Stochastic transport equation
and
non-Lipshitz SDEs

Massimiliano Gubinelli

Laboratoire de Mathématiques, Orsay

The linear transport equation (classically)

Given $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ smooth vectorfield, $\overline{u}$ smooth. Consider the Cauchy problem in $\mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0 \\ u(0,x) = \overline{u}(x) \end{cases} \tag{1}$$

and the flow generated by b :

$$\begin{cases} \partial_t \Phi_{s,t}(x) = b(t, \Phi_{s,t}(x)) \\ \Phi_{s,s}(x) = x \end{cases}$$

Solutions to (**??**) are constant on the trajectories of b :

$$\frac{d}{dt} u(t, \Phi_{0,t}(x)) = \partial_t u(t, \Phi_{0,t}(x)) + \partial_t \Phi_{0,t}(x) \cdot \nabla u(t, \Phi_{0,t}(x)) = 0$$

Method of characteristics

The unique solution to (**??**) is $u(t,x) = \overline{u}(\Phi_{0,t}^{-1}(x))$.

Non-smooth vectorfields

Weak formulation

$$\begin{cases} \partial_t u + \operatorname{div}(bu) - (\operatorname{div} b)u = 0 \\ u(0,x) = \overline{u}(x) \end{cases}$$

Testing with smooth θ

$$\int \theta(x)u(t,x)dx = \int \theta(x)\overline{u}(x)dx$$
$$+ \int_0^t ds \int (u(s,x)b(s,x) \cdot \nabla\theta(x) + u(s,x)\theta(x)\operatorname{div} b(s,x))dx$$

- Existence of L^∞ weak solutions when $b \in L^p$, $\operatorname{div} b \in L^1_{\mathrm{loc}}$ and $\overline{u} \in L^\infty$
- **[DiPerna-Lions]** Renormalized solutions: uniqueness and stability of L^∞ weak solutions when $b \in L^1(W^{1,p}) \cap L^\infty$ and $\operatorname{div} b \in L^\infty$
- **[Ambrosio]** Renormalized solutions for BV vectorfields
- Use the transport equation to select a flow Φ defined *almost everywhere*

SDEs with non-smooth coefficients

Idea:

Perturb the equation of characteristics by an additive Brownian noise acting on all components.

Why?

Consider the SDE in $\mathbb{R}^d$

$$dX_t = b(t, X_t)dt + dW_t, \qquad X_0 = x_0$$

- Strong solutions for b Lipshitz (+ linear growth) by fixed point method
- **[Veretennikov]** b bounded $\Rightarrow$ uniqueness of strong solutions
- **[Krylov-Röckner]** Strong uniqueness for b in Sobolev spaces
- **[Davie]** b bounded $\Rightarrow$ unique solution for a.e. Brownian path

$\Rightarrow$ *The noise regularizes the flow of the vectorfield b* $\Leftarrow$

Stochastic flow

To implement the method of characteristics we need information on *dependence on initial conditions*.

Definition

A *stochastic flow* is a family of maps $\{\Phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d\}_{0 \leqslant s \leqslant t \leqslant T}$ such that

- $\Phi_{s,t}(x)$ is $\sigma(\{W_r - W_q\}_{s \leqslant q \leqslant r \leqslant t})$ measurable for any $x \in \mathbb{R}^d$, $0 \leqslant s \leqslant t \leqslant T$;
- $\lim_{t \to s+} \Phi_{s,t}(x) = x$, a.s. for any x, s, t;
- $\Phi_{u,t}(\Phi_{s,u}(x)) = \Phi_{s,t}(x)$

Theorem (Kunita)

If $b \in C^{1,\alpha}$ then there exists a $C^{1,\alpha'}$-stochastic flow $\Phi_{s,t}$ for any $\alpha' < \alpha$ solving the SDE

$$\Phi_{s,t}(x) = x + \int_s^t b(u, \Phi_{s,u}(x))du + W_t - W_s$$

for any $x \in \mathbb{R}^d$.

The Itô trick (I)

The regularization effect can be understood easily in the case $b(t,x) = b(x)$. Consider

$$X_t = x + \int_0^t b(X_s)ds + W_t$$

Try the *Itô trick*: interpret the integral over time as a correction in an Itô formula:

$$G(X_t) = G(x) + \int_0^t \nabla G(X_s)dW_s + \int_0^t LG(X_s)ds$$

with $L = \Delta/2 + b \cdot \nabla$. Assume that we can solve the elliptic problem

$$\lambda G - LG = b$$

for some $\lambda > 0$ (maybe very large), then

$$X_t + G(X_t) = x + G(x) + W_t + \int_0^t \nabla G(X_s)dW_s - \int_0^t \lambda G(X_s)ds$$

where G "has two derivatives more" than b. Setting $\psi(x) = x + G(x)$ we get

$$\psi(X_t) = \psi(x) + \int_0^t \nabla\psi(X_s)dW_s - \int_0^t \lambda G(X_s)ds$$

The Itô trick (II)

Theorem (Elliptic estimates)

For any $\epsilon > 0$, $\epsilon' < \epsilon$, $b \in C^{\epsilon}$, the elliptic equation $\lambda G - LG = b$ has a solution $G \in C^{2,\epsilon}$ for which $\|G\|_{2,\epsilon'} \to 0$ as $\lambda \to \infty$.

For λ large enough $\nabla\psi = 1 + \nabla G$ is invertible and ψ has inverse ψ^{-1}.
Let $Y_t = \psi(X_t)$, $y = \psi(x)$:

$$Y_t = y + \int_0^t \tilde{\sigma}(Y_s)dW_s + \int_0^t \tilde{b}(Y_s)ds$$

where $\tilde{\sigma}(y) = \nabla\psi \circ \psi^{-1}(y)$ and $\tilde{b}(y) = \lambda G \circ \psi^{-1}(y)$.

We have $\tilde{\sigma} \in C^{1,\epsilon'}$, $\tilde{b} \in C^{2,\epsilon'}$ and there exists a $C^{1,\epsilon'}$-stochastic flow φ solving

$$\varphi_{s,t}(y) = y + \int_s^t \tilde{\sigma}(\varphi_{s,u}(y))dW_u + \int_0^t \tilde{b}(\varphi_{s,u}(y))du$$

Stochastic flow for C^ϵ vectorfields

By letting $\phi_{s,t} = \psi^{-1} \circ \varphi_{s,t} \circ \psi$ we obtain a $C^{1,\epsilon'}$ stochastic flow satisfying

$$\phi_{s,t}(x) = x + \int_s^t b(\phi_{s,u}(x))du + W_t - W_s$$

- this flow is the unique strong solution to the SDE
- it does not depend on the choice of λ.
- we have an equation for $\nabla\phi_{s,t}(x)$:

$$\nabla\psi(\phi_{s,t}(x))\nabla\phi_{s,t}(x) = \nabla\psi(x) + \int_s^t \lambda\nabla G(\phi_{s,u}(x))\nabla\phi_{s,u}(x)du$$

$$+ \int_s^t \nabla^2\psi(\phi_{s,u}(x))\nabla\phi_{s,u}(x)dW_u$$

- by a stopping procedure we can assume b locally in C^ϵ (+ linear growth)

Push-forward

For smooth b we have

$$\int \theta(\phi_{s,t}(x))dx = \int \theta(x)\frac{dx}{J_{s,t}(x)}$$

where $J_{s,t}(x) = |\det \nabla\phi_{s,t}(x)|$ (Jacobian determinant) satisfy the differential equation

$$\frac{d}{dt}J_{s,t}(x) = \operatorname{div} b(\phi_{s,t}(x))\, J_{s,t}(x), \qquad J_{s,s}(x) = 1.$$

(the stochastic perturbation is solenoidal). Then

$$J_{s,t}(x) = \exp\left(\int_s^t \operatorname{div} b(\phi_{s,u}(x))du\right)$$

For $b \in C^\epsilon$ by an approximation procedure and another Itô trick we get

$$J_{s,t}(x) = \exp\left(\Gamma(\phi_{s,t}(x)) - \Gamma(x) + \int_s^t \nabla\Gamma(\phi_{s,u}(x))dW_u + \int_s^t \lambda\Gamma(\phi_{s,u}(x))du\right)$$

where $\Gamma \in C^{1,\epsilon'}$ solve $\lambda\Gamma - L\Gamma = \operatorname{div} b$ in the sense of distributions.

Stochastic transport equation

The simplest stochastic perturbation which is compatible with the method of characteristics leads to the Stratonovich SPDE

$$\begin{cases} d_t u_t + b_t \cdot \nabla u_t \, dt + \sum_{i=1}^{d} \nabla_i u_t \circ dW_t^i = 0 \\ u_0(x) = \overline{u}(x) \end{cases}$$

and to the related SDE for the flow of characteristics:

$$\begin{cases} d_t \Phi_{s,t}(x) = b(t, \Phi_{s,t}(x))dt + dW_t \\ \Phi_{s,s}(x) = x \end{cases}$$

Euristically we must have again $u_t(x) = \overline{u}(\Phi_{0,t}^{-1}(x))$.

Assume that b is locally bounded and $\operatorname{div} b \in L^q_{\text{loc}}$.

Definition

Given $\overline{u} \in L^p_{\text{loc}}$, for some $p \geqslant 1$ a solution of the stochastic transport equation (STE) in L^p_{loc} is a measurable function $(u(t,x,\omega), t \geqslant 0, x \in \mathbb{R}^d, \omega \in \Omega)$ such that

(i) for P-a.e. $\omega \in \Omega$, $x \in \mathbb{R}^d$, $R > 0$, $\sup_{t\in[0,T]} \int_{B(x,R)} |u(t,x,\omega)|^p \, dx < \infty$

(ii) for any test function $\theta \in C_0^0(\mathbb{R}^d)$, the process $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ is continuous and $\mathcal{F}_t$-adapted;

(iii) for any test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ is an $\mathcal{F}_t$-semimartingale satisfying

$$\int_{\mathbb{R}^d} u(t,x)\theta(x)dx = \int_{\mathbb{R}^d} \overline{u}(x)\theta(x)dx + \sum_{i=1}^{d} \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x)dx \right) \circ dW_s^i$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} u(s,x)[b(x) \cdot \nabla\theta(x) + \operatorname{div} b(x)\theta(x)]dx$$

Main result

Theorem

Assume $b \in C^\epsilon$ and $\operatorname{div} b \in L^q$ and $\epsilon > d/q$. The STE has a unique solution u for any $\overline{u} \in L^p_{\text{loc}}$ and $u(t,x) = \overline{u}(\phi_{0,t}^{-1}(x))$.

Note that by the pushforward formula

$$\int_{\mathbb{R}^d} f(x) g \circ \phi_{s,t}(x) J_{s,t}(x) dx = \int_{\mathbb{R}^d} f \circ \phi_{s,t}^{-1}(x) g(x) dx$$

with $J_{s,t}(x) \leqslant C$ locally. So if $f \in L^p_{\text{loc}}, g \in L^q_{\text{loc}}$ we have $f \circ \phi_{s,t}^{-1} \in L^p_{\text{loc}}$ and

$$\int_A |f \circ \phi_{s,t}^{-1}(x)|^p dx = \int_{\phi_{s,t}^{-1}(A)} |f(x)|^p J_{s,t}(x) dx < \infty.$$

Existence

First we need to prove that $\int u(t,x)\theta(x)dx$ is a semimartingale.
Let $\phi_t = \phi_{0,t}$. Take a smooth test function θ, by Itô formula

$$\theta(\phi_t(y)) = \theta(y) + \int_0^t L^b\theta(\phi_s(y))ds + \int_0^t \nabla\theta(\phi_s(y)) \cdot dW_s.$$

Let $J_t^\varepsilon(y)$ the Jacobian determinant of the flow ϕ_t^ε for the regularized vectorfield b^ε. Since b^ε is smooth: $dJ_t^\varepsilon(y) = \operatorname{div} b^\varepsilon(\phi_t(y))J_t^\varepsilon(y)dt$.
Then

$$\begin{aligned}\int \overline{u}(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy = \int \overline{u}(y)\theta(y)dy &+ \int_0^t ds \int \overline{u}(y)L^b\theta(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t ds \int \overline{u}(y)\theta(\phi_s(y))\operatorname{div} b^\varepsilon(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t dW_s \cdot \int \overline{u}(y)\nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy\end{aligned}$$

In the limit $\varepsilon \to 0$ each term converges so

$$\lim_{\varepsilon\to 0} \int \overline{u}(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy = \int \overline{u}(y)\theta(\phi_t(y))J_t(y)dy = \int u(t,y)\theta(y)dy$$

is a semi-martingale.

Next we need to prove that the semimartingale $\int u(t,x)\theta(x)dx$ satisfy the stochastic transport equation.
By the Stratonovic-Itô formula

$$\theta(\phi_t(y)) = \theta(y) + \int_0^t b \cdot \nabla\theta(\phi_s(y))ds + \int_0^t \nabla\theta(\phi_s(y)) \circ dW_s.$$

Then

$$\begin{aligned}\int \overline{u}(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy &= \int \overline{u}(y)\theta(y)dy + \int_0^t ds \int \overline{u}(y)b \cdot \nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t ds \int \overline{u}(y)\theta(\phi_s(y))\operatorname{div} b^\varepsilon(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t dW_s \circ \int \overline{u}(y)\nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy\end{aligned}$$

and take the limit $\varepsilon \to 0$ to conclude.

Uniqueness

Goal

Prove that if $u(t,x)$ solve the STE then we must have $u(t,x) = \overline{u}(\phi_t^{-1}(x))$.

We start by smoothing u. Define

$$u_\varepsilon(t,y) = \int u(t,x)\vartheta_\varepsilon(y-x)\,dx, \quad u_{0,\varepsilon}(y) = \int \overline{u}(x)\vartheta_\varepsilon(y-x)\,dx.$$

Since u is a solution to STE we get

$$\begin{aligned} u_\varepsilon(t,y) = u_{0,\varepsilon}(y) &+ \int_0^t \left[\int u(s,x)b(x)\cdot\nabla_x\vartheta_\varepsilon(y-x)dx\right] ds \\ &+ \int_0^t ds \int u(s,x)\operatorname{div} b(x)\vartheta_\varepsilon(y-x)dx \\ &+ \sum_{i=1}^d \int_0^t \left[\int u(s,x)D_{x_i}\vartheta_\varepsilon(y-x)\,dx\right]\circ dW_s^i \end{aligned}$$

Let $b^\delta = \vartheta_\delta * b$ and let ϕ^δ the associated flow.

By Stratonovich version of Itô-Wentzel calculus

$$\frac{d}{dt} u_\varepsilon(t, \phi_t^\delta(x)) = \left\{ \int u(t,z) \left[(b(z) - b^\delta(y)) \cdot \nabla_z \vartheta_\varepsilon(y-z) + \operatorname{div} b(z) \vartheta_\varepsilon(y-z) \right] dz \right\}_{y=\phi_t^\delta(x)}$$

Test against $\rho \in C_0^\infty(\mathbb{R}^d)$ and perform a change of variables

$$\frac{d}{dt} \int u_\varepsilon(t, \phi_t^\delta x) \rho(x) dx$$
$$= \int \int u(t,x') \left[\left[b(z) - b^\delta(y) \right] \cdot \nabla_z \vartheta_\varepsilon(y-z) + \operatorname{div} b(z) \vartheta_\varepsilon(y-z) \right]_{y=\phi_t^\delta(x)} dz \rho(x) dx$$
$$= \int \int u(t,z) \left[\left[b(z) - b^\delta(y) \right] \cdot \nabla_z \vartheta_\varepsilon(y-z) + \operatorname{div} b(z) \vartheta_\varepsilon(y-z) \right] dz \rho\left((\phi_t^\delta)^{-1}(y) \right) J_t^\delta(y) dy$$

By an integration by parts this is equal to

$$= \int \left[\int \vartheta_\varepsilon(y-z) \left[b(z) - b^\delta(y) \right] \cdot \nabla_y \left[\rho\left((\phi_t^\delta)^{-1}(y) \right) J_t^\delta(y) \right] dy \right] u(t,z) dz$$
$$+ \int \int \left[\operatorname{div} b(z) - \operatorname{div} b^\delta(y) \right] \vartheta_\varepsilon(y-z) \rho((\phi_t^\delta)^{-1}(y)) J_t^\delta(y) dy\, u(t,z) dz$$

We want to show that both contributions go to zero as $\varepsilon \to 0$ and $\delta \to 0$

First term

$$A^\delta = \lim_{\varepsilon\to 0}\int \vartheta_\varepsilon\,(y-z)\,\left[b(z)-b^\delta(y)\right]\cdot\nabla_y\left[\rho\left((\phi_t^\delta)^{-1}y\right)J_t^\delta(y)\right]dy$$
$$= \left[b(z)-b^\delta(z)\right]\cdot\nabla_z\left[\rho\left((\phi_t^\delta)^{-1}(z)\right)J_t^\delta(z)\right]$$

We can prove that

$$|\nabla\left[\rho\left((\phi_t^\delta)^{-1}(\cdot)\right)J_t^\delta(\cdot)\right]|\lesssim\delta^\beta$$

locally as $\delta\to 0$ for any $\beta<-d/q$. Moreover

$$|b-b^\delta|\lesssim\delta^\epsilon$$

so $|A_\delta|\lesssim\delta^{\epsilon+\beta}\to 0$ as soon as $\epsilon+\beta>0$.

Second term

$$
\begin{aligned}
&\int\int \left[\operatorname{div} b(z) - \operatorname{div} b^{\delta}(y)\right] \vartheta_{\varepsilon}(y-z)\rho((\phi_t^{\delta})^{-1}(y))J_t^{\delta}(y)dy\, u(t,z)dz \\
&= \int \operatorname{div} b(z) \left(\int_{\mathbb{R}^d} \vartheta_{\varepsilon}(y-z)\rho((\phi_t^{\delta})^{-1}(y))J_t^{\delta}(y)dy\right) u(t,z)dz \\
&\qquad - \int \operatorname{div} b^{\delta}(y)\rho((\phi_t^{\delta})^{-1}(y))J_t^{\delta}(y)u_{\varepsilon}(t,y)dy
\end{aligned}
$$

and both terms converge, as $\varepsilon \to 0$ followed by $\delta \to 0$ to

$$
\int \operatorname{div} b(y)\rho(\phi_t^{-1}(y))J_t(y)u(t,y)dy
$$

so their difference converge to zero.

We obtained

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left[\int u_\varepsilon(t,\phi_t^\delta x)\rho\,(x)\,dx-\int_{\mathbb{R}^d}u_\varepsilon(0,x)\rho\,(x)\,dx\right]=0.$$

Now

$$\int u_\varepsilon(t,\phi_t^\delta x)\rho\,(x)\,dx=\int\int u_\varepsilon(t,y)\vartheta_\varepsilon(\phi_t^\delta(x)-y)\rho\,(x)\,dxdy$$

$$=\int\int u_\varepsilon(t,y)\vartheta_\varepsilon(z-y)\rho\left((\phi_t^\delta)^{-1}(z)\right)J_t^\delta((\phi_t^\delta)^{-1}(z))^{-1}dzdy$$

$$\to\int u(t,z)\rho\left(\phi_t^{-1}(z)\right)J_t(\phi_t^{-1}(z))^{-1}dz$$

This yields

$$\int u(t,z)\rho\left(\phi_t^{-1}(z)\right)J_t(\phi_t^{-1}(z))^{-1}dz=\int\overline{u}(x)\rho\,(x)\,dx$$

for every $\rho\,(x)\in C_0^\infty(\mathbb{R}^d)$. Choosing ρ appropriately we get

$$\int u(t,z)\rho(z)dz=\int\overline{u}(x)\rho(\phi_t(x))J_t(x)dx=\int\overline{u}(\phi_t^{-1}(y))\rho(y)dy.$$

$\square$

Counterexamples to certain extensions

Example (Random vectorfields)

Take $b(t,x) = \sqrt{|x - W_t|}$, then

$$dX_t = b(t, X_t)dt + dW_t = \sqrt{|X_t - W_t|}dt + dW_t.$$

By the change of variables $Y_t = X_t - W_t$ we obtain

$$dY_t = \sqrt{|Y_t|}dt$$

so path-wise uniqueness is impossible in general.

Not so artificial...

Consider a 2d stochastic Euler equation in vorticity variables

$$\partial_t \xi\left(t,x\right) + \left(u\left(t,x\right)\cdot\nabla\xi\left(t,x\right)\right)dt + \nabla\xi\left(t,x\right)\circ dW\left(t\right) = 0$$

where $\xi = \partial_2 u_1 - \partial_1 u_2$.
Formally equivalent to the "system" of stochastic ordinary equations

$$dX_t^a = \left[\int_{\mathbb{R}^2} K(X_t^a - X_t^{a'})\xi_0(X_t^{a'})da'\right]dt + dW_t, \qquad a\in\mathbb{R}^2$$

for a suitable kernel K, ξ_0 being the initial condition of the vorticity equation.
By the change of variable $Y_t^a = X_t^a - W_t$ we obtain

$$dY_t^a = \left[\int_{\mathbb{R}^2} K(Y_t^a - Y_t^{a'})\xi_0(X_t^{a'})da'\right]dt$$

The equation for (Y_t^a) corresponds to the classical vorticity equation

$$\frac{\partial_t \xi'\left(t,x\right)}{\partial t} + \left(u'\left(t,x\right)\cdot\nabla\xi'\left(t,x\right)\right)dt = 0 \qquad \xi' = \partial_2 u_1' - \partial_1 u_2'$$

with initial condition ξ_0.

Possible way out

Consider a more complex (infinite-dimensional) noise:

$$dX_t^a = \left[\int_{\mathbb{R}^2} K(X_t^a - X_t^{a'})\xi_0(X_t^{a'})da'\right] dt + \sum_{k=1}^{\infty} \sigma_k(X_t^a)dW_t^k, \qquad a \in \mathbb{R}^2$$

where each point X_a is moved "almost" independently of the others.

Natural assumption

$$\sum_{k=1}^{\infty} \sigma_k(x)\sigma_k(y) = a(|x-y|)$$

with $a(r) - a(0) \simeq r^\alpha$ as $r \to 0$, $\alpha \in (0, 2]$.

In order to hope some regularizing effect of the noise over the deterministic (and singular) drift we seems to need small α.
Connection with the theory of stochastic flows by Le Jan-Raimond.

Merci