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Partial Differential Equations

Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms

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Abstract

In this Note we prove an a priori estimate and the existence of a solution for a class of nonlinear elliptic problems whose model is $-\operatorname{div} A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$, when $1 < q < 2$ and $f \in L^m(\Omega)$ for some suitable m. The main interest of the result lies in the a priori estimate, the complete proof of which is given in the Note. ***To cite this article: N. Grenon et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***
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Résumé

Existence et estimation a priori pour des problèmes elliptiques avec des termes sous quadratiques par rappport au gradient. Dans cette Note nous démontrons une estimation a priori et l'existence d'une solution pour une classe de problèmes non linéaires dont le modèle est $-\operatorname{div} A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$, où $1 < q < 2$ et où $f \in L^m(\Omega)$ pour un m convenable. L'intérêt principal du résultat réside dans l'estimation a priori, dont la démonstration complète est donnée dans la Note. ***Pour citer cet article : N. Grenon et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***
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Version française abrégée

Dans cette Note, nous démontrons l'existence d'une solution de

$$u \in H_0^1(\Omega), \quad -\operatorname{div} A(x)Du + a(x)u = H(x, u, Du) \quad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

quand Ω est un ouvert borné de $\mathbb{R}^N$, $N \geqslant 3$, A une matrice bornée coercive, a une fonction bornée non négative (voir les hypothèses (2), (3) de la version anglaise) et quand $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodory qui vérifie

$$\left|H(x, s, \xi)\right| \leqslant \gamma|\xi|^q + f(x) \quad \text{avec } \gamma \geqslant 0,\ 1 + \frac{2}{N} \leqslant q < 2,\ f \in L^m(\Omega) \text{ et } m = \frac{N}{q'}. \tag{4}$$

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Un exemple modèle de fonction H qui vérifie (4) est donné par $H(x,u,Du) = c(x)Du + d(x)|Du|^r + f(x)$, où $c \in L^N(\Omega)^N$, $0 \leqslant r \leqslant q < 2$, $d \in L^s(\Omega)$ pour un s convenable et $f \in L^p(\Omega)$ avec $p \geqslant m = N/q'$.

Le problème (1) a fait l'objet de nombreux travaux quand H est à croissance quadratique ou linéaire par rapport à Du. Quand $q = 2$, la condition (4) devient $f \in L^{N/2}(\Omega)$; sous cette hypothèse, l'existence d'une solution de (1) qui vérifie de plus la condition $\mathrm{e}^{\gamma|u|} - 1 \in H_0^1(\Omega)$ a été démontrée dans [10] si $\alpha_0 = 0$ et dans [8] si $\alpha_0 > 0$ (voir aussi [4,5] dans le cas où $q = 2$ et $m > N/2$). Quant à lui, le cas $q = 1$ est classique, même si l'opérateur est non coercif lorsque γ est grand ; cette difficulté a été résolue dans [7]. Mais à notre connaissance, le cas $1 < q < 2$ est resté ouvert jusqu'à maintenant, à l'exception de l'article récent [9] (voir aussi [6]).

Dans cette Note, nous nous restreignons au cas où $q \geqslant 1 + 2/N$. En effet, dans ce cas, m vérifie $m = N/q' \geqslant 2N/(N+2) = (2^*)'$, et on a donc $f \in H^{-1}(\Omega)$, ce qui permet de chercher des solutions dans $H_0^1(\Omega)$. Dans notre futur article [11], nous traiterons aussi le cas $1 < q < 1 + 2/N$ par la méthode que nous utilisons ici ; les résultats obtenus sont similaires même si les solutions, qui ne sont plus dans $H_0^1(\Omega)$, doivent être définies au sens des solutions renormalisées (ou d'entropie).

Théorème 1. *Supposons que l'on a, outre* (2), (3) *et* (4), *l'une des deux hypothèses suivantes* :

$$\text{ou bien } \alpha_0 > 0, \text{ ou bien } \alpha_0 = 0 \text{ et } \gamma^{\frac{1}{q-1}} \, \|f\|_{L^m(\Omega)} < C_0 \, \alpha^{\frac{q}{q-1}},$$

où C_0 est une constante qui dépend seulement de N et q.

Alors il existe au moins une solution u de (1) *qui de plus vérifie*

$$|u|^\sigma \in H_0^1(\Omega) \quad \text{avec } \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \tag{7}$$

De plus, toute solution de (1) *qui vérifie la condition de régularité* (7) *vérifie l'estimation a priori*

$$\|u\|_{H_0^1(\Omega)} + \big\||u|^\sigma\big\|_{H_0^1(\Omega)} \leqslant M, \tag{8}$$

où M depend seulement de $N, q, |\Omega|, \alpha, \alpha_0, \gamma$ et f.

Quand $\alpha_0 > 0$, la constante M qui apparaît dans l'estimation a priori (8) dépend de la fonction f, non seulement par l'intermédiaire de sa norme $\|f\|_{L^m(\Omega)}$, mais aussi par l'intermédiaire du nombre k^* défini par (17). Cependant la constante M est bornée quand f varie dans un ensemble de fonctions qui sont bornées et équi-intégrables dans $L^m(\Omega)$.

La relation entre les paramètres q et m imposée dans (4) est naturelle, car les conditions nécessaires pour l'existence d'une solution de (1) démontrées dans [1,12] conduisent dans le cadre adopté ici à $m = N/q'$. De plus ces résultats montrent qu'une condition sur la taille des données est nécessaire pour l'existence d'une solution de (1) quand $\alpha_0 = 0$.

L'exposant σ défini par (7) vérifie $\sigma \geqslant 1$ quand $q \geqslant 1 + 2/N$.

Il existe des solutions de (1) qui ne vérifient pas la condition de régularité (7). Un exemple classique en est $u(x) = C_q\,(|x|^{-(2-q)/(q-1)} - 1)$, qui, lorsque $q > 1 + 2/N$, vérifie, pour un choix convenable de C_q, $u \in H_0^1(\Omega)$, $-\Delta u = |Du|^q$ in $\mathcal{D}'(B_1)$ et $|u|^\rho \in H_0^1(\Omega)$ pour tout $\rho < \sigma$, mais ne vérifie pas $|u|^\sigma \in H_0^1(\Omega)$. Par contre, des résultats d'unicité pour les solutions de (1) qui vérifient (7) ont été récemment démontrés dans [2].

Dans notre futur article [11], nous donnerons un certain nombre de généralisations du Théorème 1. Comme nous l'avons déja dit, nous traiterons tout l'intervalle $1 < q < 2$. Nous étudierons aussi le cas des conditions aux limites de Neumann et de Fourier, le cas des ouverts non bornés, et celui de fonctions f appartenant à des espaces de Lorentz ou à des espaces de Sobolev négatifs. Tout cela sera présenté dans le cadre d'opérateurs non linéaires pseudomonotones de type Leray–Lions définis dans l'espace $W^{1,p}(\Omega)$. Nous considérerons plus tard l'analogue parabolique du problème (1).

1. Introduction, main result and comments

In this Note we prove the existence of a solution of

$$u \in H_0^1(\Omega), \quad -\operatorname{div} A(x)Du + a(x)u = H(x,u,Du) \quad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

when Ω is an open bounded set of $\mathbb{R}^N$ ($N \geqslant 3$), A a bounded coercive matrix, a a bounded nonnegative function, i.e.,

$$A \in L^\infty(\Omega)^{N\times N}, \qquad A \geqslant \alpha I, \quad \alpha > 0, \tag{2}$$

$$a \in L^\infty(\Omega), \qquad a \geqslant \alpha_0, \quad \alpha_0 \geqslant 0, \tag{3}$$

and when $H:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function which satisfies

$$\bigl|H(x,s,\xi)\bigr|\leqslant\gamma|\xi|^q+f(x)\quad\text{with }\gamma\geqslant 0,\ 1+\frac{2}{N}\leqslant q<2,\ f\in L^m(\Omega)\text{ and }m=\frac{N}{q'}.\tag{4}$$

A model example of function H which satisfies (4) is $H(x,u,Du)=c(x)Du+d(x)|Du|^r+f(x)$ with $c\in L^N(\Omega)^N$, $0\leqslant r\leqslant q<2$, $d\in L^s(\Omega)$ for a suitable s and $f\in L^p(\Omega)$ with $p\geqslant m=N/q'$.

There is a wide literature concerning problem (1) when H has a quadratic or a linear growth with respect to Du. When $q=2$, condition (4) becomes $f\in L^{N/2}(\Omega)$; in this case existence of a solution of (1) which satisfies the further regularity $\mathrm{e}^{\gamma|u|}-1\in H_0^1(\Omega)$ has been proved in [10] if $\alpha_0=0$ and in [8] if $\alpha_0>0$ (previous references for the case where $q=2$ and $m>N/2$ are, e.g., [4,5]). On the other hand, the case $q=1$ is classical, but exhibits an important difficulty when γ is large, due to the fact that the operator is then non coercive; this problem has been solved in [7]. As far as we know, the case $1<q<2$ has been left open until now, except for the very recent paper [9] (see also [6]).

In the present Note, we restrict ourselves to the case where $q\geqslant 1+2/N$. Indeed in this case m satisfies $m=N/q'\geqslant 2N/(N+2)=(2^*)'$, hence $f\in H^{-1}(\Omega)$. This allows us to look for solutions which belong to $H_0^1(\Omega)$. In our forthcoming paper [11] we will also treat the case $1<q<1+2/N$ by the method used in the present Note and obtain very similar results, except for the fact that the solution is no more in $H_0^1(\Omega)$ and has to be defined as a renormalized (or entropy) solution.

Theorem 1. *Assume* (2), (3), (4), *and one of the two following hypotheses*:

$$\textit{either } \alpha_0>0,\tag{5}$$

$$\textit{or } \alpha_0=0 \textit{ and } \gamma^{\frac{1}{q-1}}\|f\|_{L^m(\Omega)}<C_0\alpha^{\frac{q}{q-1}},\tag{6}$$

where C_0 is a constant which depends only on N and q.

Then there exists at least one solution u of (1) *which further satisfies*

$$|u|^\sigma\in H_0^1(\Omega)\quad\textit{with } \sigma=\frac{(N-2)(q-1)}{2(2-q)}.\tag{7}$$

Moreover, every solution of (1) *which satisfies the regularity requirement* (7) *satisfies the estimate*

$$\|u\|_{H_0^1(\Omega)}+\bigl\||u|^\sigma\bigr\|_{H_0^1(\Omega)}\leqslant M,\tag{8}$$

where M depends only on N, q, $|\Omega|$, α, α_0, γ and f.

Remark 2. When $\alpha_0>0$, the constant M which appears in the a priori estimate (8) depends on the function f, not only through its norm $\|f\|_{L^m(\Omega)}$, but also through the number k^* defined by (17) below. However, the constant M is bounded when f varies in a set of functions which are bounded and equi-integrable in $L^m(\Omega)$.

Remark 3. The link imposed in (4) between the parameters q and m is natural. Consider indeed the model problem $u\in H_0^1(\Omega)$, $-\Delta u+u=\gamma|Du|^q+f(x)$ in $\mathcal{D}'(\Omega)$, with $1<q<2$ and $f\in L^m(\Omega)$ for some m. Since $f\in L^m(\Omega)$, one expects, in view of the $W^{2,p}$ regularity result, that $u\in W^{2,m}(\Omega)$, which in turns implies, by reading the equation, that $|Du|^q\in L^m(\Omega)$. But $W^{2,m}(\Omega)\subset W^{1,m^*}(\Omega)$ implies $|Du|\in L^{m^*}(\Omega)$. This leads to $qm=m^*$, namely $m=N/q'$ as required. On the other hand, the necessary conditions obtained in [1,12] for the existence of a solution of (1), when specialized to the present setting, lead to the condition $m=N/q'$. Moreover, these papers show that a condition like (6) on the size of the data is necessary in order to have the existence of a solution of (1) when $\alpha_0=0$.

Remark 4. The exponent σ defined by (7) satisfies $\sigma\geqslant 1$ when $q\geqslant 1+2/N$.

Remark 5. There exist solutions of (1) which do not satisfy the regularity (7). A well-known example is the function $u(x)=C_q\,(|x|^{-\frac{2-q}{q-1}}-1)$, which, when $q>1+2/N$, satisfies, for a suitable choice of C_q, $u\in H_0^1(\Omega)$, $-\Delta u=|Du|^q$ in $\mathcal{D}'(B_1)$ and $|u|^\rho\in H_0^1(\Omega)$ for every $\rho<\sigma$, but does not satisfy $|u|^\sigma\in H_0^1(\Omega)$. In contrast, uniqueness results for the solutions of (1) which satisfy the regularity requirement (7) have been recently proved in [2].

Remark 6. In our forthcoming paper [11], we will present many extensions of Theorem 1. As said before, we will consider, as far as the growth of $H(x,u,Du)$ with respect to $|Du|$ is concerned, the full range $1<q<2$. We will also treat the case of Neumann's and Robin's boundary conditions, the case of unbounded domains and the case of functions f in some Lorentz spaces and in some negative Sobolev spaces. All of this will be done for general nonlinear pseudomonotone operators of Leray–Lions type defined in $W^{1,p}(\Omega)$. We will consider the parabolic analogue of problem (1) later.

2. Proof of Theorem 1

In this proof, C_0, C_1, C_2, C_3 and the generic constant C will denote different positive constants which depend only on N and q.

We first prove the second part of Theorem 1, namely the a priori estimate (8).

Let u be any solution of (1) which satisfies the regularity requirement (7). In Eq. (1) we can take as test function any function $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, but also $v=|u|^{2\sigma-2}u$ (recall that $\sigma \geqslant 1$): to prove this assertion, take $v=|T_n(u)|^{2\sigma-2}T_n(u)$, where T_n is the truncation at height n, and pass to the limit using (4) and (7). We actually use a slight modification of the latest test function: given $k>0$, we choose as test function in (1) $v=|G_k(u)|^{2\sigma-2}G_k(u)$, where $G_k(s)=s-T_k(s)$. We obtain

$$\begin{cases} \alpha(2\sigma-1)\displaystyle\int_\Omega \big|G_k(u)\big|^{2\sigma-2}\big|DG_k(u)\big|^2\,\mathrm{d}x+\alpha_0\int_\Omega |u|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x \\ \quad\leqslant \gamma\displaystyle\int_\Omega \big|DG_k(u)\big|^q\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x+\int_\Omega |f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x. \end{cases} \tag{9}$$

The first term of the left-hand side of (9) is nothing but

$$\alpha(2\sigma-1)\int_\Omega \big|G_k(u)\big|^{2\sigma-2}\big|DG_k(u)\big|^2\,\mathrm{d}x=\alpha C\int_\Omega \big|D\big(\big|G_k(u)\big|^\sigma\big)\big|^2\,\mathrm{d}x. \tag{10}$$

We then estimate the first term of the right-hand side of (9); Hölder's inequality and the same computation as in (10) yield

$$\begin{cases} \gamma\displaystyle\int_\Omega \big|DG_k(u)\big|^q\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x=\gamma\int_\Omega \big|DG_k(u)\big|^q\big|G_k(u)\big|^{(\sigma-1)q}\big|G_k(u)\big|^{2\sigma-1-(\sigma-1)q}\,\mathrm{d}x \\ \qquad\leqslant \gamma C\left(\displaystyle\int_\Omega \big|D\big(\big|G_k(u)\big|^\sigma\big)\big|^2\,\mathrm{d}x\right)^{\frac{q}{2}}\left(\int_\Omega \big|G_k(u)\big|^{(2\sigma-1-(\sigma-1)q)\frac{2}{2-q}}\,\mathrm{d}x\right)^{1-\frac{q}{2}}. \end{cases}$$

However, $(2\sigma-1-(\sigma-1)\,q)\frac{2}{2-q}=\sigma 2^*$, and Sobolev's embedding yields

$$\gamma\int_\Omega \big|DG_k(u)\big|^q\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x\leqslant \gamma C\big\|D\big(\big|G_k(u)\big|^\sigma\big)\big\|_{L^2(\Omega)}^{q+(1-\frac{q}{2})2^*}. \tag{11}$$

We finally estimate the second term of the right-hand side of (9) by (note that $|u|>k$ when $G_k(u)\neq 0$)

$$\begin{cases} \displaystyle\int_\Omega |f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x=\int_{\{|f|\leqslant\alpha_0|u|\}}|f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x+\int_{\{|f|>\alpha_0|u|\}}|f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x \\ \qquad\leqslant \alpha_0\displaystyle\int_\Omega |u|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x+\int_{\{|f|>\alpha_0 k\}}|f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x. \end{cases} \tag{12}$$

The first term of the right-hand side of (12) is absorbed by the second term of the left-hand side of (9). Using Hölder's inequality, and since $(2\sigma-1)\,m' = \sigma\,2^*$, we estimate the second term by

$$\begin{cases} \displaystyle\int_{\{|f|>\alpha_0 k\}} |f|\big|G_k(u)\big|^{2\sigma-1}\,\mathrm{d}x \leqslant \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)}\big\|\big|G_k(u)\big|^{2\sigma-1}\big\|_{L^{m'}(\Omega)} \\ \qquad\qquad\qquad \leqslant C\|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)}\big\|D\big(\big|G_k(u)\big|^\sigma\big)\big\|_{L^2(\Omega)}^{\frac{2^*}{m'}}. \end{cases} \tag{13}$$

Set

$$Y_k = \big\|D\big(\big|G_k(u)\big|^\sigma\big)\big\|_{L^2(\Omega)}.$$

Using (10)–(13) in (9) yields, for two positive constants C_1 and C_2

$$\alpha\,C_1\,Y_k^2 \leqslant \gamma\,C_2 Y_k^{q+(1-\frac{q}{2})2^*} + \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)} Y_k^{\frac{2^*}{m'}}.$$

Dividing by $Y_k^{\frac{2^*}{m'}}$, and using $2-\frac{2^*}{m'} = \frac{2^*}{N}(q'-2)$ and $q+(1-\frac{q}{2})2^* - \frac{2^*}{m'} = \frac{2^*}{N}(q'-q)$, we finally obtain

$$\forall k \geqslant 0,\quad \alpha C_1 Y_k^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y_k^{\frac{2^*}{N}(q'-q)} \leqslant \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)}. \tag{14}$$

Define the function $F:\mathbb{R}^+ \to \mathbb{R}$ by

$$F(Y) = \alpha C_1 Y^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y^{\frac{2^*}{N}(q'-q)}.$$

Then (14) is equivalent to

$$\forall k \geqslant 0,\quad F(Y_k) \leqslant \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)}. \tag{15}$$

Since $q<2$, F is a concave function with a unique maximizer Z^* and maximum F^*, where Z^* and F^* are given by

$$Z^* = C_3\left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}} \quad\text{and}\quad F^* = C_0\frac{\alpha^{\frac{q}{q-1}}}{\gamma^{\frac{1}{q-1}}}.$$

Inequality (15) is non trivial only if its right-hand side is strictly smaller than F^*.

Here we split the proof into two cases.

(i) If $\alpha_0 = 0$, hypothesis (6) is nothing but $\|f\|_{L^m(\Omega)} < F^*$. Then equation $F(Y) = \|f\|_{L^m(\Omega)}$ has two roots Z_0^- and Z_0^+, with $0 < Z_0^- < Z^* < Z_0^+$, and inequality (15) is equivalent to

$$\forall k \geqslant 0,\quad \text{either } Y_k \leqslant Z_0^- \text{ or } Y_k \geqslant Z_0^+. \tag{16}$$

However, since $|u|^\sigma \in H_0^1(\Omega)$, the function $k \to Y_k = \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}$ is continuous and tends to zero when k tends to infinity. The alternative (16) then implies that $Y_k \leqslant Z_0^-$ *for every* k; in particular, one has

$$Y_0 = \big\|D\big(|u|^\sigma\big)\big\|_{L^2(\Omega)} \leqslant Z_0^- < Z^*.$$

(ii) If $\alpha_0 > 0$, we define k^* as

$$k^* = \inf\big\{k>0:\ \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)} < F^*\big\}. \tag{17}$$

For every $\delta>0$, one has $\|f\chi_{\{|f|>\alpha_0(k^*+\delta)\}}\|_{L^m(\Omega)} < F^*$, and equation $F(Y) = \|f\chi_{\{|f|>\alpha_0(k^*+\delta)\}}\|_{L^m(\Omega)}$ has two roots $Z_{k^*+\delta}^-$ and $Z_{k^*+\delta}^+$, with $0 < Z_{k^*+\delta}^- < Z^* < Z_{k^*+\delta}^+$. Inequality (15) implies that for every $k \geqslant k^*+\delta$, either $Y_k \leqslant Z_{k^*+\delta}^-$, or $Y_k \geqslant Z_{k^*+\delta}^+$. However, the function $k \to Y_k$ is continuous and tends to zero when k tends to infinity. We conclude that for every $k \geqslant k^*+\delta$, one has $Y_k \leqslant Z_{k^*+\delta}^-$, and in particular that

$$Y_{k^*+\delta} = \big\|D\big(\big|G_{k^*+\delta}(u)\big|^\sigma\big)\big\|_{L^2(\Omega)} \leqslant Z_{k^*+\delta}^- < Z^*.$$

We then let δ tend to zero.

In both cases, we have proved that

$$\big\|D\big(\big|G_{k^*}(u)\big|^\sigma\big)\big\|_{L^2(\Omega)} \leqslant Z^* = C_3\left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}}, \tag{18}$$

where $k^*=0$ when $\alpha_0=0$, and where k^* is defined by (17) when $\alpha_0>0$. When $k^*=0$, inequality (18) is nothing but the second part of the a priori estimate (8). Note that the constant Z^*, which plays here a role similar to the constant M, depends only on N, q, α and γ, but that k^* depends on the function f itself.

We now prove the first part of the a priori estimate (8). Since one has $DG_{k_1^*}(u)=\chi_{\{|u|\geqslant k_1^*\}}Du$, and $|D(|G_{k^*}(u)|^\sigma)| = \sigma|G_{k^*}(u)|^{\sigma-1}|DG_{k^*}(u)|$, estimate (18) provides an estimate of $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$ for $k_1^*=k^*+1$. We then use $v=T_{k_1^*}(u)$ in (1) and we get

$$\begin{cases} \alpha\int_\Omega\big|DT_{k_1^*}(u)\big|^2\,\mathrm{d}x \leqslant \int_\Omega\big(\gamma|Du|^q+f\big)\big|T_{k_1^*}(u)\big|\,\mathrm{d}x \leqslant \gamma k_1^*\int_\Omega|Du|^q + k_1^*\|f\|_{L^1(\Omega)} \\ \qquad\qquad \leqslant \gamma k_1^*\int_\Omega\big|DT_{k_1^*}(u)\big|^q\,\mathrm{d}x + \gamma k_1^*\int_\Omega\big|DG_{k_1^*}(u)\big|^q\,\mathrm{d}x + k_1^*\|f\|_{L^1(\Omega)}, \end{cases}$$

from which we deduce, using $q<2$ and the estimate on $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$, an estimate on $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$, and thus the first part of the a priori estimate (8), with a constant which depends on k^*.

Finally combining the estimates on $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$ and $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$ and estimate (18) completes the proof of the second part of the a priori estimate (8). The constant which appears in this estimate depends on k^*.

We now pass to the proof of the existence of a solution of (1) which satisfies the regularity requirement (7). This proof is classical. One considers the approximation of (1) by the problem in which the function H is replaced by the function $H_\varepsilon = T_{\frac{1}{\varepsilon}}(H)$; note that this function H_ε satisfies (4) for every $\varepsilon>0$. This equation has at least one solution u_ε which, by the weak maximum principle, belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$. Therefore u_ε satisfies the regularity requirement (7), and the a priori estimate (8) ensures that u_ε and $|u_\varepsilon|^\sigma$ are bounded in $H_0^1(\Omega)$. In view of the growth condition (4), $H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ is then bounded in $L^{2/q}(\Omega)$. Let $u\in H_0^1(\Omega)$ be such that a subsequence (still denoted by ε) u_ε weakly converges to u in $H_0^1(\Omega)$. The bound of u_ε in $H_0^1(\Omega)$ implies that $-\operatorname{div}A(x)Du_\varepsilon$ is bounded both in $H^{-1}(\Omega)$ and in $L^{2/q}(\Omega)$. These bounds imply (see, e.g., [3]) that, extracting if necessary a new subsequence, Du_ε converges to Du almost everywhere in Ω, which in turn implies that $H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ converges to $H(x,u,Du)$ strongly in $L^s(\Omega)$ for every $s<2/q$. This result easily allows one to pass to the limit in the approximate equation, which proves the existence of a solution of (1) which satisfies (7).

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