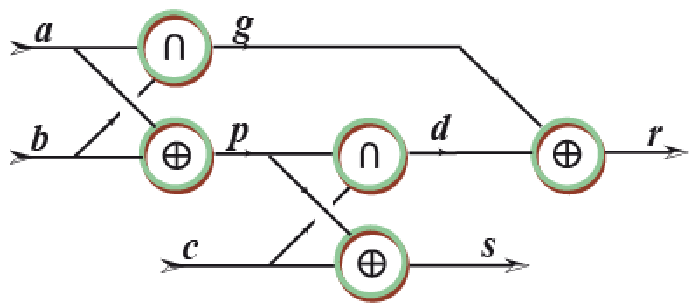
$$r=g\oplus d \quad \}$$



$$p=a\oplus b \qquad g=a\cap b$$

$$s=c\oplus p \qquad d=c\cap p$$

$$r=g\oplus d$$



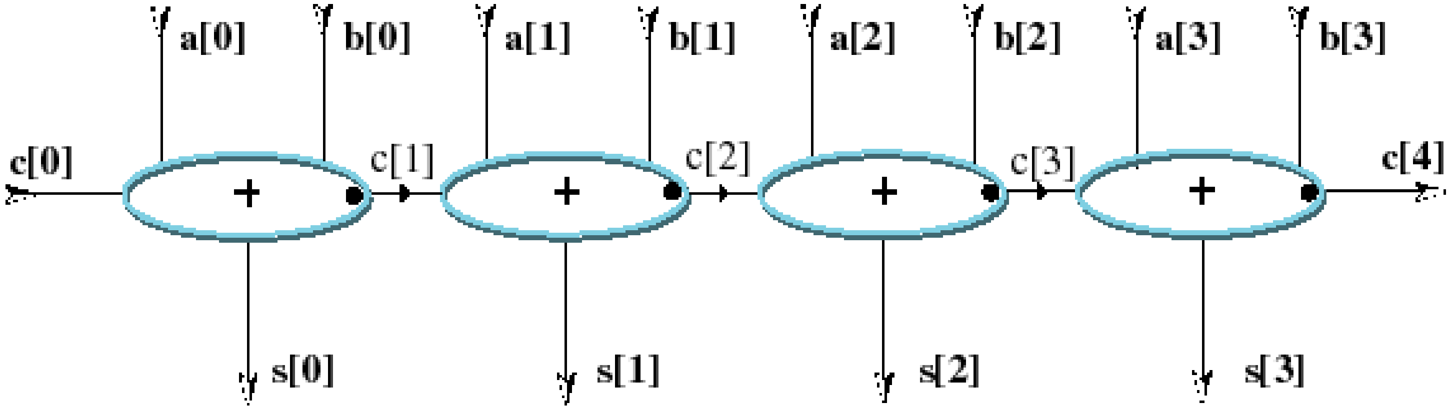
$$a+b=p+2g$$

$$c+p=s+2d$$

$$r=g+d$$

$$\boxed{a+b+c=s+2r}$$

Parallel Adder



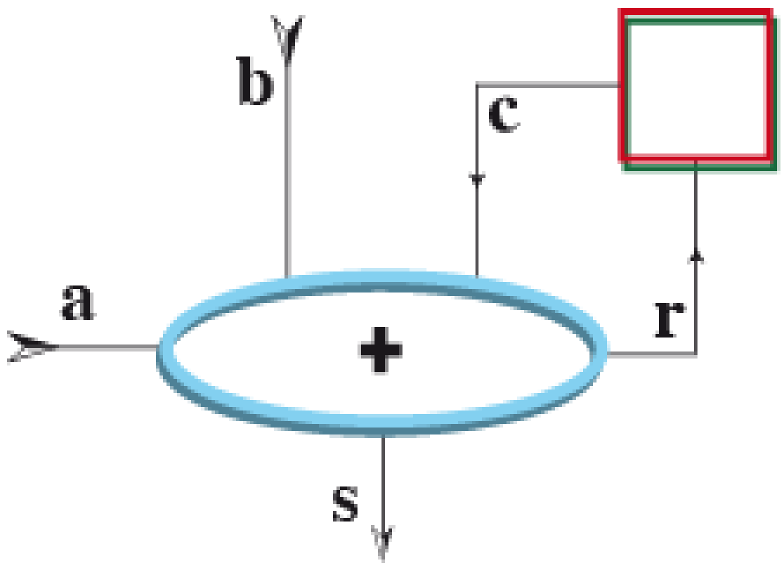
$$a_0 + b_0 + c_0 = s_0 + 2c_1$$

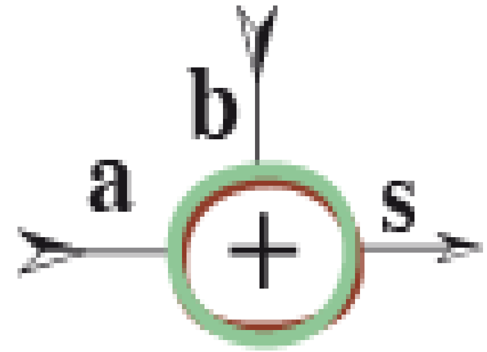
$$a_1 + b_1 + c_1 = s_1 + 2c_2$$

$$\cdots$$

$$\boxed{c_0 + \sum_{k<N} a_k 2^k + \sum_{k<N} b_k 2^k = \sum_{k<N} s_k 2^k + c_N 2^N}$$

Serial Adder





$$c = 2r$$

$$a+b+c=s+2r$$

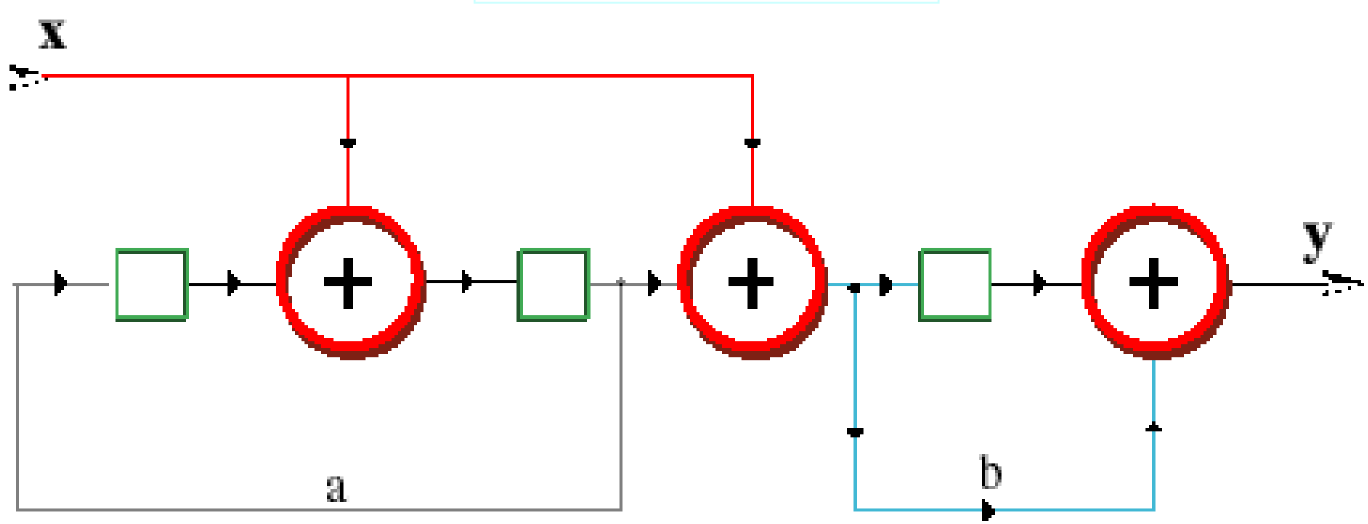
$$s=a+b$$

Beyond Addition

$$
\begin{aligned}
a &= 2(x + 2a) \\
a &= -\frac{2x}{3} \\
a &= {}_2(01)^{\infty} \times x
\end{aligned}
$$

$$
\begin{aligned}
b &= x + a \\
b &= \frac{x}{3} \\
\frac{1}{3} &= {}_2 1(10)^{\infty}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
y &= b + 2b \\
y &= x
\end{aligned}
}
$$



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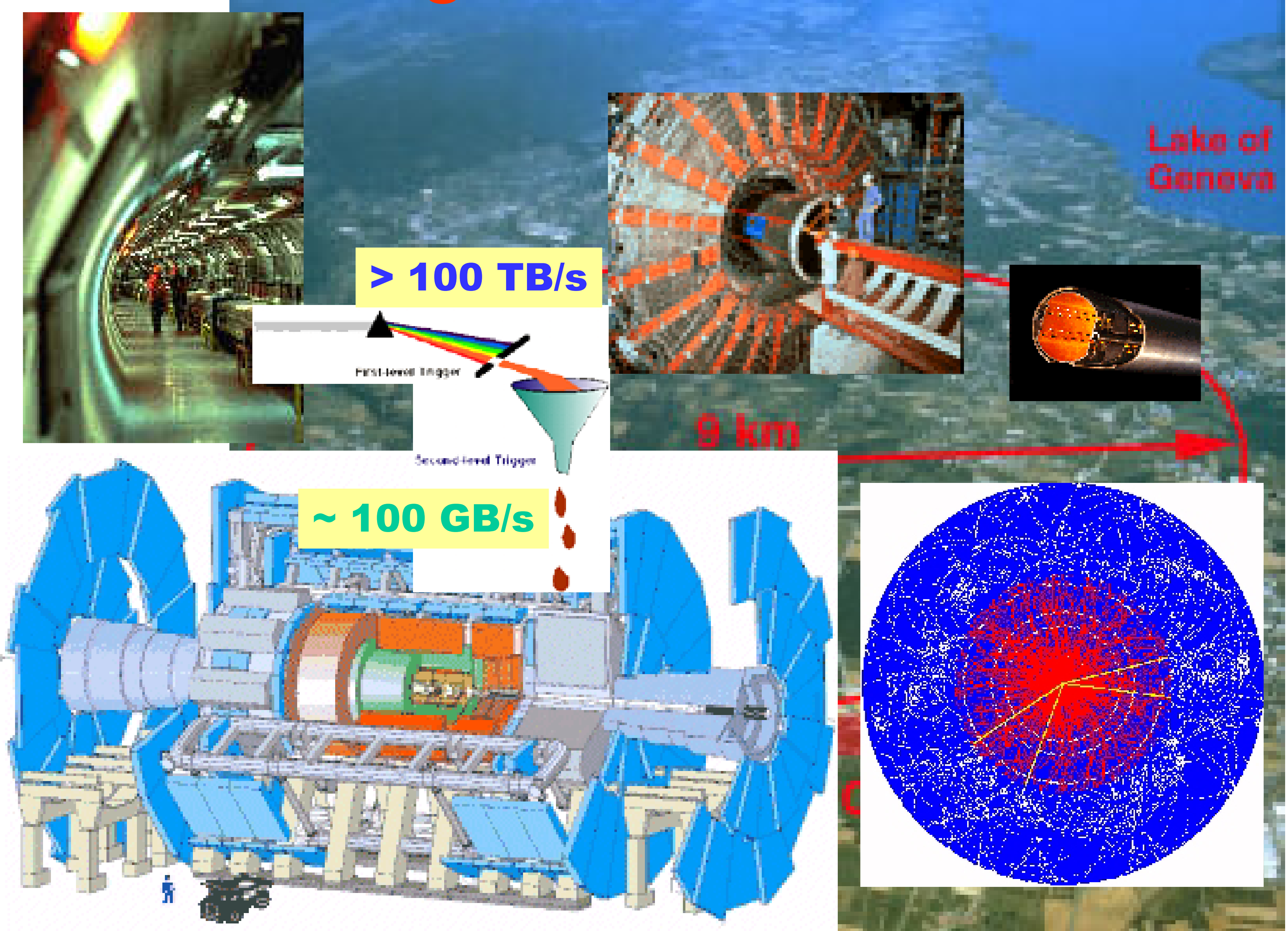
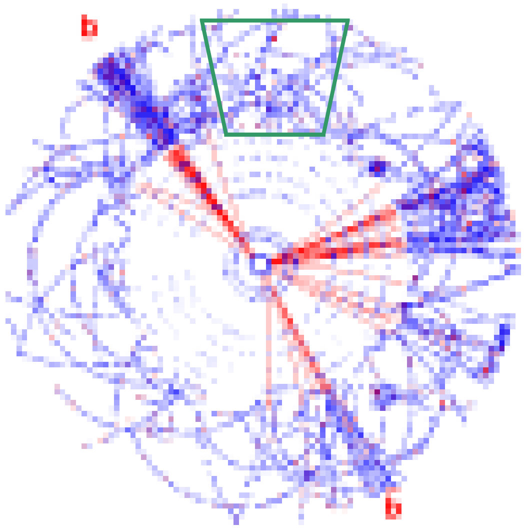
Large Hadron Collider

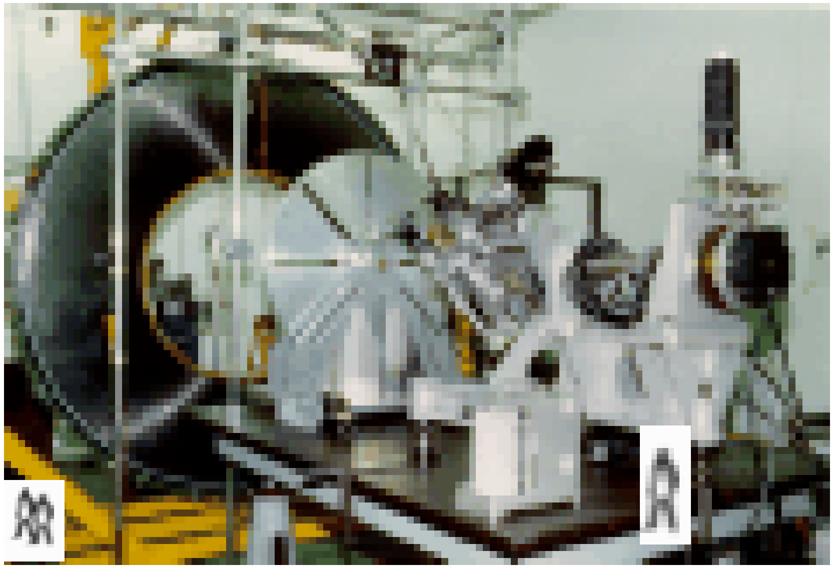


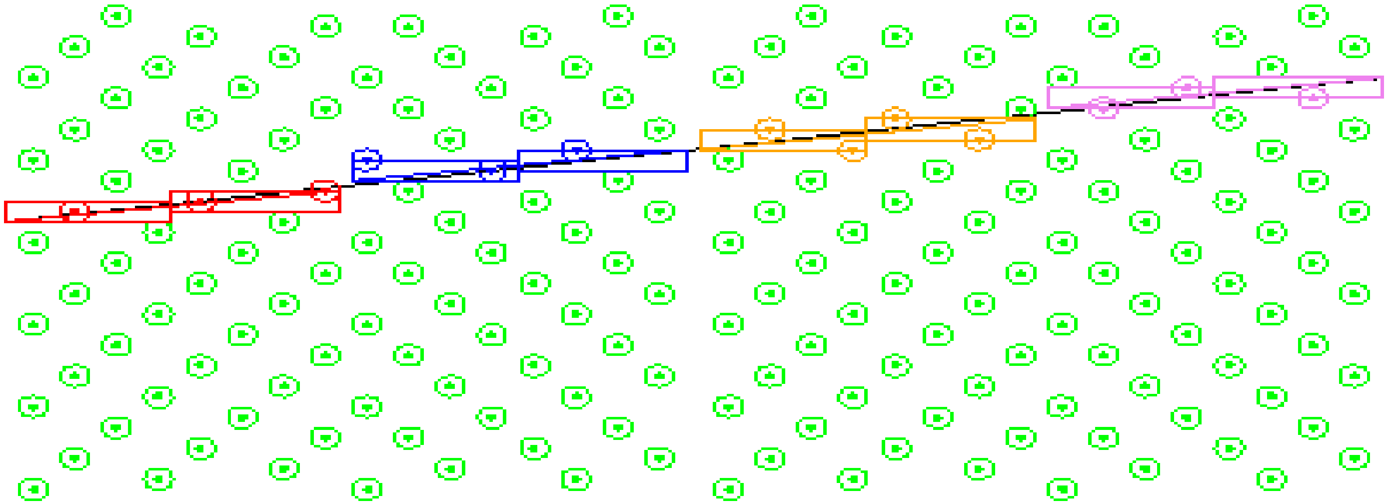
Transition Radiation Tracker

ATLAS Barrel Inner Detector

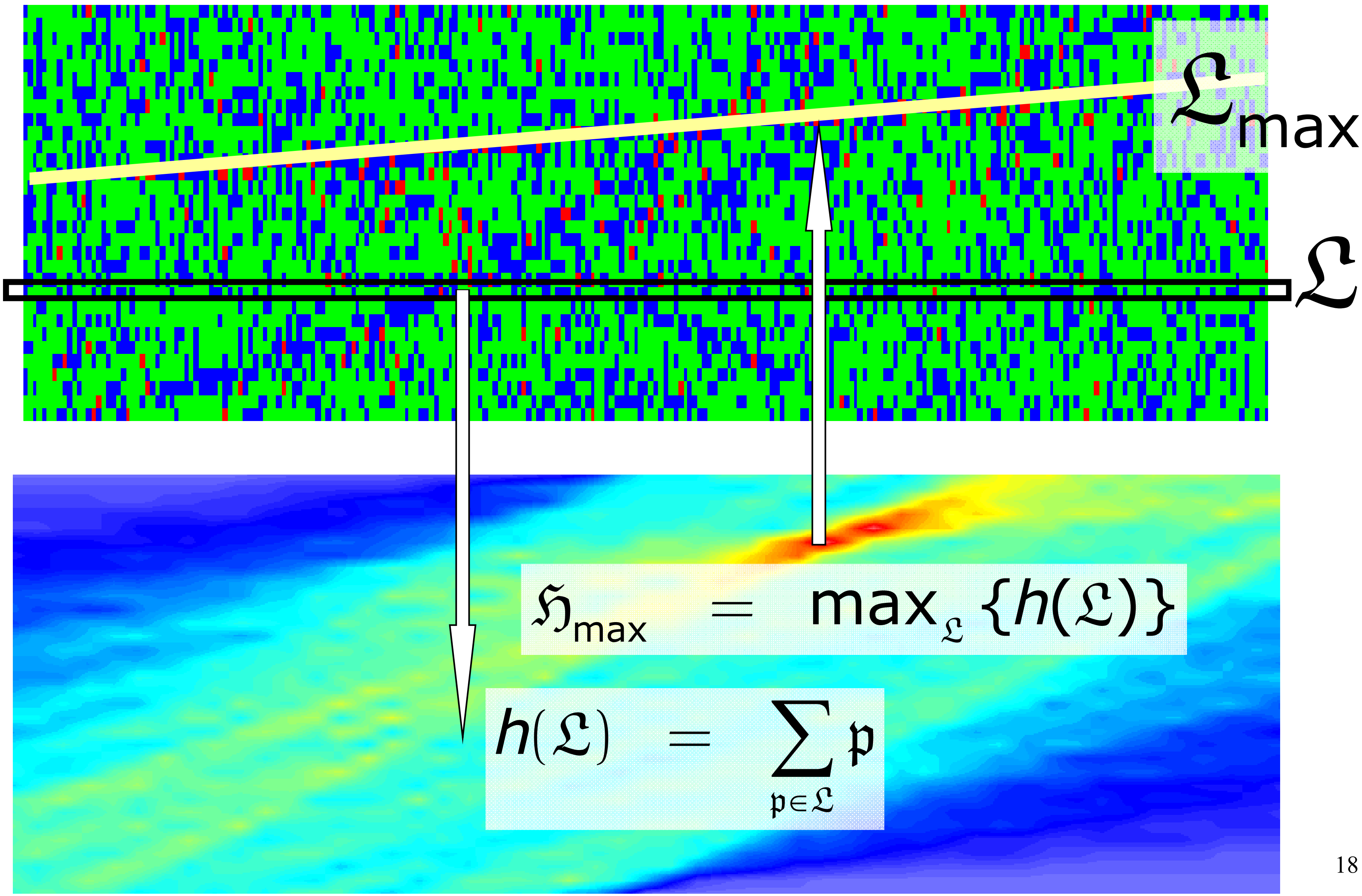
Region of interest







Linear Hough Transform



$$\mathfrak{H}_{\max} = \max_{\mathfrak{L}} \{h(\mathfrak{L})\}$$

$$h(\mathfrak{L}) = \sum_{\mathfrak{p}\in\mathfrak{L}} \mathfrak{p}$$

Input Receiver

FHT

Adders &

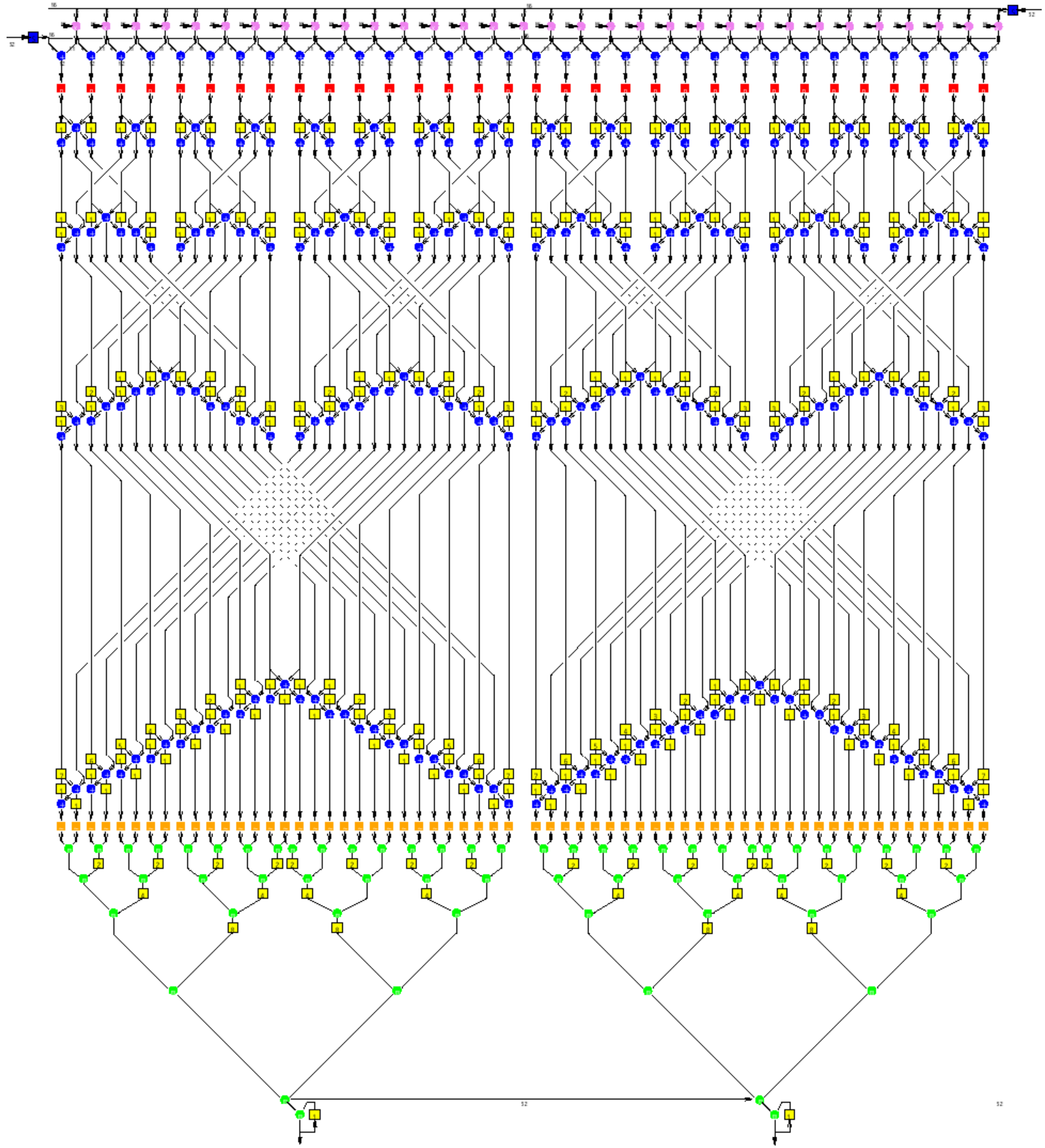
Line Delay

Network

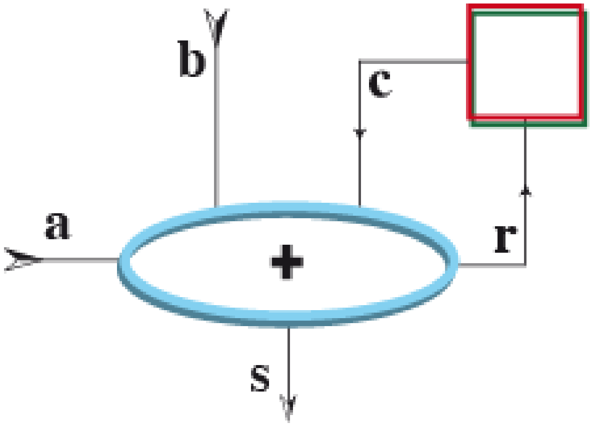
Bit Reverse

Max Column

Max Rows



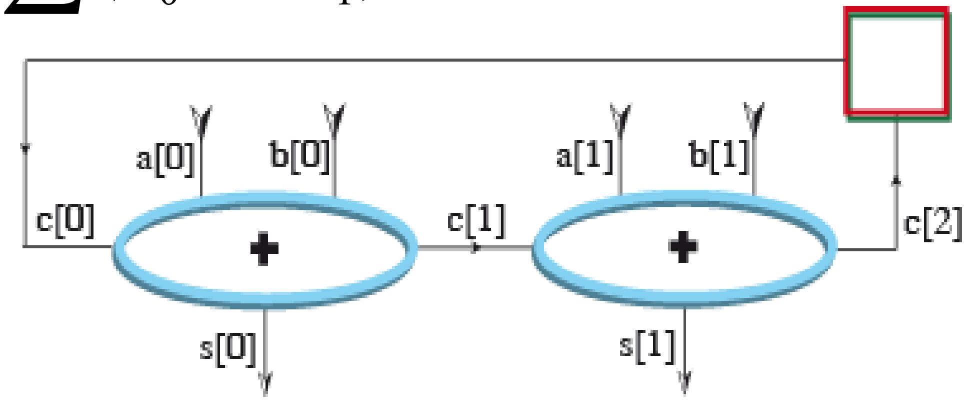
Trade Time For Space



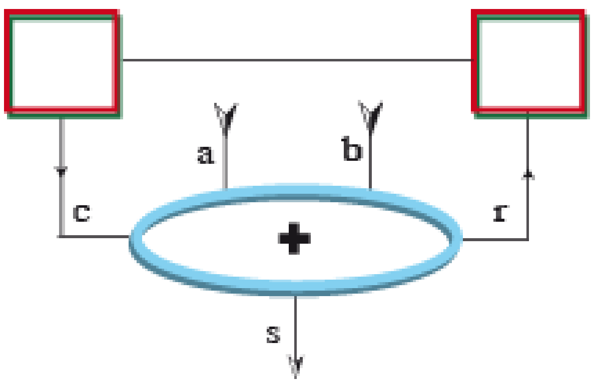
$$a = \sum a_k Z^k$$

Z=2 : 1b/cycle

$$a = \sum (a_0 + 2a_1) Z^k$$



Z=4 : 2b/cycle



$$a = \sum a_{2k} 2^k + \sqrt{2} \sum a_{1+2k} 2^k \cdots$$

Z=√2 : 0.5b/cycle

Energy is #1 design criteria for both hard & soft, hence minimal computation at minimal speed!

Questions!