New hyperbolic model of shear shallow water flows : applications to roll waves and classical hydraulic jumps

Sergey GAVRILYUK

Aix-Marseille Université
and
CNRS, UMR 7343, IUSTI, Marseille, France

Joint work with Gaël RICHARD

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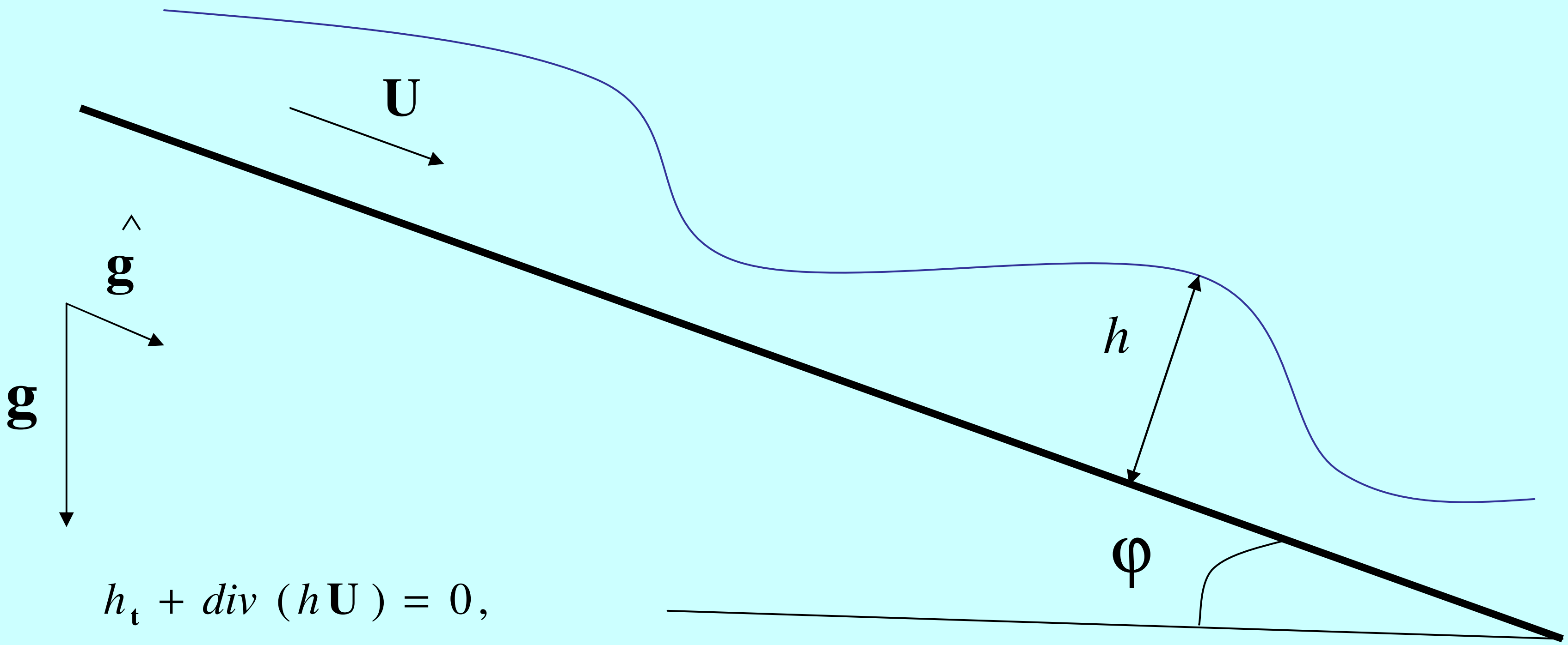
1. Introduction

Classical procedure for derivation of a reduced-dimension model of long free surface waves from the Euler equations :

- Assumption about the flow potentiality
- Introduction of a small parameter (shallowness parameter, amplitude parameter, or both) and the following asymptotic analysis (Saint Venant equations, Green-Naghdi equations, Boussinesq type equations etc.)

- The dissipation is then phenomenologically added

Saint-Venant model of shallow water flows in inclined open channels (Jeffreys, Dressler, Whitham,…)



$$h_{\mathbf{t}} + div\,(h\mathbf{U}) = 0,$$

$$(h\mathbf{U})_t + div\left(h\mathbf{U}\otimes\mathbf{U} + g\cos\varphi\frac{h^2}{2}\mathbf{I}\right) = \hat{\mathbf{g}}h - C\mathbf{U}|\mathbf{U}|$$

Consequence : energy conservation law

$$\left(h\left(\frac{|\mathbf{U}|^2}{2} + \frac{g\cos\varphi h}{2} - \hat{\mathbf{g}}\cdot\mathbf{x}\right)\right)_{\mathbf{t}} + div\left(h\mathbf{U}\left(\frac{|\mathbf{U}|^2}{2} + g\cos\varphi h - \hat{\mathbf{g}}\cdot\mathbf{x}\right)\right) = -C\,|\mathbf{U}|^3$$

Hydraulic jumps

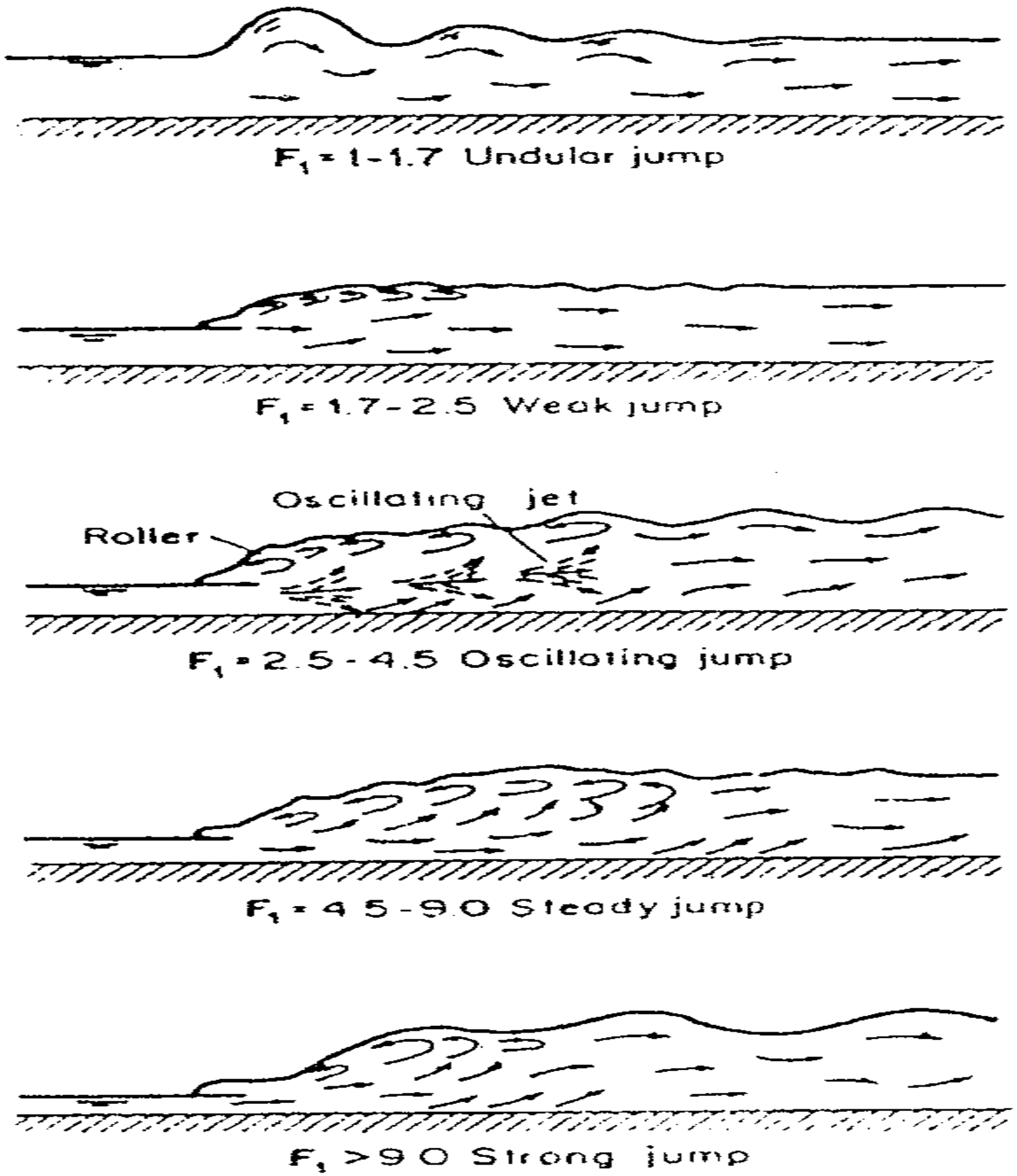
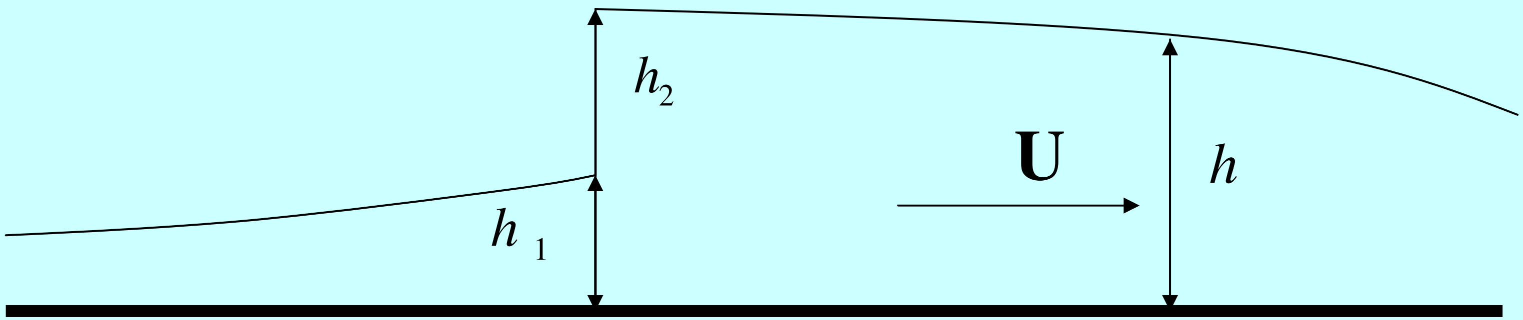


Fig.2. Various types of hydraulic jump.

Chow's classification is commonly used by engineers to describe the basic features of the hydraulic jump as a function of the upstream Froude number (Chow, 1959). This description is very intuitive. The qualitive and quantitive characteristics of the hydraulic jump can only experimentally be given. There is no mathematical model able to describe this phenomenon over all supercritical upstream Froude number range.

Saint-Venant (shallow water) equations

$$(hU)_x = 0, \quad \left(hU^2 + gh^2/2\right)_x = -CU^2$$



Bélanger equation

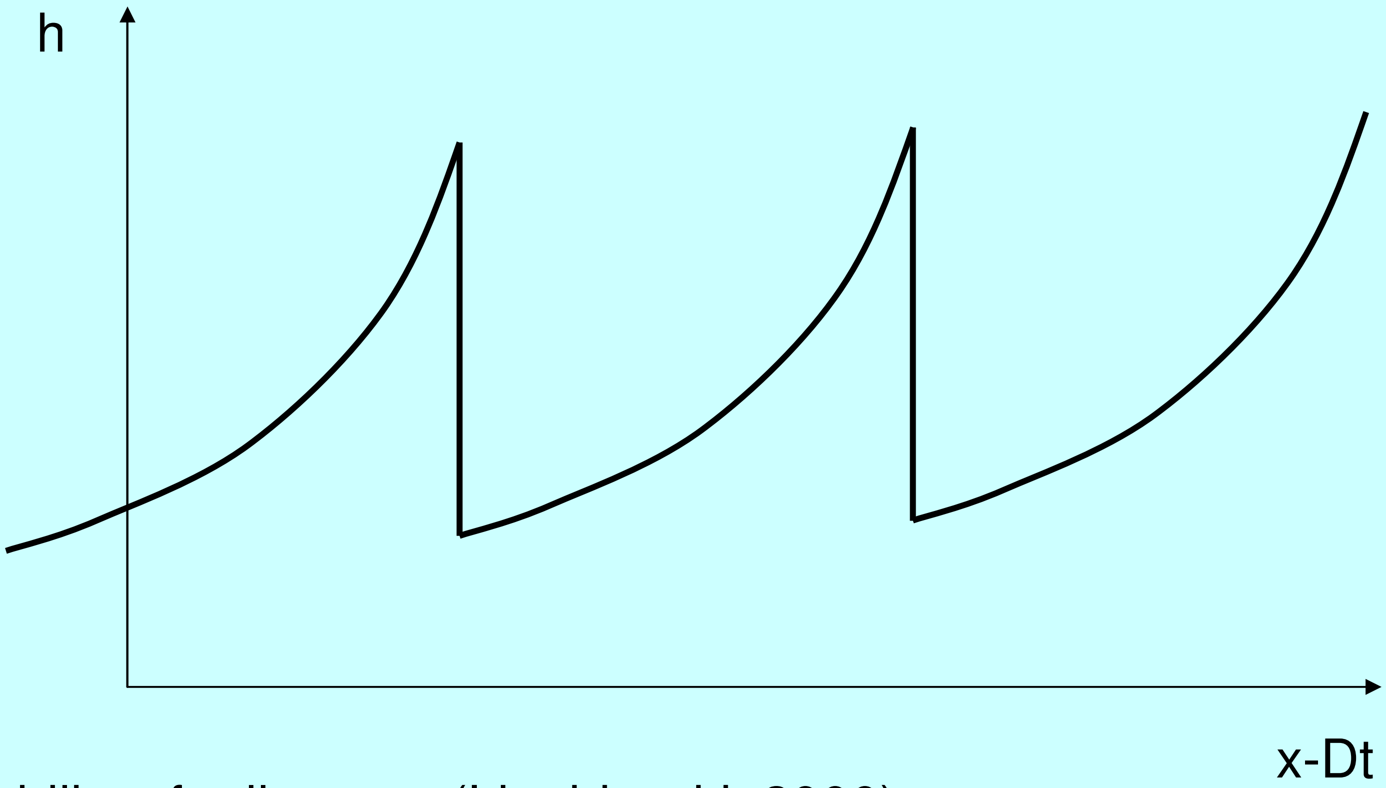
$$\frac{h_2}{h_1} = \frac{\sqrt{1+8F_1^{\,2}}-1}{2}, \quad F_1^{\,2} = \frac{U_1^2}{gh_1}$$

Hydraulic jumps can not be described by the SV equations : the length of the jump can attain 1 meter (while it is zero in the SV equations!). Even the relation between sequent depths of the jump (Bélanger equation) is not satisfied in practice. Absolutely *nothing* is working with the SV equations!

Roll waves

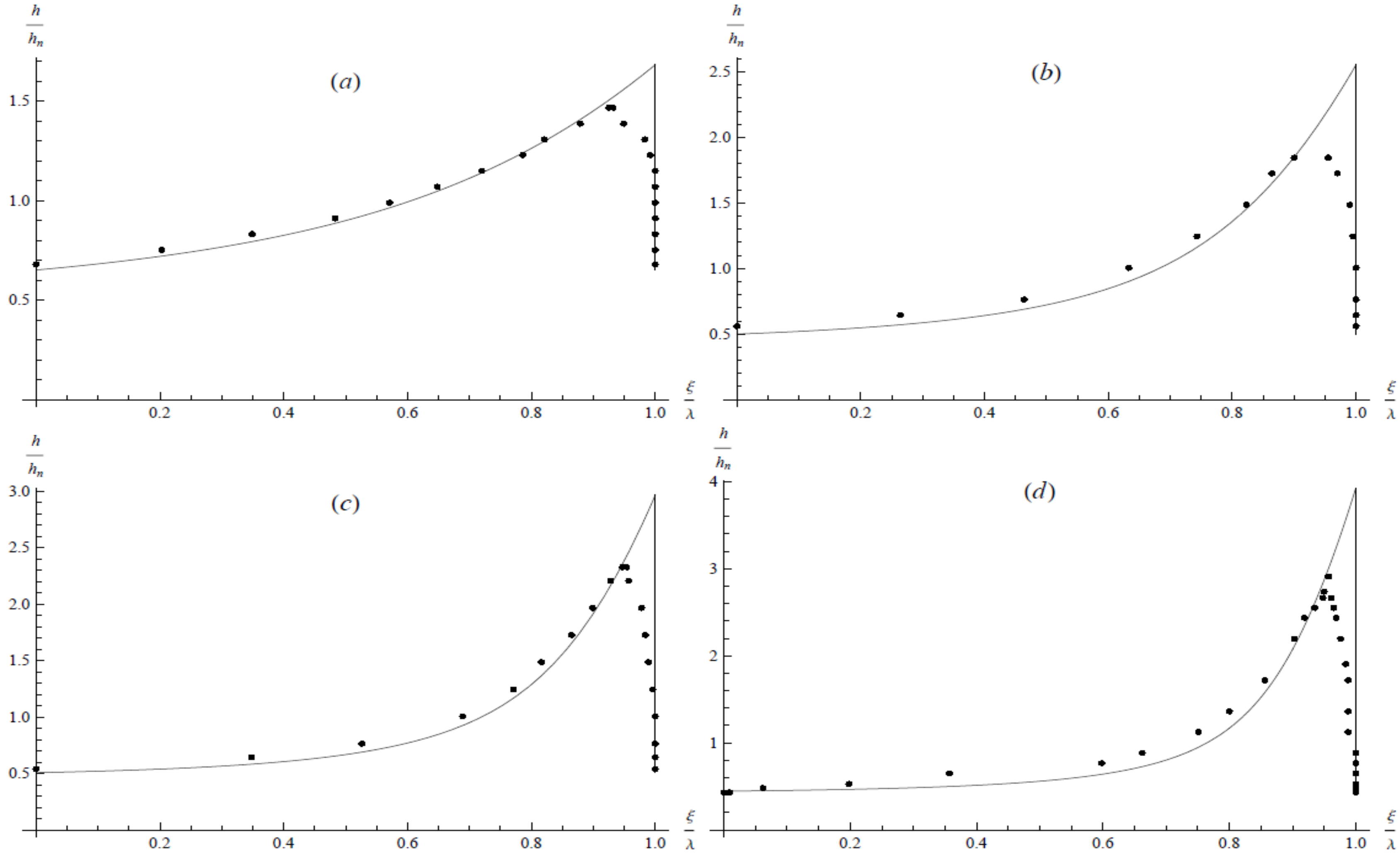
Dressler's solution (1949)

$$h_{\mathbf{t}} + div\ (h\mathbf{U}) = 0, \qquad (h\mathbf{U})_t + div\left(h\mathbf{U}\otimes\mathbf{U} + g\cos\varphi\frac{h^2}{2}\mathbf{I} \right) = \hat{\mathbf{g}}h - C\mathbf{U}\left|\mathbf{U}\right|$$

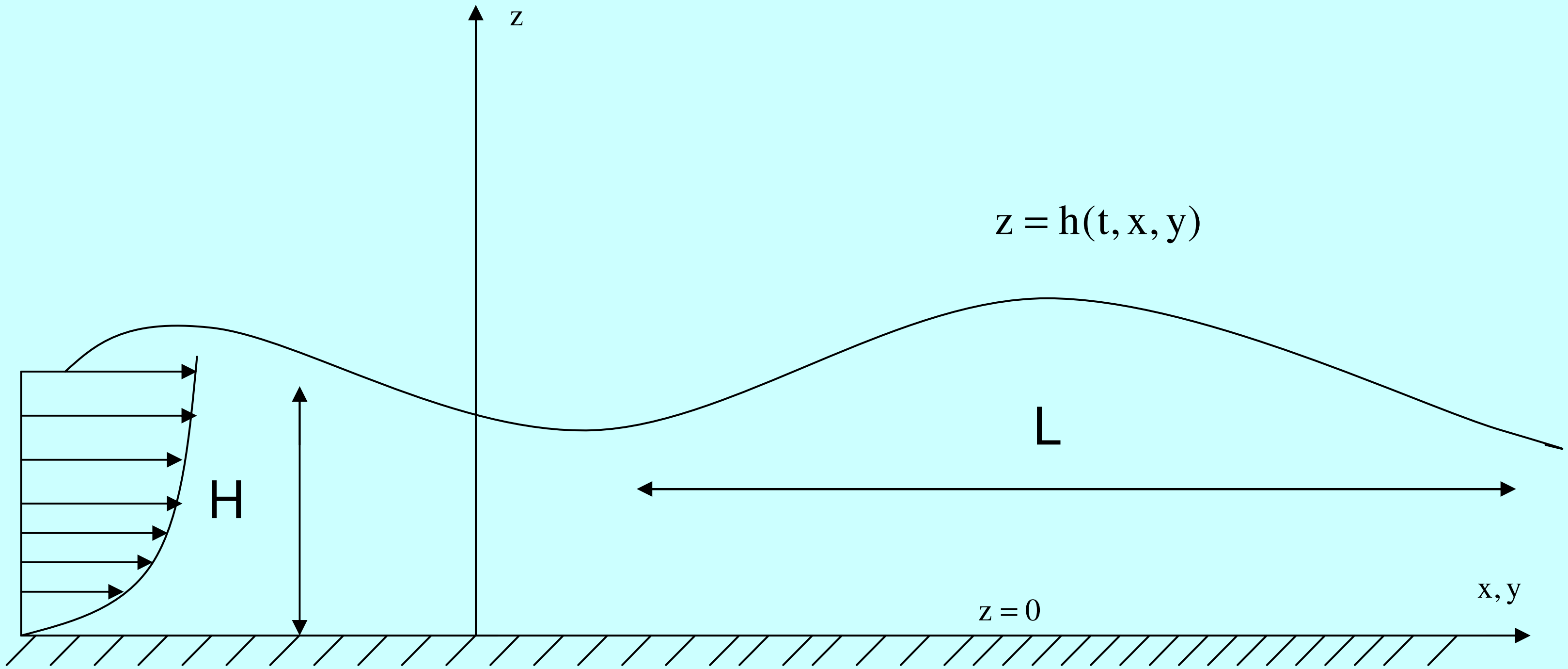


Modulation stability of roll waves (Liapidevski, 2000)
Spectral stability (Noble, 2006)
Stability analysis with viscous regularisation
(Johnson, Zumbrun, Noble, 2010)

Comparison of Brock's experiments with Dressler's solution (1967)



2. Model derivation



Small parameter : $\varepsilon = \dfrac{H}{L} << 1$

Dimensionless variables : $t \to \dfrac{L}{V} t, \quad (x, y) \to L(x, y), \quad z \to Hz, \quad (u, v) \to V(u, v),$

$w \to \varepsilon V w, \quad p \to \rho V^2 p, \quad h \to \varepsilon h$

Euler equations of incompressible fluids

$$\frac{D\mathbf{u}}{Dt} + \nabla_2 p = 0, \quad \varepsilon^2 \frac{Dw}{Dt} + p_z = -F^{-2}, \quad F^{-2} = gH / V^2$$

$$\nabla_2 \cdot \mathbf{u} + \mathrm{w}_z = 0, \quad \mathbf{u} = (\mathrm{u}, \mathrm{v})^{\mathrm{T}}, \quad \nabla_2 = \left(\frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}} \right)^{\mathrm{T}}$$

Boundary conditions

$$\mathrm{w} = 0, \quad \text{if} \quad \mathrm{z} = 0.$$

$$\mathrm{h}_{\mathrm{t}} + \mathbf{u} \cdot \nabla_2 \mathrm{h} = \mathrm{w} \quad \text{and} \quad \mathrm{p} = 0 \quad \text{if} \quad \mathrm{z} = \mathrm{h}(\mathrm{t}, \mathrm{x}, \mathrm{y}).$$

Definition of averages

$$h\mathbf{U} = \int_0^h \mathbf{u}\,dz\ , \qquad \mathbf{R} = \int_0^h (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})\,dz$$

Hydraulic system

$$h_{\mathbf{t}} + \nabla_2 \cdot (h\mathbf{U}) = 0,$$

$$(h\mathbf{U})_{\mathbf{t}} + \nabla_2 \cdot \left(h\mathbf{U} \otimes \mathbf{U} + \frac{gh^2}{2}\mathbf{I} + \mathbf{R}\right) = O\left(\varepsilon^2\right).$$

Evolution equations for the Reynolds tensor are needed.

Weakly sheared flows (Teshukov, 2007)

Let initially $u_z = O(\varepsilon^{\alpha}), \quad v_z = O(\varepsilon^{\alpha}), \quad \alpha > 0,$

then for any time $u_z = O(\varepsilon^{\beta}), \quad v_z = O(\varepsilon^{\beta}),$

$$\|\mathbf{u} - \mathbf{U}\| = O(\varepsilon^{\beta}), \quad \|\mathbf{R}\| = O(\varepsilon^{2\beta}), \quad \beta = \min(2, \alpha).$$

Proof (Barros, SG and Teshukov, 2007)
The Helmholtz equation for the vorticity implies :

$$\frac{Du_z}{Dt} + u_y v_z - v_y u_z = O(\varepsilon^2), \quad \frac{Dv_z}{Dt} + v_x u_z - u_x v_z = O(\varepsilon^2).$$

Two cases

$\beta > 1$ In this case (potential or near potential case) we obtain the classical shallow water equations or the GN model (second order)

$\beta < 1$ In this case we obtain turbulent shallow water equations

Shear shallow water equations

$$h_t + \nabla_2 \cdot (h\mathbf{U}) = 0,$$

$$(h\mathbf{U})_t + \nabla_2 \cdot \left(h\mathbf{U} \otimes \mathbf{U} + \frac{gh^2}{2}\mathbf{I} + \mathbf{R}\right) = 0,$$

$$\frac{D\mathbf{R}}{Dt} + \mathbf{R}\nabla_2 \cdot \mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\mathbf{R} + \mathbf{R}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla_2$$

Energy conservation law

$$\frac{\partial}{\partial t}\left(\frac{h|\mathbf{U}|^2}{2} + \frac{gh^2}{2} + \frac{tr(\mathbf{R})}{2}\right) + div\left(h\mathbf{U}\left(\frac{|\mathbf{U}|^2}{2} + gh + \frac{tr(\mathbf{R})}{2h}\right) + \mathbf{R}\mathbf{U}\right) = 0.$$

The equations are hyperbolic if **R** is positive definite.

General equations of turbulent compressible flows

$$\rho_{\mathbf{t}} + \nabla \cdot (\rho \mathbf{U}) = 0,$$

$$(\rho \mathbf{U})_{\mathbf{t}} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U} + p(\rho)\mathbf{I} + \mathbf{R}) = 0,$$

$$\frac{\mathrm{D}\mathbf{R}}{\mathrm{Dt}} + \mathbf{R}\nabla \cdot \mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\mathbf{R} + \mathbf{R}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^{\mathrm{T}} = \mathbf{S}.$$

Numerically, they are solved by splitting method. So, a homogeneous equation for the Reynolds stress tensor (**S**=**0**) should be first understood.

The homogeneous equation for the Reynolds stresses

$$\frac{\mathrm{D}\mathbf{R}}{\mathrm{Dt}} + \mathbf{R}\nabla\cdot\mathbf{U} + \frac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{R} + \mathbf{R}\left(\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\right)^{\mathrm{T}} = 0$$

By changement of variables this equation is equivalent to

$$\frac{\mathrm{D}\mathbf{P}}{\mathrm{Dt}} + \frac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{P} + \mathbf{P}\left(\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\right)^{\mathrm{T}} = 0, \quad \mathbf{P} = \mathbf{R}/\rho$$

What is the structure of this complicated equation ?

3. The structure of the Reynolds equations

« Symmetric» equation

$$0 = \mathbf{P}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^{\mathrm{T}} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\mathbf{P} + \frac{\mathrm{D}\mathbf{P}}{\mathrm{Dt}} \qquad \frac{\mathrm{D}\mathbf{G}}{\mathrm{Dt}} + \mathbf{G}\frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\mathbf{G} = 0$$

Equation for the Finger tensor in nonlinear elasticity

$$\mathbf{G} = \left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{F}^{-1} = \sum_{\alpha} \nabla X^{\alpha} \otimes \nabla X^{\alpha}$$

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \frac{\mathrm{D}\mathbf{F}}{\mathrm{Dt}} = \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\mathbf{F}, \quad \frac{\mathrm{D}\mathbf{F}^{-1}}{\mathrm{Dt}} = -\mathbf{F}^{-1}\frac{\partial \mathbf{U}}{\partial \mathbf{x}}$$

The equations for **P** and **G** are identical if $\frac{\partial \mathbf{U}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^{\mathrm{T}} = 0 \Leftrightarrow \mathrm{rot}\,\mathbf{U} = 0$

Spectral representation

$$\mathbf{P} = \sum_{\alpha=1}^{3} \nu_\alpha \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha, \quad \nu_\alpha > 0.$$

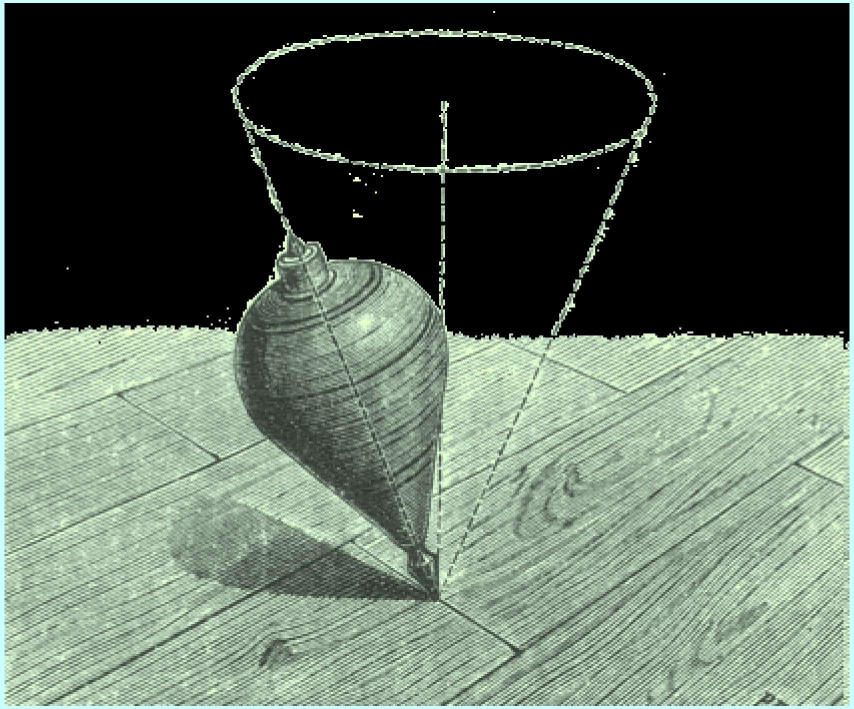
(SG & H. Gouin, 2010) The orthonormal eigenvectors and eigenvalues of the Reynolds tensor verify the following system of equations :

$$\frac{D\mathbf{e}_\alpha}{Dt} = \mathbf{i}(\boldsymbol{\kappa})\mathbf{e}_\alpha, \quad \frac{D\nu_\alpha}{Dt} + 2\nu_\alpha(\mathbf{e}_\alpha^T \mathbf{D}\mathbf{e}_\alpha) = 0, \quad \mathbf{D} = \frac{1}{2}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T\right)$$

where

$$\boldsymbol{\kappa}^T \mathbf{e}_\gamma = \frac{\mathbf{e}_\beta^T\left(\nu_\alpha \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} + \nu_\beta \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T\right)\mathbf{e}_\alpha}{\nu_\beta - \nu_\alpha}$$

Here the indicies α, β, γ correspond to a cyclic permutation of 1,2,3 **.**



Remark. Locally, each point of flow behaves like a heavy top. The eigenvalues of the Reynolds tensor are its principal moments of inertia. They are not constant !

The main question in multiD case

Equations are hyperbolic but for 6 variables (the 2D case) we have only 5 conservation laws : mass, momentum, energy and … ``entropy'' :

$$\frac{\partial}{\partial t}\left(\frac{\det \mathbf{P}}{h}\right)+div\left(\frac{\mathbf{U}\det \mathbf{P}}{h}\right)=0.$$

The same problem as in 2D and 3D compressible turbulent flow (F. Coquel).

4. Back to hydraulic jumps

1D system for the classical hydraulic jump

We have only one non-trivial component of the stress tensor $\mathbf{P}$

$$P_{11} = \Phi h^2$$

We call Φ flow enstrophy.

Dissipationless 1D system (Teshukov 2007, G. Richard and SG, 2012)

$$
\begin{aligned}
& h_t + (hU)_x = 0, \\
& (hU)_t + \left(hU^2 + p\right)_x = 0, \\
& (hE)_t + \left(hU\left(E + \frac{p}{h}\right)\right)_x = 0.
\end{aligned}
$$

Closure relations

$$
E = \frac{1}{2}U^2 + e, \quad e = \frac{1}{2}\left(gh + \Phi h^3\right), \quad p = \frac{gh^2}{2} + \Phi h^3
$$

Equation for Φ

$$
\frac{D\Phi}{Dt} = 0.
$$

The equations are hyperbolic with the sound speed given by $c = \sqrt{gh + 3\Phi h^2}$

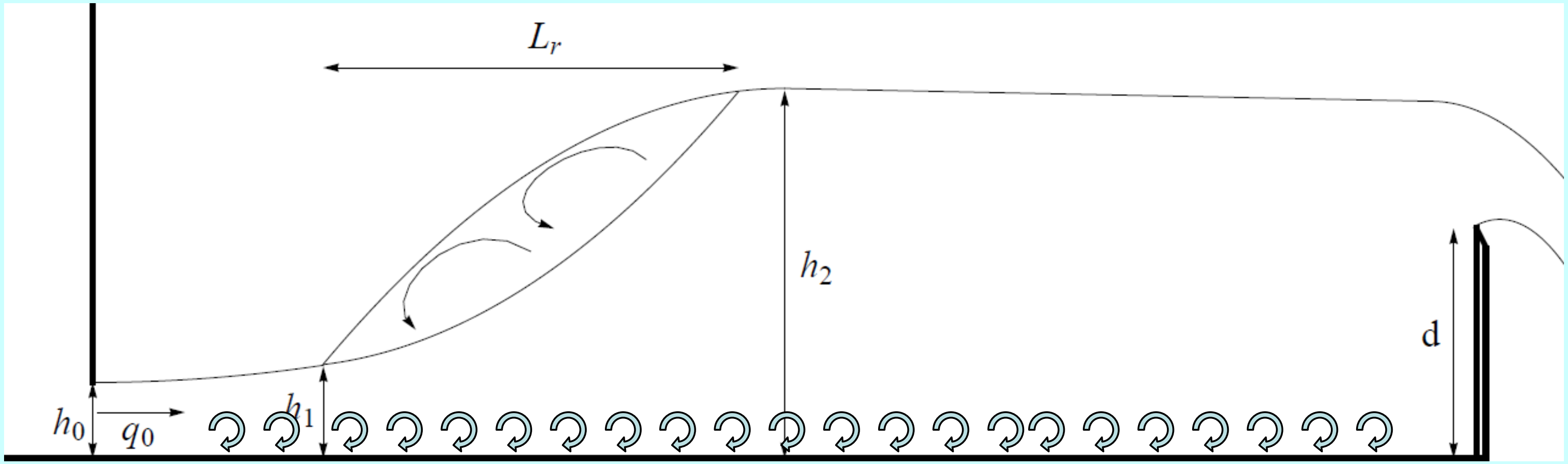
The enstrophy plays the role of the entropy.

Scale decomposition of the enstrophy

$$\Phi \rightarrow \varphi + \Phi$$

φ is the enstrophy (squared vorticity) of small scale vortices

Φ is the enstrophy (squared vorticity) of large scale vortices



φ is a constant

Φ is a varying function

Final 1D system with dissipation effects (G. Richard and SG, JFM, 2012)

$$h_t + (hU)_x = 0,$$

$$(hU)_t + \left(hU^2 + p\right)_x = -C|U|U,$$

$$(hE)_t + \left(hU\left(E + \frac{p}{h}\right)\right)_x = -C_e|U|^3,$$

$$\frac{D\varphi}{Dt} = 0$$ φ is the entropy of small vortices

Closure relations

$$E = \frac{1}{2}U^2 + e, \quad e = \frac{1}{2}\left(gh + (\Phi + \varphi)h^3\right)$$

$$p = \frac{gh^2}{2} + (\Phi + \varphi)h^3$$

Consequence

$$h^3\frac{D\Phi}{Dt} = -2(C_e - C)|U|^3, \; C_e > C.$$

Dissipation coefficients (G. Richard and SG, JFM, 2012)

C is a classical friction coefficient well defined in the literature.
C_e is more larger than C when the roller is present.
C_e conciders with C when the roller is absent.
The most simple formula could be

$$C_e = C + C_r \frac{\Phi}{\Phi + \varphi}$$

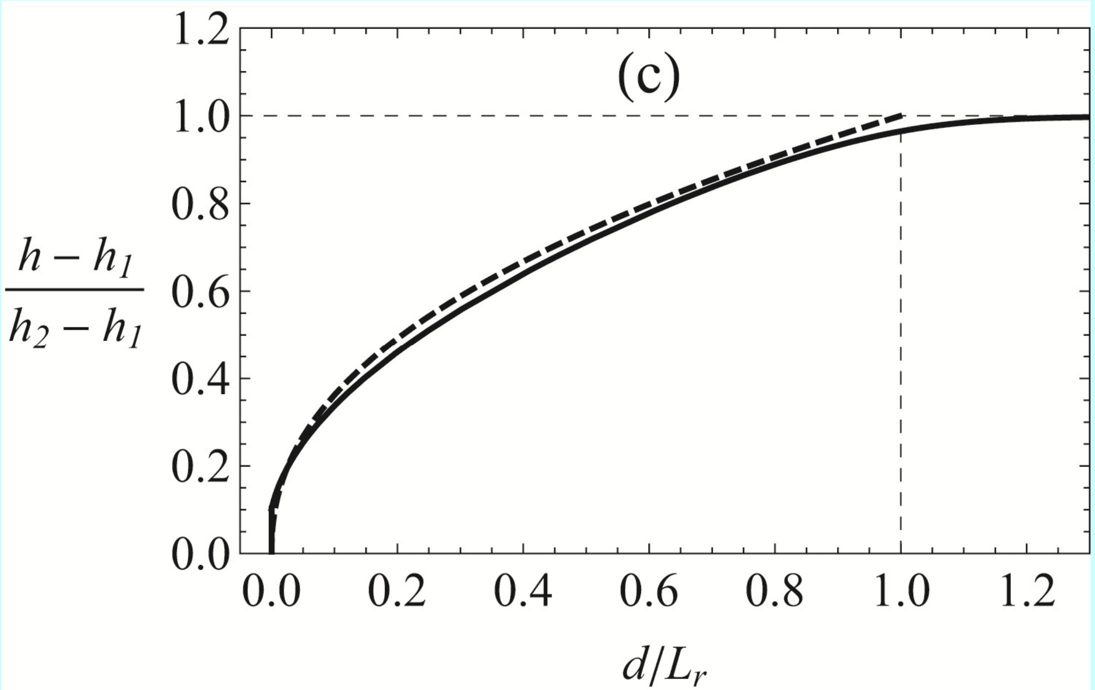
Finally, the model contains two parameters : small scale enstrophy φ and the coefficient C_r (roller dissipation coefficient).
The coefficients will be determined from the experimental data.

Experimental data about hydraulic jumps

1. The length of the hydraulic jump as a function of the upstream Froude number can experimentally be defined. It is directly related to the roller dissipation coefficient C_r . The coefficient is determined as a function of the upstream Froude number by using the experimental data by Hager et al. 1989, 1990.

2. The deviation of the experimentally measure sequent depth ratio (the relation between downstream and upstream jump depths) can experimentally be measured (Hager et al. 1989, 1990). It determines the value of the small scale enstrophy.

Stationary solution



Stationary free-surface profile of the hydraulic jump (thick curve) and the experimental law of Chanson (2011) (dashed curve).

$$F_1 = 8.25, \quad C = 1.65\times 10^{-3}, \quad C_r = 1.16, \quad \tilde{h}_2 = 10.6, \quad \tilde{L}_r = 50.5$$

What we finally have?

A system of conservation laws with a right-hand side

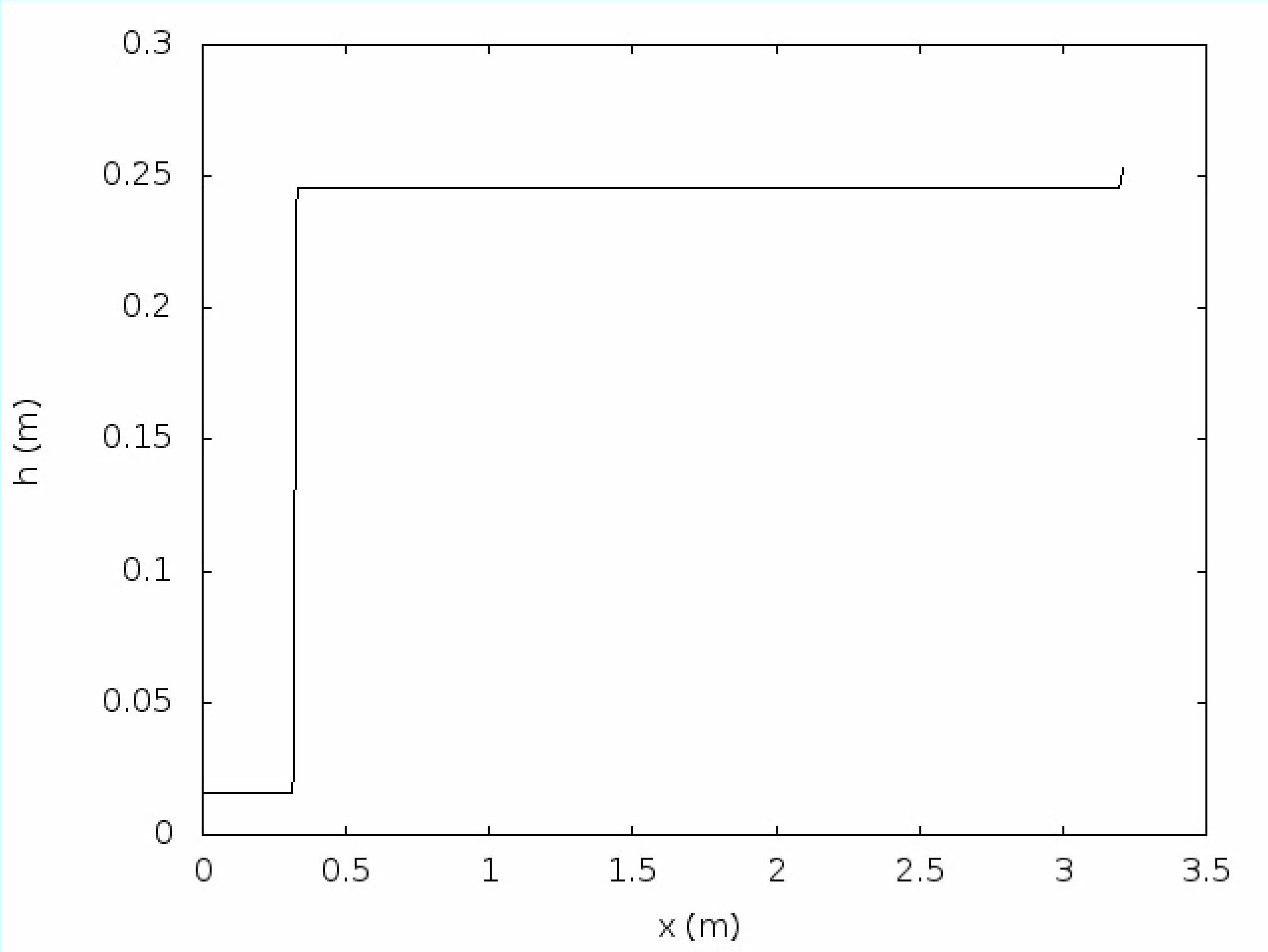
$$\mathbf{V}_t + \mathbf{F}(\mathbf{V})_x = \mathbf{f}(\mathbf{V}), \ \mathbf{V} = \left(h, hU, h\left(\frac{1}{2}U^2 + e\right)\right)^T$$

Initial and boundary conditions

$$\mathbf{V}\big|_{t=0} = \mathbf{V}_0(x)$$

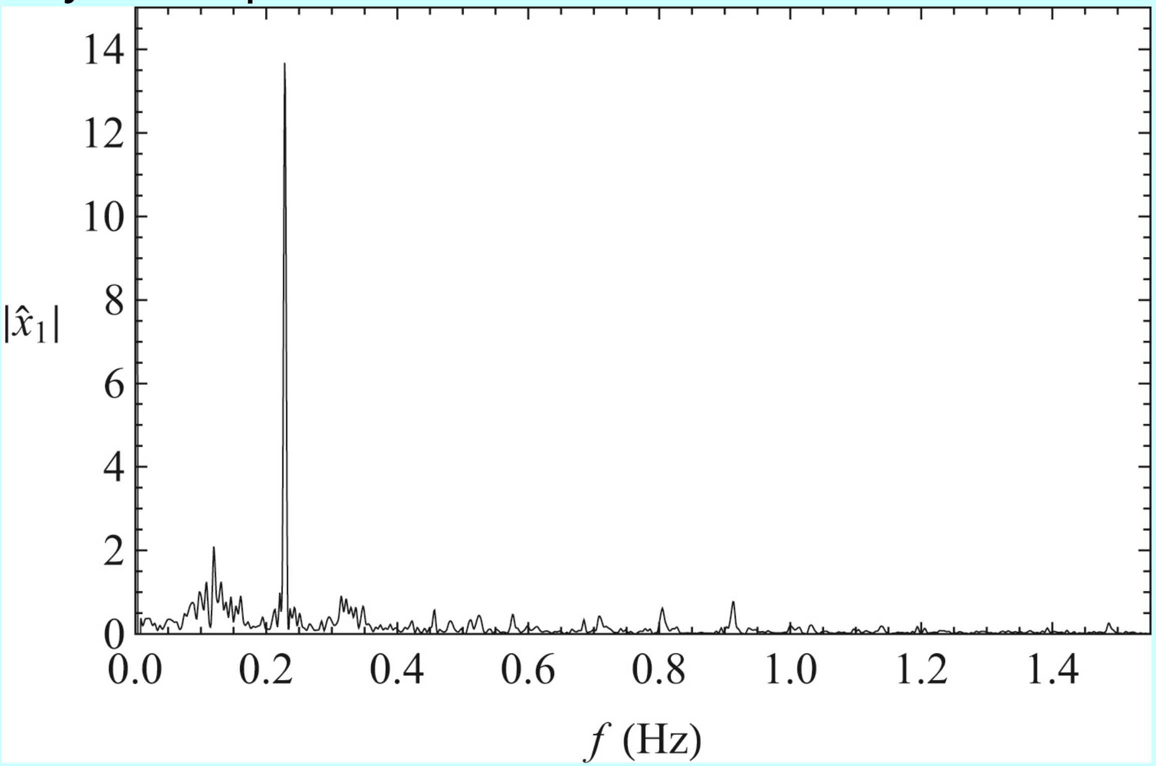
$\Phi\big|_{x=0} = 0, \quad h\big|_{x=0}, \quad hU\big|_{x=0}$ are given constants.

$$q = hU\big|_{x=L} = g(h)$$


0.3
0.25
0.2
0.15
0.1
0.05
0
h (m)
0
0.5
1
1.5
2
2.5
3
3.5
x (m)

Mok experiments (2004)

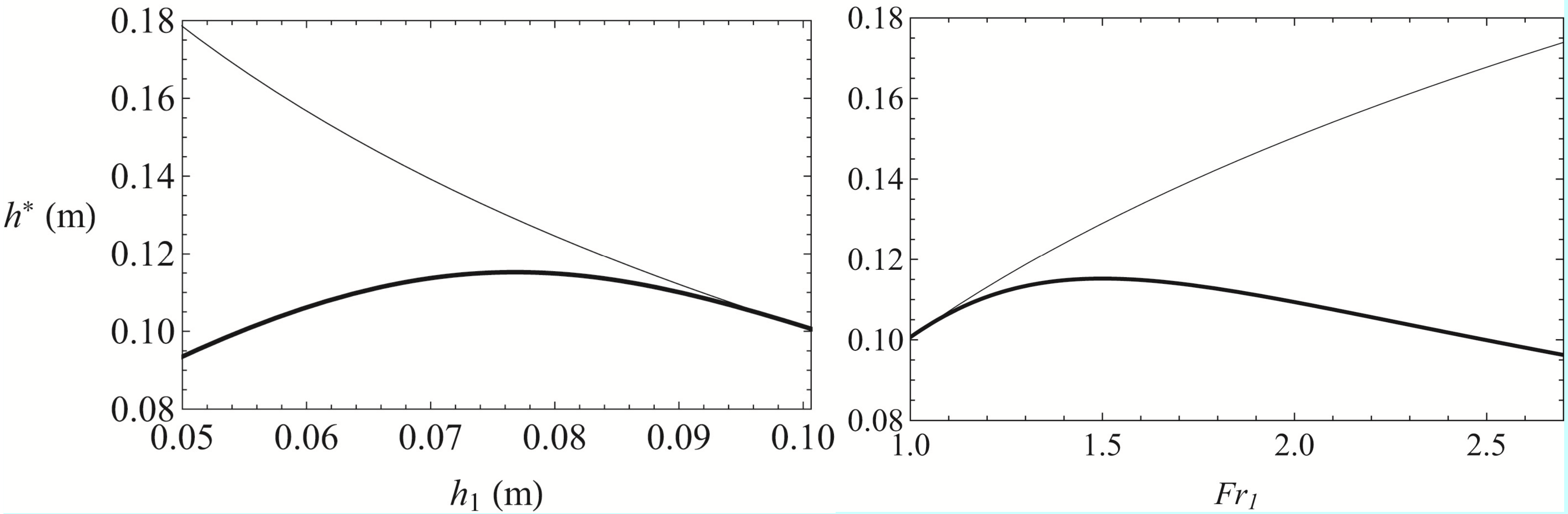
K. M. Mok (2004) measured energy spectra of water surface fluctuations. The energy spectra measurements show bands of energy centered at particular frequencies when the upstream Froude number is larger than about 1.5, while for jumps with a Froude number smaller that 1.5 such a particular frequency is not present.



Fourier spectrum of the oscillations of the jump toe position $x_1(t)$ for $F_1 = 11.25$. The main frequency is 0.228 Hz.

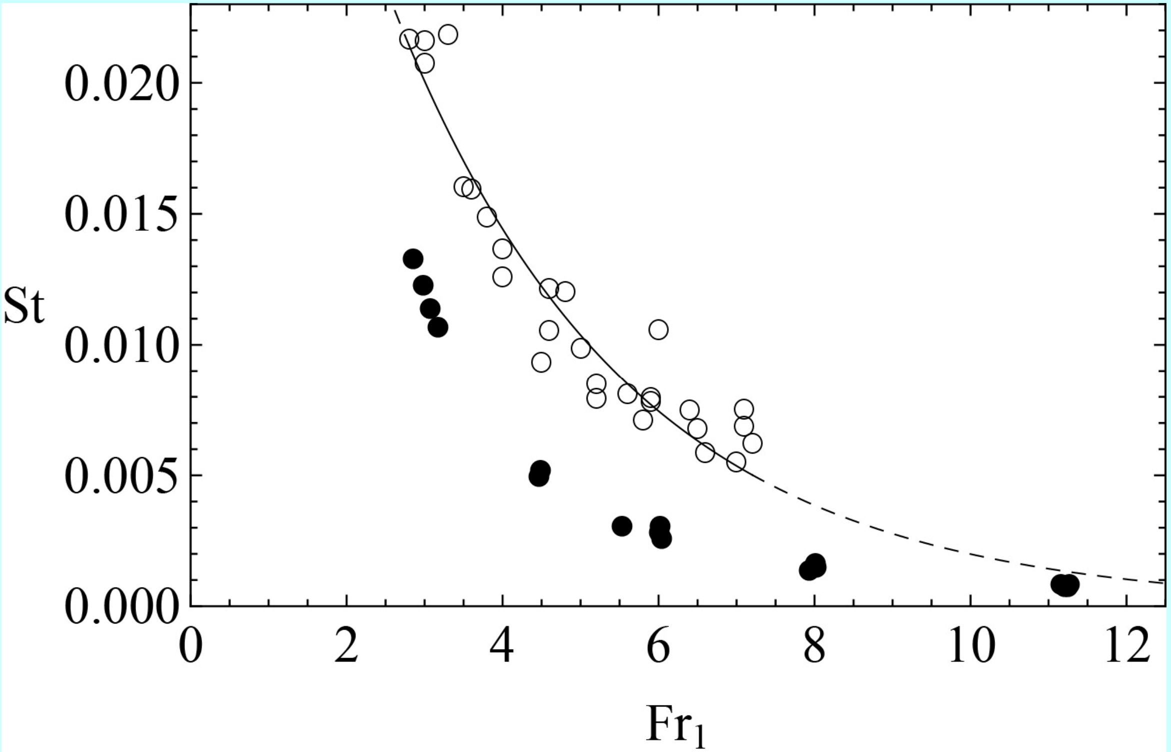
But why this magic number 3/2 ???!!!

Stationary shock relations for a given flow discharge



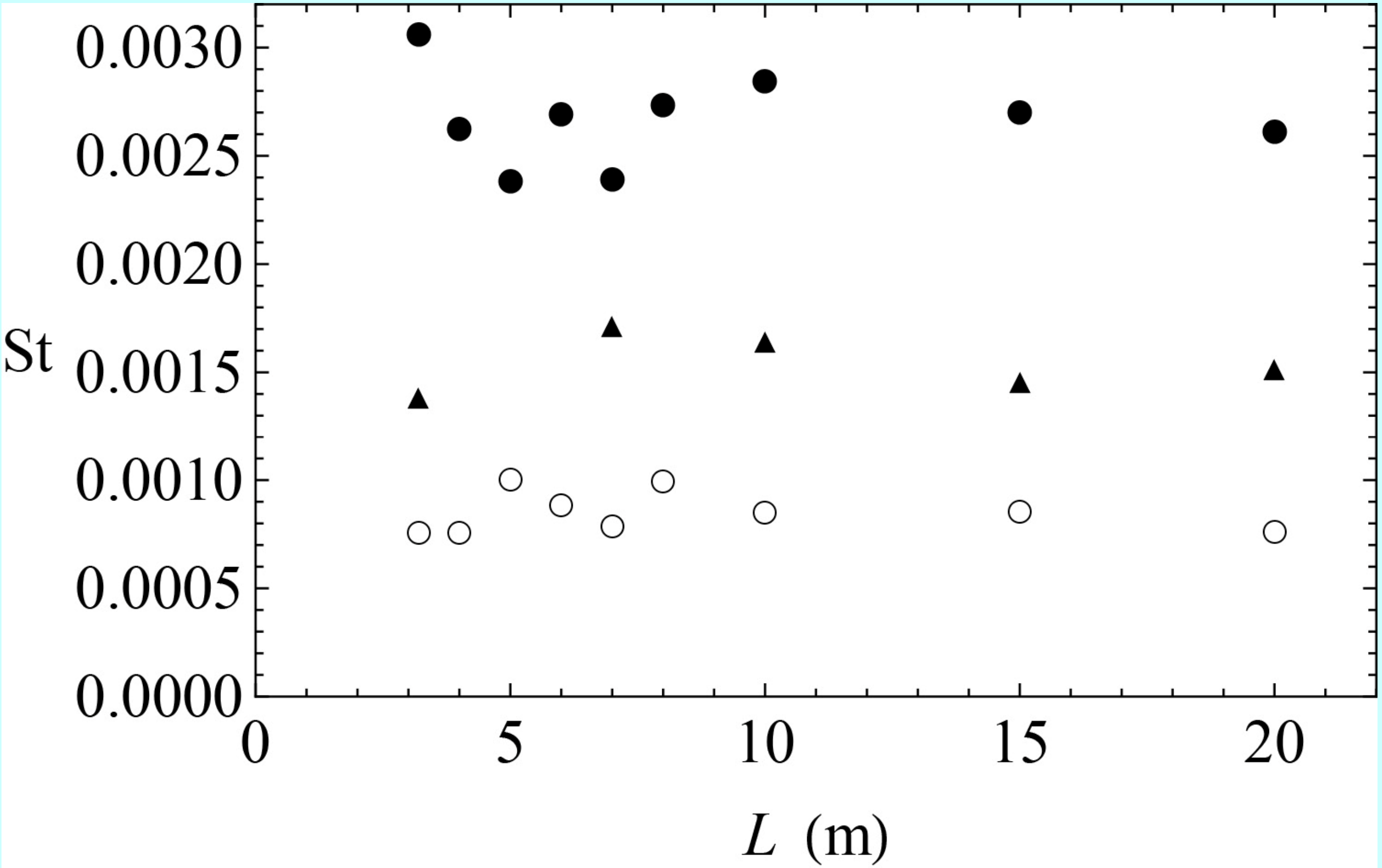
The thin line is the Bélanger equation (RH relations corresponding to SV equations), and the bold line is the RH relations for the new model ($\varphi = 0$) .

Oscillation frequency of the jump toe as a function of the Froude number



The Strouhal number $St = fh_1^2 / q$ as a function of the Froude number is shown. Here f is the principal frequency (in Hz), h_1 is the average fluid depth before the jump, and q is the flow discharge per unit width ($q = 0.0835\, m^2 / s$). These results are also in good agreement with recent experimental results by Chanson et al. (2010, 2011).

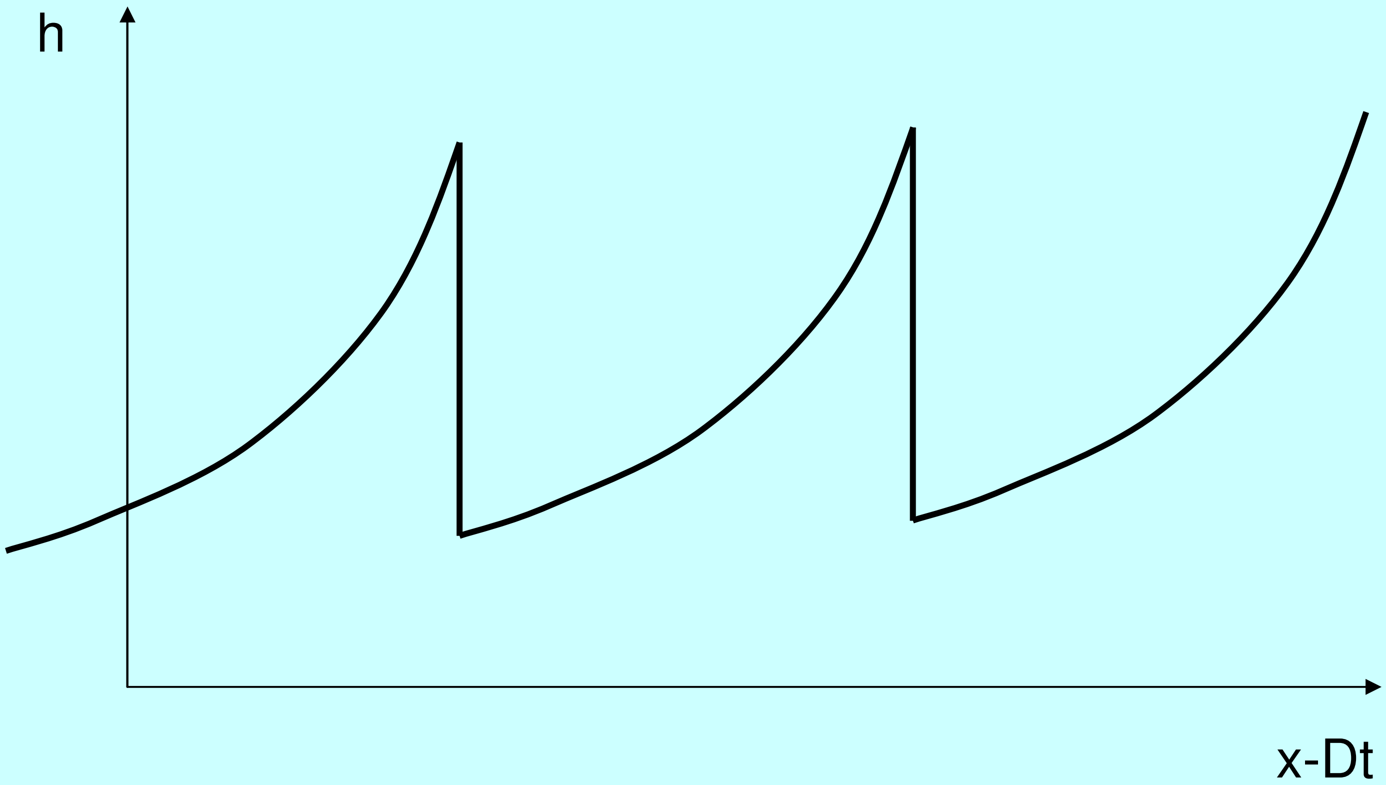
The toe oscillation frequency does not depend on the length channel -- it is a function only of the Froude number ($F_1 = 6,\ 8,\ 11.25$). Hopf's bifurcation ???



5. Back to roll waves

Dressler's solution (1949)

$$h_{\mathbf{t}} + div\,(h\mathbf{U}) = 0, \qquad (h\mathbf{U})_{\mathbf{t}} + div\left(h\mathbf{U}\otimes\mathbf{U} + \frac{1}{2}g'h^2\mathbf{I}\right) = \hat{\mathbf{g}}\,h - C\mathbf{U}\left|\mathbf{U}\right|$$



Comparison with Brock's experiments (1967)

A good agreement was obtained for all 11 experiments !

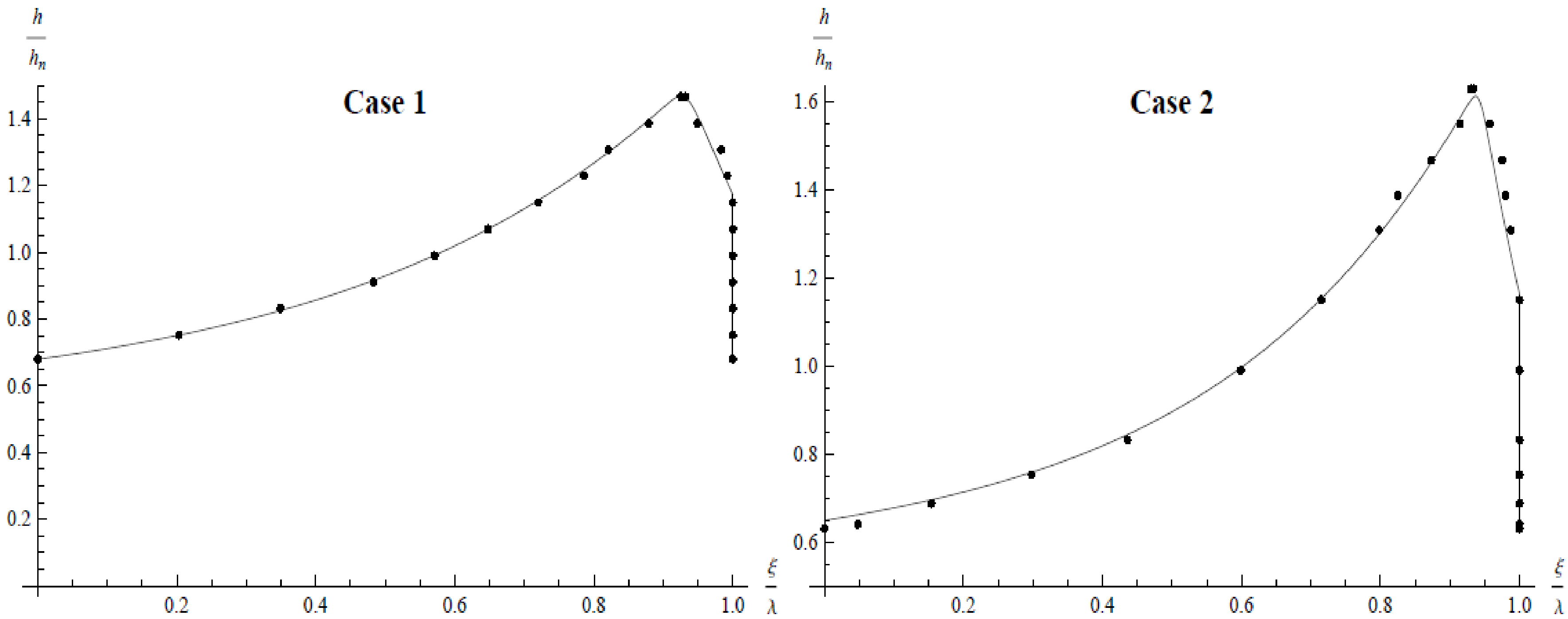
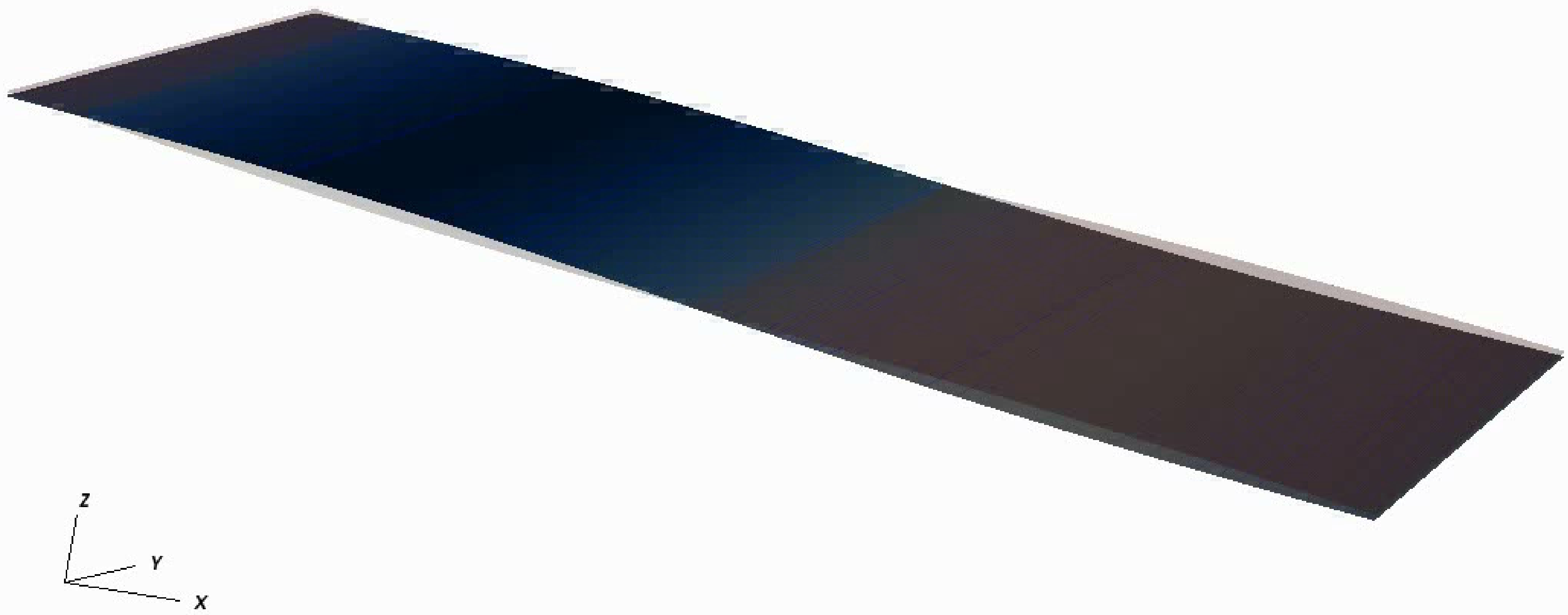


FIGURE 8. Comparison of the theoretical solution to the model (4.7)- (4.10) (thin line) and Brock's experimental results (dots) for $tg\theta = 0.0502$. The flow parameters are given in Table 2.


Z
Y
X

References

1. JFM, 2012, 2013

6. Conclusions and perspectives

1. A new model of shear shallow water flows has been derived.

2. An excellent agreement with experimental results on hydraulic jumps for the Froude numbers larger than 1.5 has been found.

3. The model predicts the frequency of oscillations of the jump toe.

4. An excellent agreement with the experiments on roll waves was obtained.

5. 2D analytical and numerical study will follow (joint work with B. Nkonga)