Optimal locomotion at low Reynolds number

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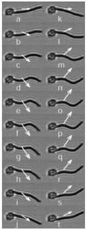
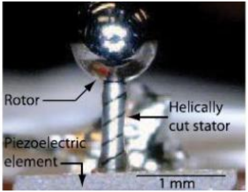
Motivation

Aim

Understanding swimming at microscopic scale

$\Longrightarrow$ For the design of micro-robots (medical applications)
$\Longrightarrow$ For biological purposes





- Swimming micro-robot : ESPCI (2005)
- Micromotor - Monash University (Australia 2008)

Swimming problems

Definition: Ability to move on or under water with appropriate movements leading to periodic shape changes (strokes) and without external forces

1st Problem: For a given deformable shape, is it possible to find an internal force law which produces a periodic shape change (a stroke) and a net displacement ?

2nd Problem: If it is possible to swim, how to swim the most efficiently possible ?

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Navier-Stokes equations

$$\left[\begin{array}{l}\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)-\nu\Delta u+\nabla p=f,\\ \mathrm{div}u=0\end{array}\right.$$

become at low $Re=\frac{\rho UL}{\nu}$ Stokes equations

$$\left[\begin{array}{l}-\nu\Delta u+\nabla p=f,\\ \mathrm{div}u=0\end{array}\right.$$

Obstruction:[Purcell]

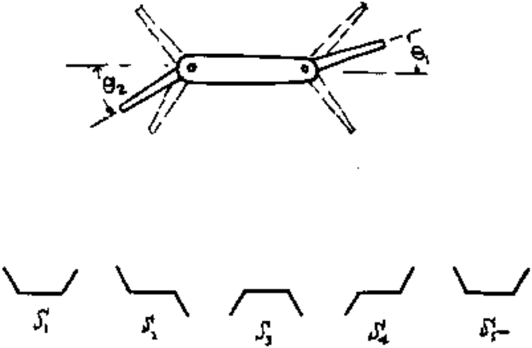
At low Reynolds number, a reciprocal motion induces no net displacement

(Movie: G. Blanchard, S. Calisti, S. Calvet, P. Fourment, C. Gluza, R. Leblanc, M. Quillas-Saavedra)

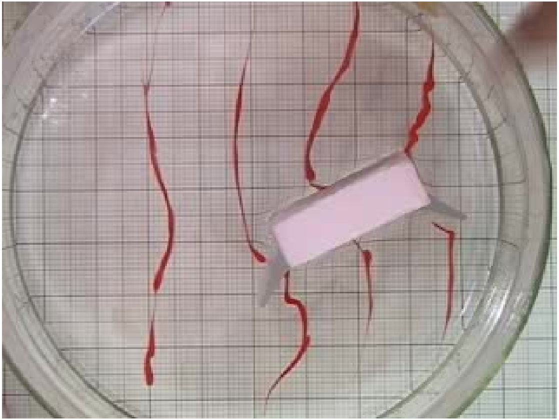
Evidence of scallop theorem

(Movie: G. I. Taylor)

Edward Mills Purcell
(1912 - 1997)



Example of swimming robot (Purcell)



Modelization

- The state of the system is given by shape and position $X = (\xi, p)$
- Shapes ξ are parameterized by a finite number of variables $\xi = (\xi_1, \cdots, \xi_N)$
- Typically the position $p = (c, R)$ where $c \in \mathbb{R}^3$, $R \in SO(3)$

The swimmer changes its shape $\Longrightarrow \xi(t)$ and pushes the fluid... which reacts (following Stokes equations) and moves (and turns) the swimmer.

Questions

- How to compute $c(t)$ and $R(t)$ knowing $\xi(t)$?
- Is it possible to find $\xi(t)$ periodic such that c and/or R is not?

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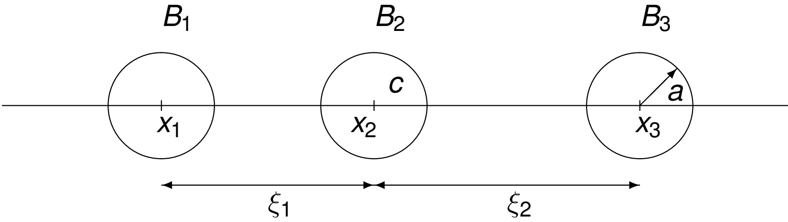
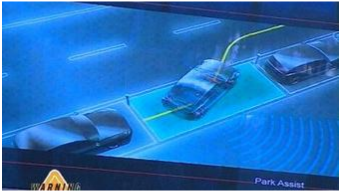
Easiest example: Najafi et Golestanian (2001)



- By changing ξ_1 and ξ_2, the spheres impose forces f_1, f_2, f_3 to the fluid $f_1 + f_2 + f_3 = 0$ (self-propulsion)
- 3 variables ξ_1, ξ_2, c and only two control parameters

Analogy



The car parking problem

- 2 controls (forward/backward motion + turn wheels to the left/right)
- Car position and orientation
- 3 variables to control and only 2 controls...

Linearity of Stokes equations

Self-propulsion

The total force applied to the fluid by the swimmer vanishes.

Here, $v = (\dot{c} - \dot{\xi_1}, \dot{c}, \dot{c} + \dot{\xi_2})$.

The total force is given by

$$F_x = A(\xi(t))\dot{c}(t) + B(\xi(t))\dot{\xi_1}(t) + C(\xi(t))\dot{\xi_2}(t) = 0$$

from which $\dot{c} = V_1(\xi)\dot{\xi_1} + V_2(\xi)\dot{\xi_2}$.

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ c\end{pmatrix} = \dot{\xi_1}\begin{pmatrix}1\\ 0\\ V_1(\xi)\end{pmatrix} + \dot{\xi_2}\begin{pmatrix}0\\ 1\\ V_2(\xi)\end{pmatrix} = \dot{\xi_1}F_1(\xi) + \dot{\xi_2}F_2(\xi)$$

- At each point $X = (\xi_1, \xi_2, c)$, the trajectory is tangent to the plane generated by $(F_1(\xi), F_2(\xi))$
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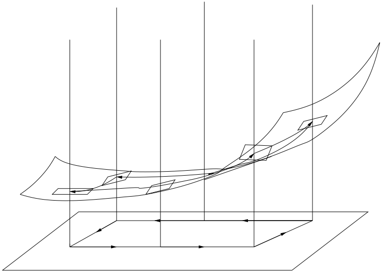
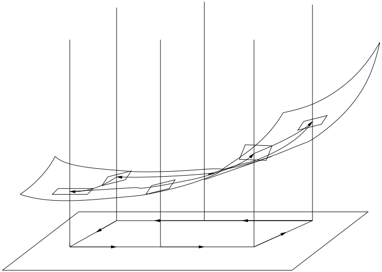
Holonomic and nonholonomic constraints



$$c = W(\xi_1, \xi_2)$$

Holonomic constraint



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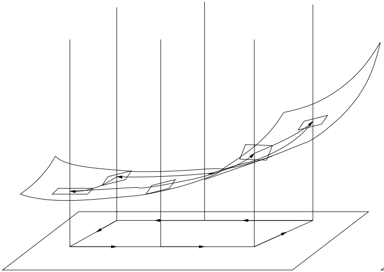

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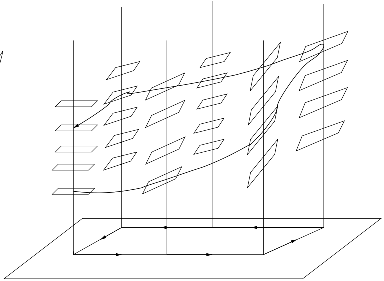

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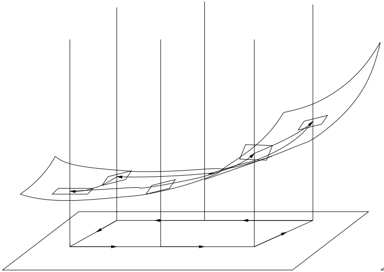

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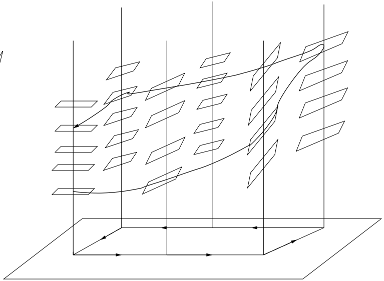

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The scallop theorem

The scallop has only one degree of freedom ξ

$$\begin{aligned}\dot{\xi} &= \alpha(t)\\ \dot{c} &= V(\xi)\dot{\xi}\end{aligned}$$

and $c = \int^{\xi} V(y)dy =: W(\xi)$

If ξ is periodic, so is c...
The constraint is always holonomic.

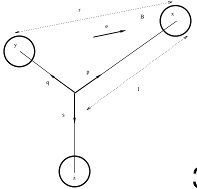
A control theorem

Theorem

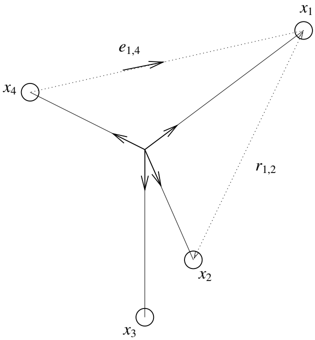
Najafi and Golestanian's 3-sphere system is globally controllable

From any state (ξ_i, c_i), one can reach any other state (ξ_f, c_f) with a suitable law force $(f_i(t))_i$ such that $\sum_i f_i(t) = 0$ (or equivalently with suitable functions $\alpha_i(t)$).

Other controllable systems:



3 controls, 3 first order Lie brackets



4 controls, 6 first order Lie brackets

Lighthill : $\mathrm{Eff}^{-1}(\xi) = C \int_0^1 \int_{\partial\Omega(t)} f \cdot v \, d\sigma \, dt$ for shape paths with fixed extremities (ξ^i, c^i) and (ξ^f, c^f)

On $\partial\Omega$, forces and velocities are linearly expressed in terms of $\dot{\xi}_i$

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Optimal swimming

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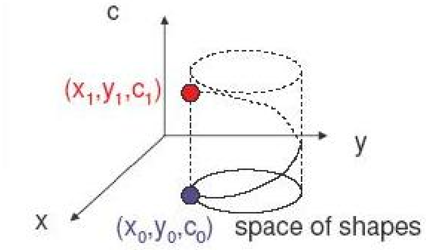
At (ξ, c) the tangent space is only bidimensional (instead of 3-dimensional) on which there is a metric
$\longrightarrow$ sub-Riemannian geometry



Optimal strokes are optimal geodesics in a sub-Riemannian space

sub-Riemannian geodesics solve a 2nd order ODE which coefficients depend on $\xi = (\xi_1, \cdots, \xi_N)$, through Stokes equation

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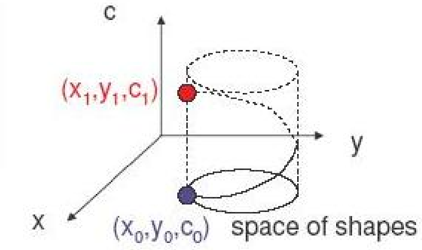

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Numerical computation of geodesic strokes

Numerical solution of Stokes problem

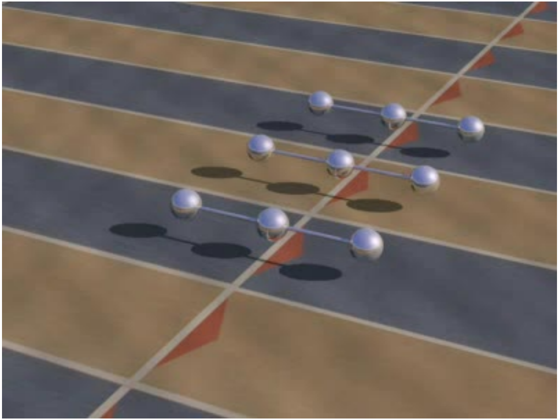
- Finite elements (axisymmetric, FREEFEM)
- BEM (axisymmetric and union of spheres)
- `C++`, written using `deal.II` library

Optimal strokes

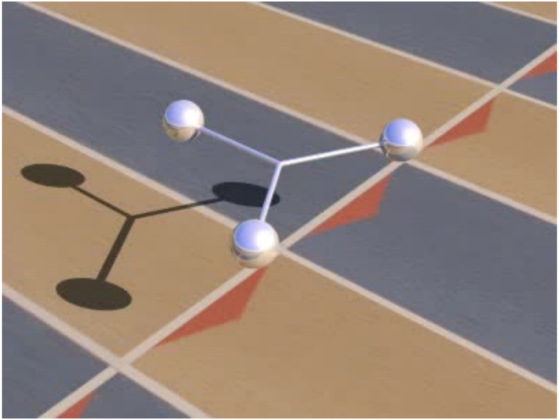
- shooting method
- global minimization using Trilinos software

Movies done with `POVRAY, BLENDER`

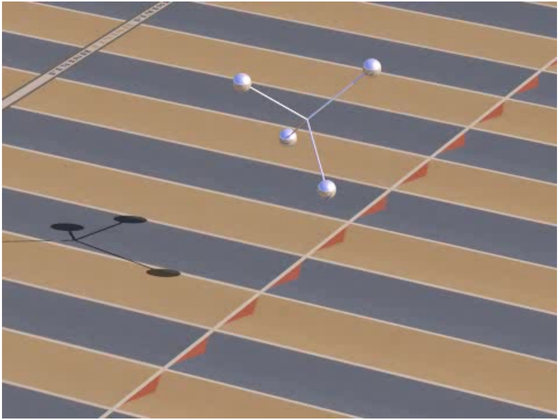
Comparison Between Square and Optimal Strokes



Plane Swimmers



Space Swimmers - Translation



Future directions

- Swimming in a bounded domain
- Stochastic forcing
- Advanced graphical tools (Blender)

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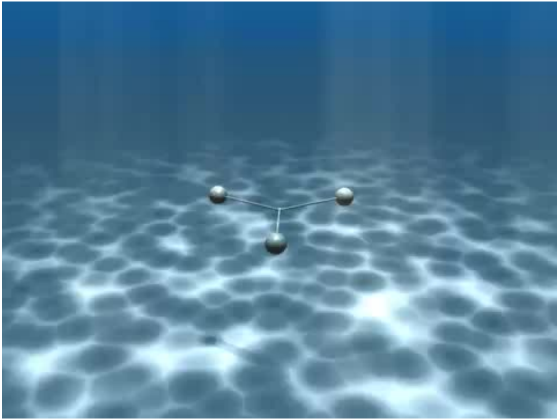

Link between both problems

PRL **103,** 078101 (2009) PHYSICAL REVIEW LETTERS week ending 14 AUGUST 2009

Accumulation of Microswimmers near a Surface Mediated by Collision and Rotational Brownian Motion

Guanglai Li and Jay X. Tang*

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(Received 23 December 2008; published 12 August 2009)

In this Letter we propose a kinematic model to explain how collisions with a surface and rotational Brownian motion give rise to accumulation of microswimmers near a surface. In this model, an elongated microswimmer invariably travels parallel to the surface after hitting it from an oblique angle. It then swims away from the surface, facilitated by rotational Brownian motion. Simulations based on this model reproduce the density distributions measured for the small bacteria *E. coli* and *Caulobacter crescentus*, as well as for the much larger bull spermatozoa swimming between two walls.

DOI: 10.1103/PhysRevLett.103.078101 PACS numbers: 47.63.Gd, 05.40.Jc, 87.17.Jj