# Quasi-Variational Inequalities and Backward SDEs with constrained jumps

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# Outline

1. Introduction
   - Reminder on BSDEs, Feynman-Kac formulae and PDEs
   - Quasi-variational inequalities and impulse controls
2. Backward SDEs with constrained jumps
   - Formulation of the problem
   - Existence and approximation via penalization
3. Connection with quasi-variational inequalities
4. Numerical issues
   - Probabilistic method based on BSDE representation of QVI
   - Numerical tests
5. Conclusion

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# Basic level : Linear parabolic PDEs

• **Linear PDE** (Heat equation) :

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f = 0, \quad \text{on } [0, T) \times \mathbb{R}^d \tag{1}$$

$$v(T, .) = g \quad \text{on } \mathbb{R}^d, \tag{2}$$

where $\mathcal{L}$ is the Dynkin operator :

$$\mathcal{L}v(t,x) = b(x).D_x v(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)D_x^2 v(t,x))$$

▶ Example of applications : European option pricing in finance

# Feynman-Kac formula and Backward Stochastic Equation

- (Forward) diffusion process : $dX_s = b(X_s)ds + \sigma(X_s)dW_s$, $X_t = x$,

$\rightarrow$ Itô's formula assuming that $v$ is a smooth solution to (1)-(2) :

$$v(t, X_t) = g(X_T) + \int_t^T f(X_s)ds - \int_t^T \sigma'(X_s)D_x v(s, X_s)dW_s$$

$\rightarrow$ By taking expectation : **linear Feynman-Kac formula**

$$v(t, x) = \mathbf{E}\Big[g(X_T) + \int_t^T f(X_s)ds \Big| X_t = x\Big]$$

$\rightarrow$ Direct computations by Monte-Carlo simulations of $X$

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$\rightarrow$ Direct computations by Monte-Carlo simulations of $X$

- Notice that the pair of adapted processes $(Y, Z)$ defined by

$$Y_t := v(t, X_t), \quad Z_t := \sigma'(X_t)D_x v(t, X_t)$$

solves the **Backward stochastic equation** (Bismut 76) :

$$Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T.$$

# Level 1 : Semilinear PDEs and BSDEs

- **Semilinear PDEs** :

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f(x, v, \sigma' D_x v) \; = \; 0, \quad v(T, .) \; = \; g,$$

- **Backward SDE and nonlinear Feynman-Kac formula** (Pardoux-Peng 90) :

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s$$

and

$$Y_t \; = \; v(t, X_t) = \mathbf{E}\Big[ g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds \Big| \mathcal{F}_t \Big].$$

$\longrightarrow$ Direct simulation of the expectation is not possible !

$\longrightarrow$ But simulation of the BSDE is possible ...

# Simulation of BSDE : time discretization

- Time grid $\pi = (t_i)$ on $[0, T]$ : $t_i = i\Delta t$, $i = 0, \ldots, N$, $\Delta t = T/N$
- **Forward Euler scheme** $X^\pi$ **for** $X$ : starting from $X^\pi_{t_0} = x$,

$$X^\pi_{t_{i+1}} := X^\pi_{t_i} + b(X^\pi_{t_i})\Delta t + \sigma(X^\pi_{t_i})\big(W_{t_{i+1}} - W_{t_i}\big)$$

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- **Backward Euler scheme** $(Y^\pi, Z^\pi)$ **for** $(Y, Z)$ : starting from $Y^\pi_{t_N} = g(X^\pi_{t_N})$,

$$Y^\pi_{t_i} = Y^\pi_{t_{i+1}} + f(X^\pi_{t_i}, Y^\pi_{t_{i+1}}, Z^\pi_{t_i})\Delta t \; - Z^\pi_{t_i}.\big(W_{t_{i+1}} - W_{t_i}\big) \tag{3}$$

and take conditional expectation :

$$Y^\pi_{t_i} = \mathbf{E}\Big[Y^\pi_{t_{i+1}} + f(X^\pi_{t_i}, Y^\pi_{t_{i+1}}, Z^\pi_{t_i})\Delta t \Big| X^\pi_{t_i}\Big]$$

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and take conditional expectation :

$$Y^\pi_{t_i} = \mathbf{E}\Big[Y^\pi_{t_{i+1}} + f(X^\pi_{t_i}, Y^\pi_{t_{i+1}}, Z^\pi_{t_i})\Delta t \Big| X^\pi_{t_i}\Big]$$

To get the $Z$-component, multiply (3) by $W_{t_{i+1}} - W_{t_i}$ and take expectation :

$$Z^\pi_{t_i} = \frac{1}{\Delta t}\mathbf{E}\Big[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})\Big| X^\pi_{t_i}\Big]$$

# Simulation of BSDE : numerical methods

How to compute these conditional expectations ! several approaches :

• **Regression based algorithms** (Longstaff, Schwartz)
Choose $q$ deterministic basis functions $\psi_1, \ldots, \psi_q$, and approximate

$$Z^\pi_{t_i} = \mathbf{E}\Big[ Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \Big| X^\pi_{t_i} \Big] \simeq \sum_{k=1}^q \alpha_k \psi_k(X^\pi_{t_i})$$

where $\alpha = (\alpha_k)$ solve the least-square regression problem :

$$\arg \inf_{\alpha \in \mathbb{R}^q} \bar{\mathbf{E}} \Big[ Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) - \sum_{k=1}^q \alpha_k \psi_k(X^\pi_{t_i}) \Big]^2$$

Here $\bar{E}$ is the empirical mean based on Monte-Carlo simulations of $X^\pi_{t_i}$, $X^\pi_{t_{i+1}}$, $W_{t_{i+1}} - W_{t_i}$.
$\rightarrow$ Efficiency enhanced by using the *same set* of simulation paths to compute all conditional expectations.

- Malliavin Monte-Carlo approach (P.L. Lions, Regnier)

- Quantization methods (Pagès)

▶ Important literature : Kohatsu-Higa, Pettersson (01), Ma, Zhang (02), Bally and Pagès (03), Bouchard, Ekeland, Touzi (04), Gobet et al. (05), Soner and Touzi (05), Peng, Xu (06), Delarue, Menozzi (07), Bender and Zhang (08), etc ...

# Level 2 : Free boundary problems and reflected BSDEs

- **Variational inequalities** : given an obstacle $\phi$,

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f\ ,\ v - \phi\Big] = 0\ , \quad v(T,.) \ = \ g,$$

$\rightarrow$ related to optimal stopping problems (e.g. American option pricing in finance) :

$$Y_t \ = \ v(t,X_t) = \mathrm{ess}\sup_{\tau\in\mathcal{T}_{t,T}} \mathbf{E}\Big[\int_t^\tau f(X_s)ds + \phi(X_\tau)1_{\tau<T} + g(X_T)1_{\tau=T}\Big|\mathcal{F}_t\Big]$$

- **Reflected BSDEs** (El Karoui et al. 97) : Find a triple of adapted processes $(Y,Z,K)$ with $K$ nondecreasing s.t.

$$Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s + K_T - K_t \qquad (4)$$

$$Y_t \geq \phi(X_t) \qquad (5)$$

and $Y$ is minimal : for any $(\tilde{Y},\tilde{Z},\tilde{K})$ satisfying (4)-(5), we have $Y_t \leq \tilde{Y}_t$.

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# Quasi-variational inequalities

- **Quasi-variational inequalities (QVIs)** :

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f\ ,\ v - \mathcal{H}v\Big] = 0\ , \quad v(T,.) \ = \ g, \qquad (6)$$

where $\mathcal{L}$ is as before the Dynkin operator :

$$\mathcal{L}v(t,x) = b(x).D_x v(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)D_x^2 v(t,x))$$

and $\mathcal{H}$ is the nonlocal operator

$$\mathcal{H}v(t,x) = \sup_{e\in E}\mathcal{H}^e v(t,x)$$

with

$$\mathcal{H}^e v(t,x) = v(t,x+\gamma(x,e)) + c(x,e).$$

# QVIs and impulse controls

The QVI (6) is the **dynamic programming equation of the impulse control problem** (see Bensoussan, J.L. Lions 82) :

$$v(t,x) = \sup_{\alpha} \mathbf{E}\Big[g(X_T^\alpha) + \int_t^T f(X_s^\alpha)ds + \sum_{t<\tau_i\leq T} c(X^\alpha_{\tau_i^-},\xi_i)\Big]$$

with

- controls : $\alpha = (\tau_i, \xi_i)_i$ where
  - $(\tau_i)_i$ **time decisions** : nondecreasing sequence of stopping times
  - $(\xi_i)_i$ **action decisions** : sequence of r.v. s.t. $\xi_i \in \mathcal{F}_{\tau_i}$ valued in $E$,
- controlled process $X^\alpha$ defined by

$$X_s^\alpha = x + \int_t^s b(X_u^\alpha)du + \int_t^s \sigma(X_u^\alpha)dW_u + \sum_{t<\tau_i\leq s} \gamma(X^\alpha_{\tau_i^-},\xi_i)$$

# Interpretation of the dynamic programming equation

The QVI (6) divides the time-space domain into :

- **a continuation region** $\mathcal{C}$ in which $v(t,x) > \mathcal{H}v(t,x)$ and

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f = 0$$

- **an action region** $\mathcal{D}$ in which :

$$v(t,x) = \mathcal{H}v(t,x) \;\; = \;\; \sup_{e\in E} v(t, x+\gamma(x,e)) + c(x,e).$$

# Various applications of impulse controls

**Examples :**

• Financial modelling with discrete transaction dates, due e.g. to fixed transaction costs or liquidity constraints

• Optimal multiple stopping : swing options

• Project's investment and real options : management of power plants, valuation of gas storage and natural resources, forest management, ...

⋮

▶ Impulse control : widespread economical and financial setting with many practical applications
$\longrightarrow$ More generally to models with control policies that do not accumulate in time.

# Usual approach to QVIs

- **Main theoretical and numerical difficulty** in the QVI (6) :
  - The obstacle term contains the solution itself
  - It is nonlocal

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▶ Classical approach : Decouple the QVI (6) by defining by iteration the sequence of functions $(v_n)_n$ :

$$\min\Big[-\frac{\partial v_{n+1}}{\partial t} - \mathcal{L}v_{n+1} - f \ , \ v_{n+1} - \mathcal{H}v_n\Big] = 0 \ , \ v_{n+1}(T,.) = g$$

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→ Furthermore, to compute $v_{n+1}$, we need to know $v_n$ on the whole domain → heavy computations, especially in high dimension (state space discretization) : **numerically challenging !**

# Idea of our approach

• Instead of viewing the obstacle term as a reflection of $v$ onto $\mathcal{H}v$ (or $v_{n+1}$ into $\mathcal{H}v_n$),

▶ consider it as a constraint on the jumps of $v(t, X_t)$ for some suitable forward jump process $X$ :

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▶ consider it as a constraint on the jumps of $v(t, X_t)$ for some suitable forward jump process $X$ :

- Let us introduce the uncontrolled jump diffusion X :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_E \gamma(X_{t^-}, e)\mu(dt, de),$$

where $\mu$ is a Poisson random measure whose intensity $\lambda$ is finite and supports the whole space E.

$\rightarrow$ We **randomize** the state space !

# Idea of our approach (II)

Take some smooth function $v(t,x)$ and define :

$$Y_t \;:=\; v(t,X_t), \quad Z_t \;:=\; \sigma(X_{t^-})' D_x v(t,X_{t^-}),$$

$$\begin{aligned} U_t(e) &:= v(t,X_{t^-} + \gamma(X_{t^-},e)) - v(t,X_{t^-}) + c(X_{t^-},e) \\ &= (\mathcal{H}^e v - v)(t,X_{t^-}) \end{aligned}$$

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- Apply Itô's formula :

$$\begin{aligned} Y_t = Y_T &+ \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s.dW_s \\ &+ \int_t^T \int_E [U_s(e) - c(X_{s^-}, e)]\mu(ds, de), \end{aligned}$$

where

$$K_t \;:=\; \int_0^t \Big(-\frac{\partial v}{\partial t} - \mathcal{L}v - f\Big)(s, X_s)ds$$

# Idea of our approach (III)

- Now, suppose that $\min[-\frac{\partial v}{\partial t} - \mathcal{L}v - f, v - \mathcal{H}v] \geq 0$, and $v(T,.) = g$ :
  - Then $(Y, Z, U, K)$ satisfies

$$Y_t = g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s.dW_s + \int_t^T \int_E [U_s(e) - c(X_{s^-}, e)]\mu(ds, de), \quad (7)$$

$K$ is a nondecreasing process, and $U$ satisfies the nonpositivity constraint :

$$-U_t(e) \geq 0, \quad 0 \leq t \leq T, \; e \in E. \quad (8)$$

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▶ View (7)-(8) as a Backward Stochastic Equation with jump constraints

▶ We expect to retrieve the solution to the QVI (6) by solving the minimal solution to this constrained BSE.

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# Definition

Minimal Solution : find a solution
$(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$ to

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s.dW_s - \int_t^T \int_E (U_s(e) - c(X_{s^-}, Y_{s^-}, Z_s, e))\mu(ds, de) \tag{9}$$

with

$$h(U_t(e), e) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(de) \;\; a.e. \tag{10}$$

such that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ to (9)-(10) :

$$Y_t \leq \tilde{Y}_t, \quad 0 \leq t \leq T, \; a.s.$$

# Assumptions on coefficients

- **Forward SDE** : $b$ and $\sigma$ Lipschitz continuous, $\gamma$ bounded and Lipschitz continuous w.r.t. $x$ uniformly in $e$ :

$$|\gamma(x,e) - \gamma(x',e)| \leq k|x - x'| \quad \forall e \in E$$

- **Backward SDE** : $f$, $g$ and $c$ have linear growth, $f$ and $g$ Lipschitz continuous, $c$ Lipschitz continuous w.r.t. $y$ and $z$ uniformly in $x$ and $e$

$$|c(x,y,z,e) - c(x,y',z',e)| \leq k_c(|y - y'| + |z - z'|)$$

- **Constraint** : $h$ Lipschitz continuous w.r.t. $u$ uniformly in $e$ :

$$|h(u,e) - h(u',e)| \leq k_h|u - u'|$$

and

$$u \mapsto h(u,e) \text{ nonincreasing.} \quad (e.g. \quad h(u,e) = -u)$$

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# Penalized BSDEs

Consider for each $n$ the BSDE with jumps :

$$Y_t^n = g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n.dW_s \\ - \int_t^T \int_E [U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)]\mu(ds, de) \qquad (11)$$

with a penalization term

$$K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e)\lambda(de)ds$$

where $h^- = \max(-h, 0)$.

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- \int_t^T \int_E [U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)]\mu(ds, de) \qquad (11)
\end{aligned}
$$

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$$
K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e)\lambda(de)ds
$$

where $h^- = \max(-h, 0)$.

$\rightarrow$ For each $n$, existence and uniqueness of $(Y^n, Z^n, U^n)$ solution to (11) from Tang and Li (94), and Barles et al. (97).

# Convergence of the penalized solutions

## Theorem

Under **(H1)**, there exists a unique minimal solution

$$(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$$

with $K$ predictable, to (9)-(10). $Y$ is the increasing limit of $(Y^n)$ and also in $\mathbf{L}^2_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, $K$ is the weak limit of $(K^n)$ in $\mathbf{L}^2_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, and for any $p \in [1, 2)$,

$$\|Z^n - Z\|_{\mathbf{L}^{\mathbf{p}}(\mathbf{W})} + \|U^n - U\|_{\mathbf{L}^{\mathbf{p}}(\tilde{\mu})} \longrightarrow 0,$$

as $n$ goes to infinity.

# Convergence of the penalized BSDEs (sketch of proof)

• Convergence of $(Y^n)$ : by comparison results (under the nondecreasing property of $h$)
$\rightarrow Y^n \leq Y^{n+1}$

• Convergence of $(Z^n, U^n, K^n)$ : more delicate !

- A priori uniform estimates on $(Y^n, Z^n, U^n, K^n)_n$ in $L^2$
  $\longrightarrow$ weak convergence of $(Z^n, U^n, K^n)$ in $L^2$
- Moreover, in general, we need some strong convergence to pass to the limit in the nonlinear terms $f(X, Y^n, Z^n)$, $c(X, Y^n, Z^n)$ and $h(U^n(e), e)$.
  $\longrightarrow$ Control jumps of the predictable process $K$ via a random partition of the interval (0,T) and obtain a convergence in measure of $(Z^n, U^n, K^n)$
  $\longrightarrow$ Convergence of $(Z^n, U^n, K^n)$ in $L^p$, $p \in [1, 2)$

# Nonmarkovian case

**Remark**

Existence and uniqueness results for the minimal solution hold true in a nonmarkovian framework :

$$
\begin{aligned}
\mathbb{F} &= \text{ filtration generated by } W \text{ and } \mu \\
g(X_T) &= \zeta \\
f(x,y,z) &= f(\omega,y,z) \\
c(x,y,z) &= c(\omega,y,z)
\end{aligned}
$$

# Related semilinear QVIs

- Markov property of $X$ $\longrightarrow$ $Y_t = v(t, X_t)$ for some deterministic function $v$

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**Asumption** (H2)

*The function $v$ has linear growth* : $\sup_{[0,T]\times\mathbb{R}^d} \frac{v(t,x)}{1+|x|} < \infty$.

**Proposition**

Under **(H2)**, the function $v$ is a viscosity solution to the semilinear QVI :

$$\min\Big[ -\frac{\partial w}{\partial t} - \mathcal{L}w - f(., w, \sigma' D_x w), \inf_{e\in E} h(\mathcal{H}^e w - w, e)\Big] = 0 \qquad (12)$$

where $\mathcal{L}$ is the second order local operator as before, and $\mathcal{H}^e$, $e \in E$, are the nonlocal operators

$$\mathcal{H}^e w(t,x) = w(t, x + \gamma(x,e)) + c(x, w(t,x), \sigma'(x) D_x w(t,x), e).$$

# Terminal condition for $v$

• Need a terminal condition to complete the PDE characterization of the function $v$.

• Condition $v(T,.) = g$ is irrelevant : discontinuity in $T^-$ due to constraints

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• Condition $v(T,.) = g$ is irrelevant : discontinuity in $T^-$ due to constraints

▶ Face-lifting terminal data : $v(T^-,.)$ is the smallest function above $g$ satisfying the $(h,\mathcal{H})$-constraint

$$\min\Big[v(T^-,.) - g, \inf_{e\in E} h(\mathcal{H}^e v(T^-,.) - v(T^-,.), e)\Big] = 0 \qquad (13)$$

# Comparison and uniqueness results for semilinear QVIs

Under suitable condition **(H3)**, we have a comparison and so a uniqueness result for the semilinear QVI (12) together with the terminal data (13) :

## Proposition

Assume that **(H3)** holds. Let $U$ (resp. $V$ ) be LSC (resp. USC) viscosity supersolution (resp. subsolution) of (12)-(13) satisfying the linear growth condition

$$\sup_{[0,T]\times\mathbb{R}^d} \frac{|U(t,x)| + |V(t,x)|}{1+|x|} < \infty$$

Then, $U \geq V$ on $[0, T] \times \mathbb{R}^d$.

**Remark** The nonincreasing property of the constraint function $h$ is also crucial here.

# PDE characterization of the function $v$

**Theorem**

Under **(H2)**, **(H3)**, the function $v$ is the unique viscosity solution to (12)-(13) satisfying the linear growth condition.

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|v(t,x)|}{1+|x|} < \infty.$$

Moreover $v$ is continuous on $[0,T)\times\mathbb{R}^d$.

$\longrightarrow$ **Probabilistic representation of semilinear QVIs, and in particular of impulse control problems by means of BSDEs with constrained jumps**.

# Outline

1. Introduction
   - Reminder on BSDEs, Feynman-Kac formulae and PDEs
   - Quasi-variational inequalities and impulse controls
2. Backward SDEs with constrained jumps
   - Formulation of the problem
   - Existence and approximation via penalization
3. Connection with quasi-variational inequalities
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   - Probabilistic method based on BSDE representation of QVI
   - Numerical tests
5. Conclusion

# One approach : approximation by the penalized BSDE

• We set $V_t^n(e) = U_t^n(e) - c(X_t, Y_{t-}^n, Z_s^n, e)$, and we rewrite the penalized BSDE for $(Y^n, Z^n, V^n)$ as :

$$
\begin{aligned}
Y_t^n &= g(X_T) + \int_t^T \int_E f_n(X_s, Y_s^n, Z_s^n, V_s^n(e), e)\lambda(de)ds \\
&\quad - \int_t^T Z_s^n dW_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(de, ds)
\end{aligned}
$$

where $\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(de)dt$ (compensated martingale measure), and

$$
f_n(x, y, z, v, e) := \frac{1}{\lambda(E)} f(x, y, z) - v + nh^-(v + c(x, y, z, e), e).
$$

• We assume for simplicity that the state space of jump size $E$ is finite : $E = \{1, \ldots, m\}$ (otherwise discretize $E$).

# Time discretization of the penalized BSDE

- Time grid $\pi = (t_i)$ on $[0, T]$ : $t_i = i\Delta t$, $i = 0, \ldots, N$, $\Delta t = T/N$
- **Forward Euler scheme** $X^\pi$ **for** $X$ : starting from $X^\pi_{t_0} = x$,

$$X^\pi_{t_{i+1}} := X^\pi_{t_i} + b(X^\pi_{t_i})\Delta t + \sigma(X^\pi_{t_i})\big(W_{t_{i+1}} - W_{t_i}\big) + \sum_{e=1}^{m} \gamma(X^\pi_{t_i}, e)\mu((t_i, t_{i+1}] \times \{e\}).$$

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- **Backward Euler scheme** $(Y^{n,\pi}, Z^{n,\pi}, V^{n,\pi})$ **for** $(Y^n, Z^n, V^n)$

$$Y^{n,\pi}_{t_N} = g(X^\pi_{t_N})$$

$$Y^{n,\pi}_{t_i} = Y^{n,\pi}_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e) f_n(X^\pi_{t_i}, Y^{n,\pi}_{t_{i+1}}, Z^{n,\pi}_{t_i}, V^{n,\pi}_{t_i}(e), e)$$

$$- Z^{n,\pi}_{t_i}.\big(W_{t_{i+1}} - W_{t_i}\big) - \sum_{e=1}^{m} V^{n,\pi}_{t_i}(e)\tilde\mu((t_i, t_{i+1}] \times \{e\})$$

and take conditional expectation :

# Time discretization of the penalized BSDE (II)

$$Y_{t_i}^{n,\pi} = \mathbf{E}\Big[ Y_{t_{i+1}}^{n,\pi} + \Delta t \sum_{e=1}^{m} \lambda(e) f_n(X_{t_i}^{\pi}, Y_{t_{i+1}}^{n,\pi}, Z_{t_i}^{n,\pi}, V_{t_i}^{n,\pi}(e), e) \Big| X_{t_i}^{\pi} \Big]$$

• To get the $Z$-component, multiply by $W_{t_{i+1}} - W_{t_i}$ and take expectation in the Backward Euler scheme :

$$Z_{t_i}^{n,\pi} = \frac{1}{\Delta t} \mathbf{E}\Big[ Y_{t_{i+1}}^{n,\pi} (W_{t_{i+1}} - W_{t_i}) \Big| X_{t_i}^{\pi} \Big]$$

• To get the $V$-component, multiply by $\tilde{\mu}((t_i, t_{i+1}] \times \{e\})$ and take expectation in the Backward Euler scheme :

$$V_{t_i}^{n,\pi}(e) = \frac{1}{\lambda(e)\Delta t} \mathbf{E}\Big[ Y_{t_{i+1}}^{n,\pi} \tilde{\mu}((t_i, t_{i+1}] \times \{e\}) \Big| X_{t_i^-}^{\pi} \Big], \quad e = 1, \ldots, m.$$

# Simulation of the penalized BSDE

• Monte-Carlo simulations of the jump-diffusion process $X$ (via the Brownian motion and Poisson random measure) at dates $t_i$, $i = 1, \ldots, n$

▶ Notice that in impulse control problem, the state process depends on the control choice, and so cannot be directly simulated : we usually construct a fixed grid in the state space

▶ Here, by introducing the Poisson random measure with **a given intensity**, we **randomize** the state space : the constraint on the jump component of the backward equation "selects" the "good" points.

• Using this set of simulations, compute all the conditionals expectations arising in the Backward algorithm

# Other approach : simulate directly the constrained BSDE

For simplicity, consider $h(u,e) = -u$ (nonpositive jumps constraints) and $c = 0$

- **Approximation scheme** $(\bar{Y}^\pi, \bar{Z}^\pi, \bar{U}^\pi)$ **for the minimal solution** $(Y, Z, U)$ :

$$
\begin{aligned}
\bar{Y}^\pi_{t_N} &= g(X^\pi_{t_N}) \\
Y^{0,\pi}_{t_i} &= \mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e) f_0(X^\pi_{t_i}, \bar{Y}^\pi_{t_{i+1}}, \bar{Z}^\pi_{t_i}, U^{0,\pi}_{t_i}(e), e) \Big| X^\pi_{t_i}\Big] \\
\bar{Z}^\pi_{t_i} &= \frac{1}{\Delta t}\mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \Big| X^\pi_{t_i}\Big] \\
U^{0,\pi}_{t_i}(e) &= \frac{1}{\lambda(e)\Delta t}\mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}} \tilde{\mu}((t_i, t_{i+1}] \times \{e\}) \Big| X^\pi_{t_i^-}\Big], \quad e = 1, \ldots, m
\end{aligned}
$$

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\bar{Z}^\pi_{t_i} &= \frac{1}{\Delta t}\mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \Big| X^\pi_{t_i}\Big] \\
U^{0,\pi}_{t_i}(e) &= \frac{1}{\lambda(e)\Delta t}\mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}} \tilde{\mu}((t_i, t_{i+1}] \times \{e\}) \Big| X^\pi_{t_i^-}\Big], \quad e = 1, \ldots, m \\
\bar{U}^\pi_{t_i}(e) &= U^{0,\pi}_{t_i}(e) 1_{U^{0,\pi}_{t_i}(e) \leq 0}, \quad e = 1, \ldots, m \\
\bar{Y}^\pi_{t_i} &= \mathbf{E}\Big[\bar{Y}^\pi_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e) f_0(X^\pi_{t_i}, \bar{Y}^\pi_{t_{i+1}}, \bar{Z}^\pi_{t_i}, \bar{U}^\pi_{t_i}(e), e) \Big| X^\pi_{t_i}\Big]
\end{aligned}
$$

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# An optimal forest management

(Example taken from the book by Øksendal and Sulem 07)

- Biomass of a forest :

$$dX_s = bds + \sigma dW_s, \quad X_t = x.$$

At any times $(\tau_i)_i$, we can decide to cut down the forest and replant it, i.e. $X_{\tau_i} = 0$ with a cost $c + \theta X_{\tau_i^-}$, $\theta \in (0,1)$ :

$$v(t,x) = \sup_{(\tau_i)} \mathbf{E}\left[ \sum_{t<\tau_i\leq T} e^{-\rho\tau_i}(X_{\tau_i^-} - c - \theta X_{\tau_i^-}) \right]$$

$\longleftrightarrow$ QVI :

$$\min\left\{ -\frac{\partial v}{\partial t} + \rho v - \mathcal{L}v ; v(t,x) - [v(t,0) + (1-\theta)x - c] \right\} = 0$$

# Explicit solution on infinite horizon

For $T = \infty$, the solution is explicitly given by :

$$v(x) = \begin{cases} \frac{1-\theta}{r} e^{-r(x^*-x)}, & \text{if } x < x^* \\ \frac{1-\theta}{r} e^{-rx^*} + (1-\theta)x - c, & \text{if } x \geq x^* \end{cases}$$

where

$$r = \frac{1}{\sigma^2}\left(\sqrt{b^2 + 2\rho\sigma^2} - b\right) \; > \; 0,$$

and $x^*$ is the unique solution in $(0, \infty)$ to :

$$e^{-rx^*} + rx^* - 1 - \frac{rc}{1-\theta} = 0.$$

# Explicit optimal strategy

- This means that the optimal strategy is :

- As long as the biomass is below $x^*$, do nothing

- Whenever the biomass reaches the critical level $x^*$, cut down the forest and replant

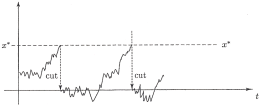



Figure 11.2: The optimal forest management

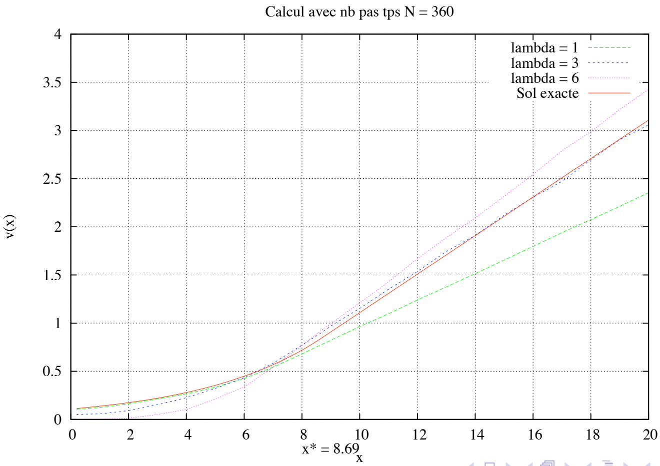
# Numerical experiments on finite horizon (Marie Bernhart)

Computation according to our algorithm of the finite horizon problem with :

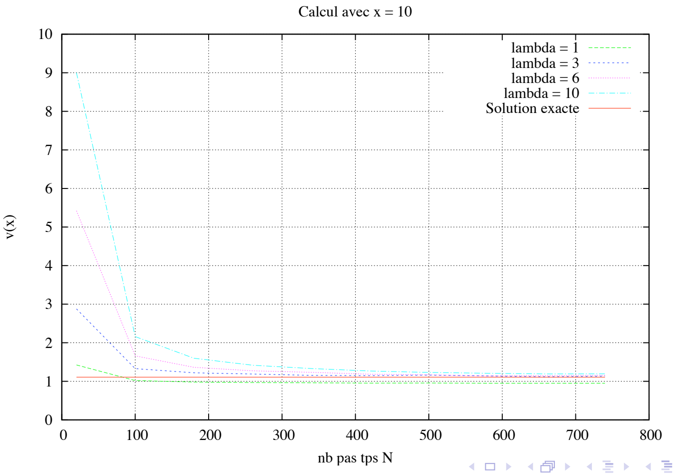
- $c = 1$, $\theta = 0.8$, $b = 2$, $\sigma = 1$,
- $T = 18$, $\rho = 0.5$

Recall that the algorithm depends on the choice of the intensity $\lambda$ of jumps, although the limiting value does not.

## Graph of $v(0,x)$ for time step $\Delta t = 1/20$ (i.e. $N = T/\Delta t = 360$), and for different values of $\lambda$



## For fixed $x = 10$, convergence of $v(0, x)$ as the number of time discretization $N = T/\Delta t$ increases, and for different values of $\lambda$



# Conclusion

- New insight into impulse control problems, and more generally into semilinear QVIs :
  - Probabilistic representation by means of BSDEs with constrained jumps
  - This provides direct (without iteration) probabilistic numerical procedure
- Current investigation and further questions on the numerical aspects
  - Analysis of the convergence of these approximation schemes
  - Computational implementation for various problems of interest with good choice of the intensity of jumps