Socially efficient discounting under ambiguity aversion

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Abstract

In an economy with uncertain consumption growth and an ambiguity-averse representative agent, we provide sufficient conditions under which ambiguity aversion decreases the social discount rate. We identify two effects. The first is an ambiguity prudence effect, similar to the effect of prudence under expected utility. We show that it decreases the rate if and only if preferences satisfy decreasing ambiguity aversion. The second effect is observationnally equivalent to a deterioration of beliefs. This pessimism effect requires joint restrictions on the growth process and the agent's preferences to be signed. The calibration of the model suggests that the effect of ambiguity aversion on the way we should discount distant cash flows is potentially large.

Keywords: Decreasing ambiguity aversion, ambiguity prudence, Ramsey rule, sustainable development.

1 Introduction

The emergence of public policy problems associated with the sustainability of our development has raised considerable interest for the determination of a socially efficient discount rate. This debate has culminated in the publication of two reports about the evaluation of different public investments. The Copenhagen Consensus (Lomborg, 2004) put top priority on public programs yielding immediate benefits, like fighting AIDS and malnutrition and rejected measures to fight climate change as not being cost-effective. The Stern Review, on the other hand, (Stern, 2007) argues for decisive and immediate action against climate change.

Because global warming will really affect our economies in a relatively distant time horizon, the choice of the rate at which these costs are discounted plays a key role in reaching either conclusion. While Stern applies an implicit rate of 1.4% per year, the Copenhagen Consensus is based on a rate around 5%. For the sake of illustrating the power of discounting, consider a project which yields its benefits in t years time. For a horizon $t = 100$ the Copenhagen Consensus would require a rate-of-return 36 times higher than Stern.

As stated by the well-known Ramsey rule (Ramsey (1928)), the socially efficient discount rate (net of the rate of pure preference for the present) is equal to the product of relative risk aversion and the growth rate of consumption. With future generations likely being richer, the return on investment needs to be large enough to compensate for the increased intertemporal inequality that it generates. If we assume that the growth rate of wealth is 2% and relative risk aversion equals 2, this yields a discount rate of 4%.

However, this basic reasoning does not take into account the riskiness affecting the long-term growth of consumption. Hansen and Singleton (1983), Gollier (2002) and Weitzman (2007a), among others, have extended the Ramsey rule in this spirit. An exogenously given stochastic growth process adds a precautionary term to the Ramsey rule which reduces the discount rate under prudence: The willingness to save increases with risk since marginal utility is convex (Leland, 1968; Drèze and Modigliani, 1972).

The present paper goes one step further in recognizing the potential uncertainty on the growth process itself. Such parameter uncertainty on priors is typically referred to as statistical ambiguity or Knightian uncertainty. We believe that this assumption is realistic, especially for long-term forecasts.

Departing from the standard Subjective Expected Utility paradigm (SEU, Savage (1954)), we also assume that the representative agent is ambiguity-averse, i.e., that she dislikes mean-preserving spreads over prior beliefs. Indeed, starting with the pioneering work by Ellsberg (1961), ample evidence in favor of this hypothesis has been accrued.[1] All of which suggests that it is behaviorally

[1] The *Ellsberg Paradox* goes back to a thought experiment by Daniel Ellsberg (1961). Suppose two urns containing 90 balls of three colors. The first contains 30 red balls, with the remaining black and yellow in unknown proportion. The second one contains 30 balls of each color. Imagine a bet on drawing the color black. Ellsberg conjectured that a majority of bettors prefers to place it on the unambiguous urn. However, when asked to bet on the

meaningful to distinguish lotteries over prior distributions from lotteries over final outcomes. In what follows, we will consider a representative agent who displays "smooth ambiguity preferences", as recently proposed by Klibanoff, Marinacci and Mukerji (KMM, 2005, 2009). Accordingly, the agent computes the expected utility of future consumption conditional on each possible value of the uncertain parameter. She then evaluates her future felicity by computing the certainty equivalent of these conditional expected utilities, using an increasing and concave function ϕ. The concavity of this function implies that she dislikes any mean-preserving spread in the set of plausible beliefs, i.e. that she is ambiguity-averse. It can be shown that the smooth ambiguity family entails the well-known max-min criterion as a special case.

Intuitively, one might expect that ambiguity aversion should raise the agent's willingness to save in order to compensate for its adverse effect on the present value of future welfare. In this paper we show that this is not true in general: ambiguity aversion may increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utility. On the one hand, there is an ambiguity prudence effect, similar to the prudence effect in the expected utility framework. Independent of risk preferences, decreasing absolute ambiguity aversion (DAAA) has a positive effect on the willingness to save for the ambiguous future. This effect is related to future felicity being measured by the ϕ-certainty equivalent rather than the expectation of ϕ.

On the other hand ambiguity aversion acts like an implicit pessimistic shift in beliefs with respect to the expected utility benchmark. This has been observed by KMM (2005, 2007). It is as if probability weights were shifted towards unfavorable priors in the sense of the Monotone Likelihood Ratio order (MLR). However, pessimism does not in general imply a reduction of the interest rate. We derive pairs joint conditions on the risk attitude and the stochastic ordering of plausible distributions to guarantee that, under DAAA, the socially efficient discount rate will indeed be lower under ambiguity aversion.

This paper is related to Weitzman (2007a) and Gollier (2007b) in recognizing uncertainty as a determinant feature of the discounting problem. Weitzman (2007a) shows that uncertainty about the volatility of the growth process may yield a term structure of the discount rate that tends to minus infinity for very long time horizons. Gollier (2007b) provides a general typology for parameter uncertainty. He shows that the sign of the third or fourth derivative of the utility function are necessary to sign the effect on the efficient discount rate, depending upon its type. The present paper departs strongly from these works in acknowledging evidence in favor of ambiguity-averse preferences.

Jouini, Marin and Napp (2008) and Gollier (2007) consider the related question of how to aggregate diverging beliefs in a SEU framework. Jouini, Marin and Napp show that an aggregation bias might cause a rich evolution of the discount rate than in the representative agent models. In particular, the dis-

contrary (*not black*), they would also prefer to place the opposite bet on the unambiguous urn. This pattern would be incompatible with any stable belief in the SEU model. Experimental data confirms Ellsberg's "paradoxical" predictions (see e.g. Camerer and Weber, 1992).

count rate might be first increasing and only then approach its limit, namely the smallest individual rate.

The most active branch of the literature on uncertainty studies phenomena on financial markets. Methodologically, our paper is most closely related to Gollier (2006). He investigates the effect of ambiguity aversion on the demand for risky assets. Like in the present model, the pessimism effect on beliefs is shown to affect behavior differently than risk aversion.

Ju and Miao (2007) and Collard, Mukerji, Sheppard and Tallon (2008) investigate the quantitative effects of KMM preferences on asset prices. Using tractable functional forms, they are able to replicate several empirical phenomena, like low risk-free rates, which are difficult to explain within the standard expected utility model. The present paper shows, however, that the relation between the degree of ambiguity aversion and the risk-free rate needs not be negative in the general case.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. We decompose the effect of ambiguity aversion into its two components in Section 4, whereas Sections 5 and 6 are devoted to respectively the ambiguity prudence effect and the pessimism effect. Section 7 investigates under which conditions our findings extend to any increase in ambiguity aversion. Finally, before concluding, we calibrate the model using two different specifications in Section 8.

2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\widetilde{c}_t$ fruits at date t, $t = 0, 1, 2, ...$ There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{r_t t}$ fruits for sure at date t. Thus, the real interest rate associated to maturity t is r_t. The distribution of $\widetilde{c}_t$ is a function of a parameter θ, $\theta = 1, 2, ..., n$. This parametric uncertainty takes the form of a random variable $\widetilde{\theta}$ whose probability distribution is a vector $q = (q_1, ..., q_n)$, where q_θ is the probability that $\widetilde{\theta}$ takes value θ. Let the cumulative distribution function of $\widetilde{c}_t$ conditional on θ be denoted by $F_{t\theta}$. The crop conditional to θ is denoted $\widetilde{c}_{t\theta}$. An ambiguous environment for $\widetilde{c}_t$ is thus fully described by $\widetilde{c}_t \sim (\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$. Conditional on θ, the expected utility of an agent who purchases α zero-coupon bonds with maturity t equals

$$U_t(\alpha, \theta) = Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) = \int u(c + \alpha e^{r_t t}) dF_{t\theta}(c).$$

We assume that u is three times differentiable, increasing and concave, so that $U(\cdot, \theta)$ is concave in the investment α, for all θ.

Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji, 2009), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. Ex ante, for a given investment α, her welfare is measured by $V_t(\alpha)$, which is the certainty equivalent of the conditional expected utilities:

$$\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(U_t(\alpha,\theta)) = \sum_{\theta=1}^{n} q_\theta \phi\left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})\right). \tag{1}$$

Function ϕ describes the investor's attitude towards ambiguity (or parameter uncertainty). It is assumed to be three times differentiable, increasing and concave. A linear function ϕ means that the investor is neutral to ambiguity as her preferences simplify to the subjective expected utility functional $V_t^{SEU}(\alpha) = Eu(\widetilde{c}_t + \alpha e^{r_t t})$. In contrast, a concave ϕ is equivalent to ambiguity aversion. In other words, she dislikes mean-preserving spreads over candidate levels of $U_t(\alpha,\theta)$.

An interesting particular case arises when absolute ambiguity aversion $A(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -A^{-1}\exp(-AU)$. As proven by Klibanoff, Marinacci and Mukerji (2005), the ex-ante welfare $V_t(\alpha)$ tends to the max-min expected utility functional $V_t^{MEU}(\alpha) = \min_\theta Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})$ when the degree of absolute ambiguity aversion ϕ tends to infinity. Thus, the max-min criterion à la Gilboa and Schmeidler (1989) is a special case of this model.

The optimal investment α^* maximizes the intertemporal welfare of the investor,

$$\alpha^* \in \arg\max_{\alpha} \qquad u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha), \tag{2}$$

where parameter δ is the rate of pure preference for the present.

At this stage, it is important to point out that the basic assumptions underlying KMM models do not guarantee that the maximization problem (2) is convex. To see why, it suffices to recall that certainty equivalent functions need not be concave. Indeed, even if we imposed ϕ and u to be strictly concave, the solution to program (2), when it exists, need not be unique. To handle this problem we prove the following.

Proposition 1 *Suppose that ϕ has a concave absolute ambiguity tolerance, i.e., $-\phi'(U)/\phi''(U)$ is concave in U. This implies that V_t is concave in α.*

Proof. Relegated to the Appendix.

If the inverse of absolute ambiguity aversion increases at a linear or decreasing rate in U, then the KMM functional is concave in α. The above proposition includes the specifications which are most widely used in the literature, in particular the families of exponential and power functions.

Henceforward we will consider the following assumption satisfied.

Assumption 1 *The function ϕ exhibits a concave absolute ambiguity tolerance, i.e. $-\phi'(U)/\phi''(U)$ is concave in U everywhere.*

Thanks to Assumption 1, the necessary and sufficient condition to solve program (2) can be written as

$$u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*).$$

Fully differentiating equation (1) with respect to α yields

$$V_t'(\alpha) = e^{r_t t} \frac{\sum_{\theta=1}^n q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) \right) Eu'(\widetilde{c}_{t\theta} + \alpha e^{r_t t})}{\phi'(V_t(\alpha))}.$$

Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity t is $\alpha^* = 0$. Combining the above two equations implies the following equilibrium condition:

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{\sum_{\theta=1}^n q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right) Eu'(\widetilde{c}_{t\theta})}{\phi'(V_t(0)) u'(c_0)} \right]. \tag{3}$$

This is also the socially efficient rate at which sure benefits and costs occurring at date t must be discounted in any cost-benefit analysis at date 0.

Under ambiguity-neutrality, the standard bond pricing formula $r_t = \delta - t^{-1} \ln \left[Eu'(\widetilde{c}_t)/u'(c_0) \right]$ obtains.[2] The riskiness of future consumption reduces the social discount rate if and only if $Eu'(\widetilde{c}_t)$ is larger than $u'(E\widetilde{c}_t)$,, that is to say, if and only if u' is convex and the agent displays *prudence* (see Leland, 1968; Drèze and Modigliani, 1972; or Kimball, 1990).

Our goal in this paper is to determine the conditions under which ambiguity aversion reduces the discount rate. An ambiguous environment $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$ is said to be acceptable if the respective supports of the $\widetilde{c}_{t\theta}$ are in the domain of u, and if all $Eu'(\widetilde{c}_{t\theta})$ are in the domain of ϕ. The set of acceptable ambiguous environments is denoted Ψ.

3 An analytical solution

Let us consider the following specification:

- The plausible distributions of $\ln \widetilde{c}_{t\theta}$ are all normal with the same variance $\sigma^2 t$, and with mean $\ln c_0 + \theta t$.[3]
- The parameter θ is normally distributed with mean μ and variance σ_0^2.[4]
- The representative agent's preferences exhibit constant relative risk aversion $\gamma = -cu''(c)/u'(c)$, such that $u(c) = c^{1-\gamma}/(1-\gamma)$.

[2] See for example Cochrane (2001).

[3] In continuous time, this would mean that the consumption process is a geometric Brownian motion $d \ln c_t = \theta dt + \sigma dw$.

[4] We consider the natural continuous extension of our model with a discrete distribution for $\widetilde{\theta}$.

- The representative agent's preferences exhibit constant relative ambiguity aversion $\eta = -|u|\,\phi''(u)/\phi'(u) \geq 0$. This means that $\phi(U) = k(kU)^{1-\eta k}/(1-\eta k)$, where $k = sign(1-\gamma)$ is the sign of u.

As is well-known, the Arrow-Pratt approximation is exact under CRRA and lognormally distributed consumption. Therefore, conditional to each θ, we have that

$$Eu(\widetilde{c}_{t\theta}) = (1-\gamma)^{-1}\exp(1-\gamma)(\ln c_0 + \theta t + 0.5(1-\gamma)\sigma^2 t).$$

We can again use the same trick to compute the ϕ-certainty equivalent V_t, since $\phi(Eu(\widetilde{c}_{t\theta}))$ is an exponential function and the random variable $\widetilde{\theta}$ is normal, which is another case where the Arrow-Pratt approximation is exact. It yields

$$V_t(0) = (1-\gamma)^{-1}\exp(1-\gamma)\Big(\ln c_0 + \mu t + 0.5(1-\gamma)\sigma^2 t + 0.5(1-\gamma)(1-k\eta)\sigma_0^2 t^2\Big).$$

However, in order to solve for the pricing rule (3) we are really interested in $V_t'(0)$. A convenient way to structure the algebra is to decompose $V_t'(0)$ in the following way: again exploiting the Arrow-Pratt approximation, we have on the one hand

$$\frac{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)}{\phi'(V_t(0))} = \exp\Big(\frac{1}{2}(1-\gamma)^2 k\eta\sigma_0^2 t^2\Big), \tag{4}$$

and on the other hand

$$\begin{aligned}\frac{E[\phi'\left(Eu(\widetilde{c}_{t\theta})\right)Eu'(\widetilde{c}_{t\theta})]}{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \quad = \quad & \exp -\Big(\gamma(\ln c_0 + \mu t) \quad -\frac{1}{2}\gamma^2(\sigma^2 t + \sigma_0^2 t^2) - \\ & -(\gamma(1-\gamma)k\eta)\sigma_0^2 t^2\Big). \end{aligned} \tag{5}$$

Finally, multiplying (4) by (5) and plugging the result into (3), yields the desired analytical expression:

$$r_t = \delta + \gamma\mu - \frac{1}{2}\gamma^2(\sigma^2 + \sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{6}$$

Let g be the growth rate of expected consumption. It is easy to check that $g = \mu + 0.5(\sigma^2 + \sigma_0^2 t)$. It implies that the above equation can be rewritten as

$$r_t = \delta + \gamma g - \frac{1}{2}\gamma(\gamma+1)(\sigma^2 + \sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{7}$$

The first two terms on the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the growth rate of expected consumption g. When g is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\widetilde{c}_t)$ under prudence, this has a negative impact on the discount rate.[5] Notice that the variance of consumption at date t equals $\sigma^2 t + \sigma_0^2 t^2$, so that it increases at an increasing rate with respect to the time horizon. Therefore, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2008) to justify a decreasing discount rate in an expected utility framework.

The final term reduces the discount rate under positive ambiguity aversion ($\eta > 0$). It is increasing with ambiguity aversion η, the degree of uncertainty σ_0, and with the time horizon t.

Thanks to the above specifications, absent ambiguity the term structure is flat. The mere presence of ambiguity (i.e. $\sigma_0^2 > 0$) causes the rates to decrease linearly over time. If in addition, the agent displays ambiguity aversion ($\eta > 0$), this decline steepens.

The following sections investigate whether it is true in general, that ambiguity aversion decreases the socially efficient discount rate for any maturity. Contrary to the example presented above, the next section reveals that ambiguity aversion might even decrease the willingness to save.

4 The two effects of ambiguity aversion

The usual bond-pricing formula under SEU yields a benchmark expression for the social discount rate

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{Eu'(\widetilde{c}_t)}{u'(c_0)} \right], \tag{8}$$

where the random variable $\widetilde{c}_t$ describes future consumption, distributed as $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$. Like in the analytical example from above, the effect of ambiguity-aversion on $V_t'(0)$ can be decomposes such that

$$r_t = \delta - \frac{1}{t} \ln \left[a \frac{Eu'(\overset{\circ}{\widetilde{c}_t})}{u'(c_0)} \right], \tag{9}$$

where the constant a is defined as

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\phi'(V_t(0))}, \tag{10}$$

and where $\overset{\circ}{\widetilde{c}_t}$ is a distorted probability distribution $(\widetilde{c}_{t1}, \overset{\circ}{q}_1; ...; \widetilde{c}_{tn}, \overset{\circ}{q}_n)$ with the property that for any $\theta = 1, ..., n$,

$$\overset{\circ}{q}_\theta = \frac{q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\sum_{\tau=1}^{n} q_\tau \phi' \left(Eu(\widetilde{c}_{t\tau}) \right)}. \tag{11}$$

[5] This precautionary effect is equivalent to reducing the growth rate of consumption g by the precautionary premium (Kimball (1990)) $0.5(\gamma+1)(\sigma^2 + \sigma_0^2 t)$. Indeed, $\gamma + 1 = -cu'''(c)/u''(c)$ is the index of relative prudence of the representative agent.

Notice the similarity between pricing formula (8) and (9). It implies that ambiguity aversion reduces the discount rate if

$$aEu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t). \tag{12}$$

Moreover, Observe that this condition simplifies to $a \geq 1$ when the agent is risk neutral. Because we don't constrain the risk attitude in any way except risk aversion, condition $a \geq 1$ is necessary to guarantee that ambiguity aversion reduces the discount rate. For reasons that will be clarified in the next section, we will refer to $a \geq 1$ as the ambiguity prudence effect.

In the absence of an ambiguity prudence effect ($a = 1$), condition (12) becomes $Eu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t)$, which is referred to as the pessimism effect. At this stage, it is enough to say that it comes from a distortion of the beliefs $(q_1, ..., q_n)$ on the likelihood of the different plausible probability distributions $(\widetilde{c}_1, ..., \widetilde{c}_n)$.

5 The ambiguity prudence effect

In this section, we focus on whether the constant a, defined by equation (10), is larger than unity. As stated above, this is necessary to guarantee that the discount rate is reduced and it becomes necessary and sufficient in the special case of risk-neutrality. Notice that in the latter case, a can be interpreted as the sensitiveness of the $\phi-$certainty equivalent of $\overline{c}_{\widetilde{\theta}} = E\left[\widetilde{c}_{t\widetilde{\theta}} \mid \widetilde{\theta}\right]$ with respect to an increase in saving.[6] We seek to determine whether one more unit saved yields an increase in the $\phi-$certainty equivalent future consumption. More generally, condition $a \geq 1$ can be rewritten as

$$\sum_{\theta} q_\theta \phi(u_\theta) = \phi(V_t) \quad \Longrightarrow \quad \sum_{\theta=1}^{n} q_\theta \phi'(u_\theta) \geq \phi'(V_t). \tag{13}$$

Similar questions have been raised in risk theory: Do expected-utility-preserving risks raise expected marginal utility? Along the same lines, we are able to conclude from expected utility theory that condition (13) requires ϕ to satisfy decreasing absolute ambiguity aversion (see e.g. Gollier, 2001, Section 2.5). Indeed, defining function ψ such that $\psi(\phi(U)) = \phi'(U)$ for all U, the above condition can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \psi(\phi_\theta) \geq \psi(\Sigma_\theta q_\theta \phi_\theta),$$

where $\phi_\theta = \phi(u_\theta)$ for all θ. This is true for all distributions of $(\phi_1, q_1; ...; \phi_n, q_n)$ if and only if ψ is convex. Because $\psi'(\phi(U)) = \phi''(U)/\phi'(U)$, this is true iff $A(U) = -\phi''(U)/\phi'(U)$ be non-increasing. This proves the following results.

Lemma 1 $a \geq 1$ *(resp.* $a \leq 1$*) for all acceptable ambiguous environments* $\widetilde{c} \in \Psi$ *if and only if absolute ambiguity aversion is non-increasing (resp. non-decreasing).*

[6] Define $V(s, \overline{c}_{\widetilde{\theta}})$ such that $\phi(s+V) = E\phi(s+\overline{c}_{\widetilde{\theta}})$. We have that $a = \partial V(s, \overline{c}_{\widetilde{\theta}})/\partial s$ at $s = 0$.

Proposition 2 *Suppose that the representative agent is risk-neutral. The socially efficient discount rate is smaller (resp. larger) than under ambiguity neutrality for all ambiguous environments $\widetilde{c}$ if and only if ϕ exhibits non-increasing (resp. non-decreasing) absolute ambiguity aversion.*

Rather than the extent of ambiguity aversion itself, what drives the result is how the degree of ambiguity aversion relates to conditional expected utility U. For instance, in the limit case with constant absolute ambiguity aversion, ambiguity has *no* effect on the equilibrium interest rate. The intuition for these results is easy to derive from the observation that the period-t felicity V_t is approximately equal to expected consumption minus the ambiguity premium. Moreover, the premium is itself proportional to ambiguity aversion A, which makes the willingness to save decreasing in A'. Thus, ambiguity aversion raises the willingness to save – therefore reducing the equilibrium interest rate – if absolute ambiguity aversion is decreasing.

Exactly as decreasing absolute risk aversion is unanimously accepted as a natural assumption for risk preferences, we believe that decreasing absolute ambiguity aversion (DAAA) is a reasonable property of uncertainty preferences. It means that a local mean-preserving spread in conditional expected utility has an impact on welfare that is decreasing in the level of utility where this spread is realized.

We call this the *ambiguity prudence effect* because it emerges as a consequence of the uncertainty of the future conditional expected utility. This raises the willingness to save exactly as the risk on future income raises savings in the standard expected utility model under "risk prudence". But contrary to risk prudence, which is characterized by $u''' \geq 0$, ambiguity prudence is described by decreasing absolute uncertainty aversion, which is weaker than $\phi''' \geq 0$. This is because, in the intertemporal KMM model, the future felicity is represented by the $\phi-$certainty equivalent of the conditional expected utilities, rather than by the expected $\phi-$valuation of the conditional expected utilities. Had we used this alternative model, ϕ' convex (concave) would have been the property to determine the sign of the effect.

However, once we allow for risk aversion, non-increasing ambiguity aversion is no longer sufficient to sign the impact of ambiguity on the discount rate, as shown by the following counterexample.

Counterexample 1. Let c_0 equal 2 and $\widetilde{c}_t$ shall take two olausible distributions $\widetilde{c}_{t1} \sim (1, 1/3; 4, 1/3; 7, 1/3)$ and $\widetilde{c}_{t2} \sim (3, 2/3; 4, 1/3)$, both equally plausible, i.e., $q_1 = q_2 = 1/2$. Further, preferences shall display constant relative risk aversion (CRRA) with $\gamma = 2$ such that $u(c) = -c^{-1}$, and $\delta = 0$. It is easy to check that absent ambiguity aversion the interest rate equals 9.24%. However, under constant absolute ambiguity aversion of $A = 2.11$, i.e., $\phi(U) = -\exp(-2.11U)$, tedious computations yield a discount rate of exactly zero: $r_t = 0$. ■

6 The pessimism effect

Counter-example 1 can be explained by the presence of a second effect, the pessimism effect. In the pricing formula (9), ambiguity has an effect which is equivalent to compute marginal expected utility using the distorted random variable $\widetilde{c}_t^\circ$ instead of $\widetilde{c}_t$. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (11). This section is devoted to characterize how the distortion affects the discount rate. If we find that it is pessimistic in the sense of FSD, then we are able to unambiguously sign the effect on the discount rate.

To examine this question we begin by comparing of the distorted probabilities $q^\circ = (q_1^\circ, ..., q_n^\circ)$ to the original probabilities $q = (q_1, ..., q_n)$.

Let the priors θ be ordered such that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$ and the agent prefers θ to be large. We hereafter show that ambiguity aversion is equivalent to a distortion of the prior beliefs on parameter $\widetilde{\theta}$ in the sense of the Monotone Likelihood Ratio Order (MLR). By definition, a shift of beliefs from q to q° entails a deterioration in the sense of the *monotone likelihood ratio ordering* (MLR) if $q_{\widetilde{\theta}}^\circ / q_{\widetilde{\theta}}$ and $\tilde{\theta}$ are anti-comonotonic. Observe from (11) that $q_\theta^\circ / q_\theta$ is proportional to $\phi'(Eu(\widetilde{c}_{t\theta}))$. Thus, since ϕ' is decreasing, we know that $q_{\widetilde{\theta}}^\circ / q_{\widetilde{\theta}}$ and $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ are anti-comonotonic, establishing the following property.

Lemma 2 *The subsequent conditions are equivalent:*

1. *Beliefs q° are dominated by q in the sense of the monotone likelihood ratio order for any set of marginals $(\tilde{c}_{t1}, ..., \tilde{c}_{tn})$ such that $Eu(\tilde{c}_{t1}) \leq ... \leq Eu(\tilde{c}_{tn})$.*

2. *ϕ is concave.*

The intuitive interpretation is that ambiguity aversion is characterized by an MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\widetilde{c}_{t\theta}$ to marginal $\widetilde{c}_{t\theta'}$, then, the ambiguity-averse representative agent increases the implicit prior probability q_θ° relatively more than the implicit prior probability $q_{\theta'}^\circ$. This result gives some flesh to our pessimism terminology. It also generalizes – and builds a bridge to – the max-min case where all the weight is transferred to the worst θ.[7]

However, the MLR deterioration of the distribution $\widetilde{\theta}$ of priors is not enough to ensure a negative pessimism effect on the discount rate, as shown by Counterexample 1. Instead, the crucial requirement would be that the distortion overweights scenarios which yield larger conditional expected *marginal* utility. The above lemma says something different, namely that the distortion overweights scenarios which yield larger conditional expected utility. Therefore, to

[7] This equivalence between concave transformation functions and stochastic orderings is well known (see Lehmann, 1955). Economic applications notably include rank-dependent utility models (see Quiggin, 1995).

obtain the desired result, we need to find conditions such that u and $-u'$ agree on the ranking of scenarios.

Lemma 3 *The following two conditions are equivalent:*

1. *The pessimism effect reduces the discount rate, i.e.,* $Eu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t)$, *for all* ϕ *increasing and concave;*

2. $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ *and* $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ *are anti-comonotonic.*

Proof: To prove that $2 \Rightarrow 1$, suppose that $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ be anti-comonotonic. Since ϕ' is decreasing, our assumption implies that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are comonotonic. By the covariance rule, it implies that

$$\begin{aligned}
Eu'(\overset{\circ}{\widetilde{c}_t}) &= \frac{\sum_{\theta=1}^n q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right) Eu'(\widetilde{c}_{t\theta})}{\sum_{\theta=1}^n q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&\geq \frac{\left[\sum_{\theta=1}^n q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)\right]\left[\sum_{\theta=1}^n q_\theta Eu'(\widetilde{c}_{t\theta})\right]}{\sum_{\theta=1}^n q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&= \sum_{\theta=1}^n q_\theta Eu'(\widetilde{c}_{t\theta}) = Eu'(\widetilde{c}_t).
\end{aligned}$$

In order to prove that $1 \implies 2$, suppose by contradiction that $Eu(\widetilde{c}_{t1}) < Eu(\widetilde{c}_{t2}) < ... < Eu(\widetilde{c}_{tn})$, but there exists $\theta \in [1, n-1]$ such that $Eu'(\widetilde{c}_{t\theta}) \leq Eu'(\widetilde{c}_{t\theta+1})$. Then, consider any increasing and concave ϕ that is locally linear for all $U \leq Eu(\widetilde{c}_{t\theta})$ and for all $U \geq Eu(\widetilde{c}_{t\theta+1})$, and has a strictly negative derivative in between these bounds. For any such function ϕ, we have that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic. Using the covariance rule as above, that implies that $Eu'(\overset{\circ}{\widetilde{c}_t}) < Eu'(\widetilde{c}_t)$, a contradiction.■

6.1 The CARA case

By consequence of Lemma 3, in order to sign the pessimism effect, we need to look for conditions such that u and $-u'$ indeed "agree" on a ranking of lotteries $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$. Consider first an agent who satisfies constant absolute risk aversion (CARA), i.e.,

$$u(c) = -\frac{1}{A}\exp(-Ac)$$

Since $u'(c) = \exp(-Ac)$, functions u and $-u'$ represent the same preferences over priors. Hence the following result.

Proposition 3 *Under CARA preferences, the pessimism effect always reduces the socially efficient discount rate.*

6.2 The general case

One might conjecture that the previous result extends to any concave u. That is, if both u and $-u'$ are increasing, it may seem natural that their expectations agree on the ranking of lotteries. Yet, the theory of stochastic dominance tells us that the solution is not that simple. Indeed, a necessary and sufficient condition for any two increasing utility functions on agree on the ranking of two lotteries is that they be ranked along first-degree stochastic dominance (FSD).

However, ranking the priors according to FSD is rather restrictive. It would be desirable to extend this result to a weaker stochastic order. For instance, consider the second-degree stochastic dominance order (SSD). It guarantees that $Ef(\widetilde{c}_{t\theta})$ is increasing in θ for all increasing and concave functions f. Indeed, prudence means precisely that $f = u$ and $f = -u'$ are increasing and concave. Using these conditions, we are able to once more apply Lemma 3 to obtain the following.

Proposition 4 *The pessimism effect reduces the socially efficient discount rate if*

1. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to FSD.*
2. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to SSD and u exhibits prudence.*

Essentially, the previous result exploits results on how changes in risk affect savings decisions under ambiguity-neutrality. Indeed, being risk-averse (prudent) means that an SEU agent would like to save more if an unfair (zero-mean) risk is added to her wealth (Leland, 1968; Drèze and Modigliani, 1972).

In the following proposition, we put forward a third pair of sufficient conditions. Compared to the SSD/prudence requirement, we impose a stronger restriction on utility functions as we replace prudence by the stronger DARA condition. In return we are able to relax SSD to the weaker stochastic order introduced by Jewitt (1989).

Definition 1 *We say that $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of Jewitt if the following condition is satisfied: for all increasing and concave u, if agent u prefers $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$, then all agents more risk-averse than u also prefer $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$.*

Of course, from the definition itself, if $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of SSD, this preference order also holds in the sense of Jewitt, thereby showing that this order is weaker than SSD. Jewitt (1989) shows that distribution function $F_{t\theta'}$ dominates $F_{t\theta}$ in this sense if and only if there exists some w in their joint support $[a, b]$, such that

$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \qquad \text{for all} \qquad x \in [a, w],$$
$$\int_a^w (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0,$$
$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \qquad \text{is non-increasing} \quad \text{on } [w, b].$$

Hence the result.[8]

Proposition 5 *The pessimism effect reduces the socially efficient discount rate if the set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to Jewitt's stochastic order and* u *exhibits decreasing absolute risk aversion.*

Proof: Decreasing absolute risk aversion means that $v = -u'$ is more concave than u in the sense of Arrow-Pratt. By definition of Jewitt's stochastic order, it implies that $Eu(\widetilde{c}_{t\theta'}) \geq Eu(\widetilde{c}_{t\theta})$ implies that $Ev(\widetilde{c}_{t\theta'}) \geq Ev(\widetilde{c}_{t\theta})$, or equivalently, that $Eu'(\widetilde{c}_{t\theta'}) \leq Eu'(\widetilde{c}_{t\theta})$. Using Lemma 3 concludes the proof. ■

Finally, combining Lemma 1 with Propositions 3, 4 and 5 yields our main result.

Proposition 6 *Suppose that the representative agent exhibits non increasing absolute ambiguity aversion (DAAA). Then, ambiguity aversion reduces the socially efficient discount rate if one of the following conditions holds:*

1. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to FSD and* u *is increasing and concave.*
2. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to SSD and* u *is increasing, concave, and exhibits prudence.*
3. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to Jewitt (1989) and* u *is increasing and concave, and exhibits DARA.*
4. u *exhibits constant absolute risk aversion.*

Observe that the result in our analytical example in Section 3 fits condition *1*: A mere translation in the distribution constitutes a first-degree stochastic dominance. Yet, in many circumstances, the degrees of riskiness also differ across the plausible distributions, usually implying that the plausible prior distributions cannot be ranked according to FSD. Condition *2* provides a sufficient condition on risk attitudes if marginals can only be ranked according to second-degree stochastic dominance, which contains Rothschild-Stiglitz's increases in risk as a particular case. It turns out that in this case, in addition to risk-aversion, the representative agent should also be prudent. Note that even the weaker Jewitt-ordering from condition *3* only requires decreasing absolute risk aversion. This property is widely accepted in the economic literature. It is in particular compatible with the observation that more wealthy individuals tend to take more portfolio risk.[9]

[8] Two random variables fulfill Definition 1 if there exists a consumption level w in their support such that, conditional on the outcome being lower than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of SSD, whereas conditional on the outcome being higher than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of FSD.

[9] In counterexample 1, the two random variables $\widetilde{c}_{t1}$ and $\widetilde{c}_{t2}$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

7 The comparative statics of an increase in ambiguity aversion

Our results up to now characterize the effect of smooth ambiguity aversion on the equilibrium interest rate, starting from the ambiguity-neutral benchmark. A natural question to ask is whether our results hold for any increase in ambiguity aversion.

For this purpose, consider two economies, $i = 1, 2$, identical up to the level of ambiguity aversion, with the agent in $i = 2$ more ambiguity-averse. That is to say that $\phi_2(U) = \psi(\phi_1(U))$ for all U, with the property that ψ is increasing and concave. According to the adjusted pricing formula in (9) an increase in ambiguity aversion decreases the social discount rate if and only if

$$a_2 Eu'(\tilde{c}_t^2) \geq a_1 Eu'(\tilde{c}_t^1), \tag{14}$$

where a_i is defined as in (10) with ϕ being replaced by ϕ_i, and where $\tilde{c}_t^i$ is random future consumption distorted by weights q_θ^i, as in (11). Naturally, taking ϕ_1 linear, we retrieve condition (12) from the SEU benchmark.

At the outset, we are able to generalize our findings about the pessimism effect to any increase in ambiguity aversion.

Lemma 4 *The following two conditions are equivalent:*

1. *Beliefs q^2 are dominated by q^1 in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*

2. *Agent $i = 2$ is more ambiguity-averse than $i = 1$.*

Proof: Note that we need to find that $q_{\tilde{\theta}}^2 / q_{\tilde{\theta}}^1$ and $\tilde{\theta}$ are anti-comonotonic. Using (11), we can rewrite the ratio as

$$\frac{q_\theta^2}{q_\theta^1} = \psi'\left(\phi_1(Eu(\widetilde{c}_{t\theta}))\right) \frac{\sum_{\tau=1}^n q_\tau \phi_1'\left(Eu(\widetilde{c}_{t\tau})\right)}{\sum_{\tau=1}^n q_\tau \phi_2'\left(Eu(\widetilde{c}_{t\tau})\right)}.$$

The fraction on the RHS does not change with θ. Furthermore, ψ' is decreasing in its argument. Finally, since the argument $\phi_1(Eu(\widetilde{c}_{t\theta}))$ is itself increasing with θ by assumption, we get the desired result.■

Hence, under the stochastic order conditions from Proposition 6, more ambiguity aversion reinforces the pessimism effect, which makes saving more attractive. However, it is clear from section 5 that an increase of ambiguity aversion need not reinforce the ambiguity *prudence* effect a.[10] For small amounts of ambiguity, the relation between ambiguity aversion to ambiguity on a can be approximated in the following way.

[10] For instance, introducing *increasing* absolute ambiguity aversion will in fact raise the interest rate if the representative agent is risk neutral.

Lemma 5 *Consider a family of ambiguous environments parametrized by $k \in R$ and a vector $(u_1, ..., u_n) \in R^n$ such that $Eu(\widetilde{c}_{t\theta}(k)) = u_0 + ku_\theta$ for all θ. Let us define $a(k) = \Sigma_\theta q_\theta \phi'(Eu(\widetilde{c}_{t\theta}(k)))/\phi'(V(k))$, where $\phi(V(k)) = \Sigma_\theta q_\theta \phi(Eu(\widetilde{c}_{t\theta}(k)))$. We have that*

$$a(k) = 1 - \frac{1}{2} Var(ku_{\widetilde{\theta}}) \frac{\partial}{\partial u_0} \left(\frac{-\phi''(u_0)}{\phi'(u_0)} \right) + o(k^2), \tag{15}$$

where $\lim_{k \to 0} o(k^2)/k^2 = 0$.

Proof: Observe first that $V(0) = u_0$, $V'(0) = Eu_{\widetilde{\theta}}$, and $V''(0) = Var(u_{\widetilde{\theta}})\phi''(u_0)/\phi'(u_0)$. Notice also that $a(0) = 1$. We have in turn that

$$a'(k) = \frac{E\left[u_{\widetilde{\theta}}\phi''(u_0 + ku_{\widetilde{\theta}})\right]\phi'(V(k)) - E\left[\phi'(u_0 + ku_{\widetilde{\theta}})\right]\phi''(V(k))V'(k)}{\phi'(V(k))^2}.$$

It implies that $a'(0) = 0$. Differentiating again the above equality at $k = 0$ yields

$$\begin{aligned} \phi_0'^2 a''(0) &= E\left[u_{\widetilde{\theta}}^2\right]\phi_0'\phi_0''' + \left(Eu_{\widetilde{\theta}}\right)^2 \phi_0''^2 - \left(Eu_{\widetilde{\theta}}\right)^2 \phi_0''^2 - \\ &\quad - \left(Eu_{\widetilde{\theta}}\right)^2 \phi_0'\phi_0''' - \phi_0'\phi_0''V''(0) \\ &= \left(E\left[u_{\widetilde{\theta}}^2\right] - \left(Eu_{\widetilde{\theta}}\right)^2\right)\left(\phi_0'\phi_0''' - \phi_0''^2\right), \end{aligned}$$

where $\phi_0^{(i)} = \phi^{(i)}(u_0)$. This implies that

$$a''(0) = -Var(u_{\widetilde{\theta}}) \frac{\partial}{\partial u_0} \left(\frac{-\phi''(u_0)}{\phi'(u_0)} \right).$$

The Taylor expansion of a yields $a(k) = a(0) + ka'(0) + 0.5k^2 a''(0) + o(k^2)$. Collecting the successive derivatives of a concludes the proof.■

Accordingly, for small degrees of ambiguity, a_2 is larger than a_1 if and only if

$$\frac{\partial}{\partial u_0} \left(\frac{-\phi_2''(u_0)}{\phi_2'(u_0)} \right) \geq \frac{\partial}{\partial u_0} \left(\frac{-\phi_1''(u_0)}{\phi_1'(u_0)} \right). \tag{16}$$

In others words, if locally, at the ambiguity-free expected utility level u_0, absolute ambiguity aversion decreases more rapidly under ϕ_2 than under ϕ_1 the ambiguity prudence effect is more negative in economy 2.

Unfortunately, as the following example shows, even if condition (16) holds for all u_0, this is not sufficient to guarantee $a_2 \geq a_1$.

> **Counterexample 2.** Let $\phi(U) = U^{1-\eta}/(1-\eta)$ be defined on R^+. Observe that $-\phi''(U)/\phi'(U) = \eta/U$ is positive and decreasing in its domain. Moreover, an increase in η raises both ambiguity aversion, and the speed at which absolute ambiguity aversion decreases with

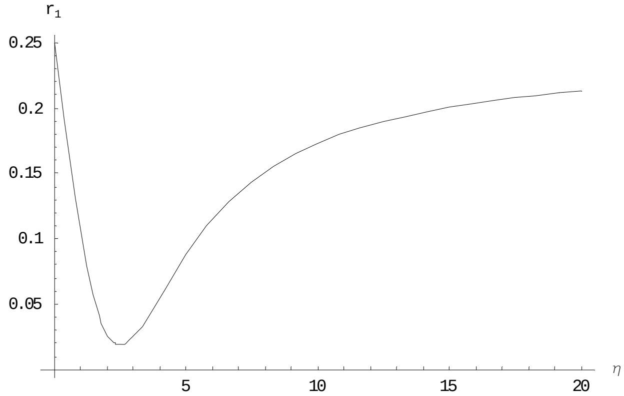

Figure 1: The discount rate as a function of relative ambiguity aversion. We assume that $\phi(U) = U^{1-\eta}/(1-\eta)$, $u(c) = c$, $\delta = 0.25$, $\overline{c}_1 = 0.5$, $\overline{c}_2 = 1.5$ and $p = 0.5$.

U. Proposition 5, yields that a is increasing in η when the risk on U is small. To show that this is not true for large degrees of ambiguity suppose risk-neutrality $u(c) = c$, $n = 2$ equally likely plausible probability distributions with $\overline{c}_1 = 0.5$ and $\overline{c}_2 = 1.5$, and let $\delta = 0.25$. In Figure 1, we draw the socially efficient discount rate r_t for $t = 1$ as a function of the degree of relative ambiguity aversion η. As stated in Proposition 2, we see that the discount rate $r_1(\eta)$ under ambiguity aversion is always smaller than under ambiguity neutrality $(r(0))$. However, the relationship between the discount rate and the degree of ambiguity aversion is not monotone. For example, increasing relative ambiguity aversion from $\eta = 3$ to any larger level *raises* the discount rate. ■

With a counter-example based on the most common family of utility functions $\phi(U) = U^{1-\eta}/(1-\eta)$, there is no hope for convincing sufficient conditions to guarantee an increase in savings. To summarize, we are left with three special cases where signing the effect on a is possible:

i) The degree of ambiguity aversion is small and condition (16) is satisfied;

ii) The initial degree of ambiguity aversion is small, so that Proposition 2 can be used as an approximation;

iii) The initial ϕ_1 function exhibits non decreasing ambiguity aversion, whereas the final ϕ_2 function exhibits non increasing ambiguity aversion. This implies that $a_1 \leq 1 \leq a_2$.

Combining any of these conditions with any of the three conditions from Proposition 6 is sufficient to guarantee that a marginal increase in ambiguity aversion reduces the socially efficient discount rate.

8 Numerical illustrations

8.1 The power-power normal-normal case

As observed in Section 3, we can solve analytically for the socially efficient discount rate by taking a "power-power" specification. That is, CRRA risk preferences and CRAA ambiguity preferences allow for an exact solution if both ambiguity and the logarithm of consumption are normally distributed. For our quantitative analysis, we parametrize the model according to the *quartet of Twos*, as put forward by Weitzman (2007b). We assume a rate of pure preference for the present $\delta = 2\%$, a degree of relative risk aversion $\gamma = 2$, a mean growth rate of consumption $g = 2\%$, and standard deviation of growth $\sigma = 2\%$. We introduce ambiguity by assuming that the growth-trend has a normal distribution with standard deviation $\sigma_0 = 1\%$. In other words, consumers believe that with a 95% probability, the growth trend lies between 0% and 4%. The Ramsey rule (7) implies

$$r_t = 5.88\% - 3\sigma_0^2 t(1 + \eta/2). \tag{17}$$

As usual, in the absence of ambiguity, the Ramsey rule prescribes a flat discount rate of 5.88%. This is no longer true under ambiguity, even for SEU agents ($\eta = 0$), as shown by Weitzman (2007a) and Gollier (2007b).

This is because ambiguity creates fatter tails in the distribution of future consumption. Indeed, ambiguity increases the volatility of log-consumption at date t by $\sigma_0^2 t^2$. Accordingly, the prudent agent wants to save more for the remote future, and the interest rate should fall with the time-horizon. If in addition, the agent exhibits ambiguity aversion, the social discount rate decreases more quickly, as seen in equation (17).

In order to calibrate the model, one needs to evaluate the degree of relative ambiguity aversion η. Consider therefore the following thought experiment.[11] Suppose that the growth rate of the economy over the next 10 years is either 20% – with probability π–, or 0%. Further, suppose that the true value of π is unknown. Rather, it is uniformly distributed on $[0, 1]$, as in the Ellsberg game in which the player has no information on the proportion of black and white balls in the urn.

Let us define the certainty equivalent growth rate $CE(\eta)$ as the sure growth rate of the economy that yields the same welfare as the ambiguous environment described above. It is implicitly defined by the following condition:

$$\left(k\frac{(1+CE)^{1-\gamma}}{1-\gamma}\right)^{1-k\eta} = \int_0^1 \left(k\left(\pi\frac{1.2^{1-\gamma}}{1-\gamma} + (1-\pi)\frac{1^{1-\gamma}}{1-\gamma}\right)\right)^{1-k\eta} d\pi,$$

where γ is set at $\gamma = 2$. In Figure 2, we plot the certainty equivalent as a function of the degree of relative ambiguity aversion. In the absence of ambiguity aversion (or if π is known to be equal to 50%), the certainty equivalent growth

[11] This is based on a 10-year version of the calibration exercise performed by Collard, Mukerji, Sheppard and Tallon (2008), who considered a power-exponential specification.

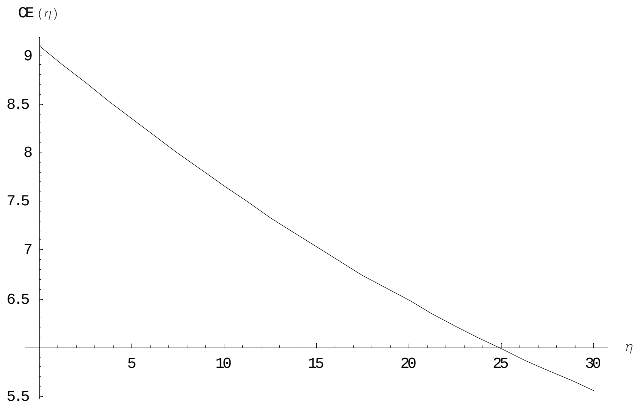

Figure 2: The certainty equivalent growth rate CE (in %) as a function of relative ambiguity aversion η. We assume that the growth rate is either 20% or 0% respectively with probability π and $1-\pi$, with $\pi \sim U(0,1)$. Relative risk aversion equals $\gamma = 2$.

rate equals $CE(0) = 9.1\%$. Surveying experimental studies, Camerer (1999) reports ambiguity premia $CE(0) - CE(\eta)$ in the order of magnitude of 10% of the expected value for such an Ellsberg-style uncertainty. This environment yields a reasonable ambiguity premium of 10%, i.e., a 1% reduction in the growth rate. Thus, ambiguity aversion should reduce the certainty equivalent from 9.1% to around 8%. From Figure 2, this is compatible with a degree of relative ambiguity aversion between $\eta = 5$ and $\eta = 10$.

Table 1 reports the values of efficient rates for projects with maturity 10 and 30 respectively.

Table 1: The social discount rate at the benchmark "quartet of twos", with $\sigma_0 = 1\%$.

t	$\boldsymbol{\eta = 0}$	$\boldsymbol{\eta = 5}$	$\boldsymbol{\eta = 10}$
10	5.58%	4.83%	4.08%
30	4.98%	2.73%	0.48%

While ambiguity aversion has no effect on the short term interest rate, its effect on the long rate is important. The discount rate for a cash flow occurring in 30 years is reduced from 4.98% to 2.73% when relative ambiguity aversion goes from $\eta = 0$ to $\eta = 5$.

The discrepancies between the settings call for an empirical separation between standard risk and ambiguity in an economy. While the former shifts the level of the yield curve, the latter determines its slope. A negative slope increases the relative importance of long-term costs and benefits.

8.2 An AR(1) process for log consumption with an ambiguous long-term trend

Clearly, in our benchmark economy, we abstract from rich consumption dynamics, notably any serial correlation. It is thus not surprising that our predictions do not fare well when confronted with the term structure of interest rates observed on financial markets. Thus, we will relax the assumption of uncorrelated growth rates and allow for persistence of shocks, as in Collard, Mukerji, Sheppard and Tallon (2008) and Gollier (2008). We hereafter show that this model can produce the desired non-linear term structure in the short run and the medium run. While, in the limit, it generates a linearly decreasing term structure in the long run.

Consider first an auto-regressive consumption process of order 1 à la Vasicek (1977), but in which the long-term growth μ of log consumption around which the actual growth mean-reverts is uncertain:

$$\begin{aligned}
\ln c_{t+1} &= \ln c_t + x_t \\
x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\
\varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\
\mu &\sim N(\mu_0,\sigma_0^2),
\end{aligned} \tag{18}$$

where $0 \leq \xi \leq 1$. That is, system (18) describes an AR(1) consumption process with unknown trend. The polar case without persistence ($\xi = 0$), amounts to the discrete time equivalent of the geometric Brownian motion considered in Section 3 and calibrated here above. In contrast, $\xi = 1$ describes shocks on the growth of log consumption that are fully persistent. Using the same techniques which led us to equation (6), we obtain the following generalization:

$$r_t = \delta + \gamma\frac{EX_t}{t} - \frac{1}{2}\gamma^2\frac{Var\left[X_t \mid \mu\right] + Var\left[E[X_t \mid \mu]\right]}{t} - \frac{1}{2}\eta\left|1-\gamma^2\right|\frac{Var\left[E[X_t \mid \mu]\right]}{t}, \tag{19}$$

where X_t is defined as

$$X_t = \ln c_t - \ln c_0 = \mu t + (x_{-1}-\mu)\frac{\xi(1-\xi^t)}{1-\xi} + \sum_{\tau=1}^{t}\frac{1-\xi^\tau}{1-\xi}\varepsilon_{t-\tau}.$$

It yields

$$\frac{EX_t}{t} = \mu_0 + (x_{-1}-\mu_0)\frac{\xi(1-\xi^t)}{t(1-\xi)},$$

$$\frac{Var\left[X_t \mid \mu\right]}{t} = \frac{\sigma^2}{(1-\xi)^2} + \sigma^2\frac{\xi(1-\xi^t)}{t(1-\xi)^3}\left[\frac{\xi(1+\xi^t)}{1+\xi} - 2\right],$$

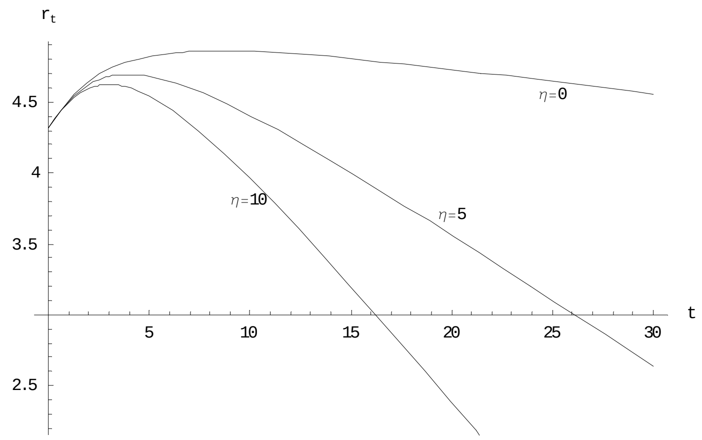

Figure 3: The term structure of discount rates in the case of an AR(1) process with ambiguous long-term trend, $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$, $x_{-1} = 1\%$, and $\xi = 0.7$.

and

$$\frac{Var\left[E[X_t \mid \mu]\right]}{t} = \frac{\sigma_0^2}{t}\left(t - \frac{\xi(1-\xi^t)}{1-\xi}\right)^2.$$

To illustrate, suppose that $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$,and $x_{-1} = 1\%$. Following Backus, Foresi and Telmer (1998), suppose also that $\xi = 0.7$ year^{-1}, such that a shock has a half-life of 3.2 years. In Figure 3, we have drawn the term structure of discount rates for 3 different degrees of ambiguity aversion: $\eta = 0$, 5, and 10. We can see that, as in the absence of persistence, the role of ambiguity aversion is to force a downward slope on the yield curve for long time horizons. This is confirmed by the following observation:

$$\lim_{t\to\infty} \frac{\partial r_t}{\partial t} = -\frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2.$$

8.3 An AR(1) process for log consumption with an ambiguous degree of mean reversion

Consider alternatively an auto-regressive consumption process of order 1 with a known long-term trend, but in which the coefficient of mean reversion is unkown:

$$\begin{aligned}
\ln c_{t+1} &= \ln c_t + x_t \\
x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\
\varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\
\xi &\sim U(\underline{\xi},\overline{\xi}).
\end{aligned}$$

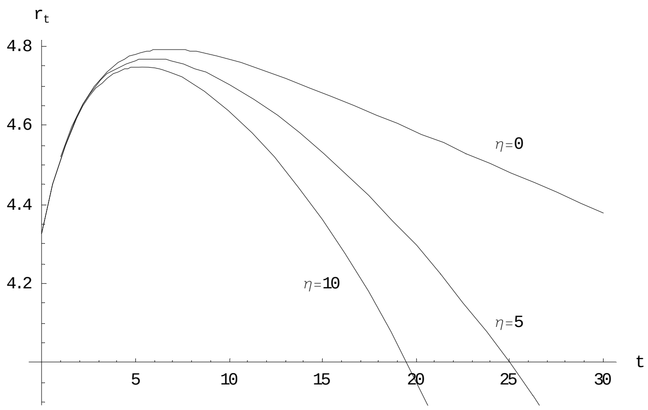

Figure 4: The term structure of discount rates in the case of an AR(1) with an ambiguous mean reversion coefficient, with $\delta = 2\%$, $\gamma = 2$, $\mu = 2\%$, $\sigma = 2\%$, $x_{-1} = 1\%$, and $\xi \sim U(0.5, 0.9)$.

There is no analytical solution for the discount rate, which must be computed numerically by estimating the following two terms, deduced from equation (3) (we normalized $c_0 = 1$):

$$\frac{E\phi'(Eu)\,Eu'}{u'(c_0)} = b(E\exp(G))$$

and

$$\phi'(V_t(0)) = b\,(E\exp(H))^{\frac{-k\eta}{1-k\eta}},$$

with

$$\begin{aligned} G &= -(\gamma + k\eta(1-\gamma))E\left[X_t \mid \xi\right] + \frac{1}{2}\left(\gamma^2 - k\eta(1-\gamma)^2\right) Var\left[X_t \mid \xi\right], \\ H &= (1-k\eta)(1-\gamma)E[X_t \mid \xi] + \frac{1}{2}(1-k\eta)(1-\gamma)^2 Var[X_t \mid \xi]. \end{aligned}$$

In Figure 4 we draw the term structure of the discount rate with the same parameter values as in the previous section, except that $\mu = 2\%$ and $\xi \sim U(0.5, 0.9)$. As before, longer time horizons yields more ambiguity in the set of plausible distributions of consumption, which implies that ambiguity aversion has a stronger negative impact on the discount rates associated to these longer durations.

9 Conclusion

The present paper has shown how ambiguity-aversion affects the efficient rate to discount future costs and benefits of investment projects. In line with recent literature, our analysis suggests that parameter uncertainty might be decisive for long-term policy appraisals. We found that, in general, it is not true that ambiguity aversion decreases the discount rate. However, we identified moderate requirements on risk-attitudes and the statistical relation among prior distributions, such that decreasing ambiguity aversion should induce us to use a smaller discount rate. Our numerical illustrations suggest the effect of ambiguity aversion on the discount rate to be large, in particular for longer time horizons.

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Appendix

Proof of Proposition 1. In order to prove this result, we need the following Lemma, which is Theorem 106 in Hardy, Littlewood and Polya (1934), Proposition 1 in Polak (1996), and Lemma 8 in Gollier (2001).

Lemma 6 *Consider a function $\phi : \mathbb{R} \to \mathbb{R}$, twice differentiable, increasing and concave. Consider a vector $(q_1, ..., q_n) \in \mathbb{R}^n_+$ with $\sum_{j=1}^n q_j = 1$, and a function f from $\mathbb{R}^n$ to $\mathbb{R}$, defined as*

$$f(U_1, ..., U_n) = \phi^{-1}(\sum_{\theta=1}^n q_\theta \phi(U_\theta)).$$

Define function T such that $T(U) = -\frac{\phi'(U)}{\phi''(U)}$. Function f is concave in $\mathbb{R}^n$ if and only if T is weakly concave in $\mathbb{R}$.

Having established the above, consider two scalars α_1 and α_2 and let us denote $U_{i\theta} = Eu(\widetilde{c}_{t\theta} + \alpha_i e^{r_t t})$. Using the notation introduced in the Lemma, it implies that $V_t(\alpha_i) = f(U_{i1}, ..., U_{in})$. Because u is concave, we have that, for any (λ_1, λ_2) such that $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} \lambda U_{1\theta} + \lambda_2 U_{2\theta} &= E\left[\lambda_1 u(\widetilde{c}_{t\theta} + \alpha_1 e^{r_t t}) + \lambda_2 u(\widetilde{c}_{t\theta} + \alpha_2 e^{r_t t})\right] \\ &\leq Eu(\widetilde{c}_{t\theta} + \alpha_\lambda e^{r_t t}) =_{def} U_{\lambda\theta}, \end{aligned}$$

for all θ, where $\alpha_\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2$. Because f is increasing in R^n, this inequality implies that

$$\begin{aligned} V_t(\lambda_1\alpha_1 + \lambda_2\alpha_2) &= f(U_{\lambda 1}, ..., U_{\lambda n}) \\ &\geq f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}). \end{aligned} \tag{20}$$

Suppose that $-\phi'/\phi''$ be concave. By the Lemma, it implies that

$$\begin{aligned} f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}) &\geq \lambda_1 f(U_{11}, ..., U_{1n}) + \lambda_2 f(U_{21}, ..., U_{2n}) \\ &= \lambda_1 V_t(\alpha_1) + \lambda_2 V_t(\alpha_2). \end{aligned} \tag{21}$$

Combining equations (20) and (21) yields $V_t(\lambda_1\alpha_1+\lambda_2\alpha_2) \geq \lambda_1 V_t(\alpha_1)+\lambda_2 V_t(\alpha_2)$, i.e., V_t is concave in α. ■