Global well-posedness and decay for the viscous surface wave problem without surface tension

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Outline

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The viscous surface wave problem

We consider:

- A viscous fluid of finite depth in $3D$ (the ocean)
- Lower boundary is fixed (the solid ocean floor)
- Upper boundary is a free surface where the fluid meets the air (surface waves)
- Air is constant pressure, zero viscous forcing
- Uniform gravitational field
- No surface tension

Main features

- Fluid evolves according to the incompressible Navier-Stokes equations: nonlinear system of PDEs
- The domain in which the fluid evolves is an unknown in the problem: free boundary problem
- Geometric evolution for the boundary (hyperbolic) is coupled to the nonlinear PDE for fluid (parabolic)
- Potential for nasty singularities in boundary geometry: self-intersections, topology changes, ...

Singularities

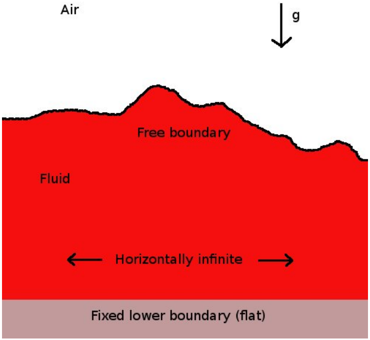
Wave breaking

Spray

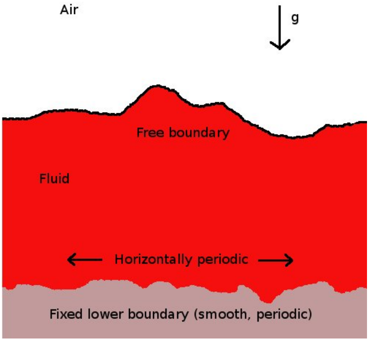
Because of these singularities, it is reasonable to only expect global-in-time (strong) solutions to exist for small initial data.

Singularity formation verified recently by Castro-Cordoba-Fefferman-Gancedo-Gomez-Serrano and Coutand-Shkoller.

Cartoons of our configurations (cross-sections)



Infinite, flat bottom



Periodic, non-flat bottom

Fluid domain and unknowns

The moving domain has a free surface given as the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$, where $\Sigma = \mathbb{R}^2$ or $\mathbb{T}^2$:

- $\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t)\}$
- $b \in (0, \infty)$ is constant in the infinite case ($\Sigma = \mathbb{R}^2$)
- $0 < b \in C^\infty(\mathbb{T}^2)$ in the periodic case ($\Sigma = \mathbb{T}^2$)

For each $t \geq 0$ the fluid is described by

- velocity $u(\cdot, t) : \Omega(t) \to \mathbb{R}^3$
- pressure $p(\cdot, t) : \Omega(t) \to \mathbb{R}$

Equations of motion

Incompressible Navier-Stokes in $\Omega(t)$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u - g e_3 \\ \operatorname{div} u = 0 \end{cases}$$

- $(u \cdot \nabla u)_i = u_j \partial_j u_i$
- $\mu > 0$ is the viscosity = fluid friction = dissipation mechanism
- $g > 0$ is the gravitational constant
- $\operatorname{div} u = 0$ means that volume is preserved along the flow

Equations of motion

Continuity of normal stress on the free surface, $\{y_3 = \eta(y_1, y_2, t)\}$:

$$(pI - \mu\mathbb{D}(u))\nu = p_{atm}\nu$$

- p_{atm} is the constant atmospheric pressure
- I is the 3×3 identity matrix
- ν is the unit normal to $\{y_3 = \eta(y_1, y_2, t)\}$
- $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ is the symmetric gradient
- $S(p, u) = (pI - \mu\mathbb{D}(u))$ is the stress tensor

Equations of motion

Surface is advected with the fluid on $\{y_3 = \eta(y_1, y_2, t)\}$:

$$\partial_t \eta + u_1 \partial_{y_1} \eta + u_2 \partial_{y_2} \eta = u_3$$

- Kinematic transport equation: free boundary is defined by where the fluid is
- No dissipation mechanism

Equations of motion

No-slip BCs on $\{y_3 = -b(y_1, y_2)\}$:

$$u = 0$$

- Required by viscosity

Initial data

$$\begin{cases} u(t=0) = u_0 \\ \eta(t=0) = \eta_0, \end{cases}$$

- Enforce compatibility conditions (ignore for now)

Full problem

- Make change of pressure $p \mapsto p + gy_3 - p_{atm}$ to shift forcing to the boundary

Then (u, p, η) satisfy:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t) \\
\operatorname{div} u = 0 & \text{in } \Omega(t) \\
\partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\
(pI - \mu \mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\
u = 0 & \text{on } \{y_3 = -b(y_1, y_2)\} \\
u(t = 0) = u_0, \eta(t = 0) = \eta_0.
\end{cases}
$$

Natural energy structure

The problem possesses a natural energy structure:

$$\frac{d}{dt}\left(\underbrace{\frac{1}{2}\int_{\Omega(t)}|u(t)|^2+\frac{g}{2}\int_{\Sigma}|\eta(t)|^2}_{\text{energy } \mathcal{E}}\right)+\underbrace{\frac{\mu}{2}\int_{\Omega(t)}|\mathbb{D}u(t)|^2}_{\text{dissipation } \mathcal{D}}=0.$$

- On one hand, $g > 0$ gives a priori control of η
- On the other hand, it seems to obstruct decay...

Decay info

In a fixed domain without gravity:

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\int_\Omega |u(t)|^2\right)}_{\mathcal{E}} + \underbrace{\frac{\mu}{2}\int_\Omega |\mathbb{D}u(t)|^2}_{\mathcal{D}} = 0.$$

$$C\mathcal{E} \le \mathcal{D} \text{ via Korn's inequality } \Rightarrow \partial_t \mathcal{E} + C\mathcal{E} \le 0$$

$$\Rightarrow \mathcal{E}(t) \le e^{-Ct}\mathcal{E}(0).$$

Decay info

In a moving domain with gravity, we can prove

$$\frac{C}{2}\int_{\Omega(t)} |u(t)|^2 \leq \frac{\mu}{2}\int_{\Omega(t)} |\mathbb{D}u(t)|^2$$

if we have some uniform control of the geometry of $\Omega(t)$, but at best

$$C\,\|\eta\|^2_{H^{-1/2}(\Sigma)} \leq \mathcal{D},$$

so

$$C\mathcal{E} \not\leq \mathcal{D} \Rightarrow \text{ decay is not clear.}$$

Outline

Beale's non-decay theorem, part 1

Beale ('81) proves three theorems for the infinite problem

Theorem (Local well-posedness)

For $u_0 \in H^{r-1}$ with $r \in (3, 7/2)$, there exists a unique solution

$$u \in H^0((0,T); H^r) \cap H^{r/2}((0,T); H^0)$$

with $T = T(\|u_0\|_r) > 0$.

Theorem (Large-but-finite-time well-posedness)

For all $T > 0$ there exists $\delta = \delta(T) > 0$ so that if $\|u_0\|_r < \delta$, then there exists a unique soln on $(0,T)$. Also, solutions are analytic in the data.

Beale's non-decay theorem, part 2

Given these, one might expect GWP + decay, but...

Theorem (No global well-posedness and decay)

There exists an initial surface ζ so that there cannot exist a curve of global-in-time solutions, $(u(\varepsilon), p(\varepsilon), \eta(\varepsilon))$ for ε near 0, so that (among other things)

$$\begin{cases} \eta_0(\varepsilon) = \varepsilon\zeta + O(\varepsilon^2), u_0(\varepsilon) = 0 \\ u(\varepsilon) \in L^1((0,\infty); H^r) \text{ for } r \in (3, 7/2), \\ \lim_{t\to\infty} \eta(\varepsilon, t) = 0 \text{ in } H^{r-1/2}. \end{cases}$$

- Proof is a reductio ad absurdum that critically uses specially chosen properties of ζ.
- Beale notes that the theorem does not preclude GWP + decay, but rather indicates that such a result must follow from different hypotheses.

Surface tension results

A way to add stability is to consider the effect of surface tension:

$$(pI - \mu\mathbb{D}u)\nu = g\eta\nu - \sigma H\nu$$

where $H = \text{div}(\nabla\eta/\sqrt{1+|\nabla\eta|^2})$ is the mean curvature on the free surface and $\sigma > 0$ is the surface tension.

- Geometric forcing: like mean curvature flow for the surface, leads to a smoothing of the surface (RHS is now an elliptic operator)
- Beale ('83): small data global well-posedness
- Beale-Nishida ('84): algebraic decay, which is sharp

Inviscid, irrotational problem

- If viscosity is neglected ($\mu = 0$) and the fluid is initially irrotational, curl $u_0 = 0$, then curl $u(t) = 0$ for $t > 0$. Hence $u = \nabla\varphi$ for φ harmonic.
- "Surface reformulation" reduces problem to PDE on horizontal cross section ($\mathbb{R}^2$) only.
- GWP: Wu ('09), Germain-Masmoudi-Shatah ('09)
- With viscosity, irrotationality is impossible: vorticity is generated at the free surface

Intriguing questions

1. Is viscosity alone capable of producing global well-posedness? (Physics: is surface tension required for global stability, or is viscosity alone enough?)
2. Do the solutions decay in time, and if so, in which spaces and at what rate? Which of the assumptions of Beale's non-decay theorem must be violated?

Outline

Answers

In joint work with Y. Guo, we answer both questions in the affirmative in a trio of papers.

- High regularity local well-posedness, using linear problems in moving domains
- Two-tier energy method: a priori estimates in the infinite case with flat bottom
- Two-tier energy method: a priori estimates in the periodic case with smooth bottom

Consequence: GWP + decay in both cases

Infinite case – rough statement of theorem, part 1

Theorem

Let $\lambda \in (0,1)$. Suppose the data (u_0, η_0) satisfy certain compatibility conditions. There exists a $\kappa > 0$ so that if

$$\|u_0\|^2_{H^{20}} + \|\eta_0\|^2_{H^{20+1/2}} + \|\mathcal{I}_\lambda u_0\|^2_{H^0} + \|\mathcal{I}_\lambda \eta_0\|^2_{H^0} < \kappa,$$

then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys various estimates...

Note: $\mathcal{I}_\lambda =$ horizontal Riesz potential $=$ negative λ horizontal derivatives (more later...)

Infinite case – rough statement of theorem, part 2

Theorem

In particular, we have the decay estimates

$$\sup_{t\geq 0}\left[(1+t)^{2+\lambda}\|u(t)\|_{C^2}^2+(1+t)^{1+\lambda}\|u(t)\|_{H^2}^2\right]\leq C\kappa,$$

$$\sup_{t\geq 0}\left[(1+t)^{1+\lambda}\|\eta(t)\|_{L^\infty}^2+\sum_{j=0}^{1}(1+t)^{j+\lambda}\left\|D^j\eta(t)\right\|_{H^0}^2\right]\leq C\kappa$$

for a universal constant $C > 0$.

Periodic case – rough statement of theorem, part 1

Theorem

Let $N \geq 3$ be an integer. Suppose the data (u_0, η_0) satisfy certain compatibility conditions and that η_0 satisfies a "zero average condition." There exists a $0 < \kappa = \kappa(N)$ so that if

$$\|u_0\|_{H^{4N}}^2 + \|\eta_0\|_{H^{4N+1/2}}^2 < \kappa,$$

then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys various estimates. In particular, we have the decay estimates

$$\sup_{t \geq 0} (1+t)^{4N-8} \left[\|u(t)\|_{H^{2N+4}}^2 + \|\eta(t)\|_{H^{2N+4}}^2 \right] \leq C\kappa$$

for a universal constant $C > 0$.

Remarks

- Infinite case: the sharp decay rates with surface tension (Beale-Nishida) correspond to $\lambda = 1$, so by taking $\lambda \approx 1$, we recover almost the same decay.
- Periodic: by making N larger, we recover arbitrarily fast algebraic decay. This is almost exponential decay. This is in contrast with a result of Nishida-Teramoto-Yoshihara ('04) with surface tension, which proves exponential decay with flat lower bottom.
- Moral: viscosity is the basic decay mechanism, surface tension just enhances the decay rate, and the rate of decay with ST can "almost" be achieved without it.

Outline

Avoiding the non-decay theorem

We avoid the hypotheses of Beale's non-decay theorem in three important ways:

- We work in a very different functional framework with higher regularity and more compatibility conditions for the data.
- In the infinite case, our framework does not require that $u \in L^1((0,\infty); H^2)$. Moreover, our decay estimates do not imply this since the best we can do has the $L^1((0,T); H^2)$ norm diverging like $\log T$.
- In the periodic case, Beale's choice of data, $\eta_0 = \varepsilon\zeta + O(\varepsilon^2)$, violates the natural "zero-average condition" for the data.

Zero average condition, periodic case

- In the periodic case we have

$$\frac{d}{dt}\int_{\mathbb{T}^2} \eta = 0 \Rightarrow \int_{\mathbb{T}^2} \eta(t) = \int_{\mathbb{T}^2} \eta_0.$$

  Then a necessary condition for the decay $\eta(t) \to 0$ in L^2 and L^∞ as $t \to \infty$ is that η_0 satisfies the "zero average condition":

$$\int_{\mathbb{T}^2} \eta_0 = 0.$$

- It turns out that the properties of ζ require that $\eta_0 = \varepsilon\zeta + O(\varepsilon^2)$ violate this.
- Note: a large class of data can be shifted to force this to be true while maintaining the condition $b > 0$ (essentially a constraint on the fluid mass to prevent pooling).

Zero average condition, infinite case

- The condition $\int_{\mathbb{R}^2} \eta_0 = 0$ need not make sense if $\eta_0 \in H^k$.
- Equivalent to $\hat{\eta}_0(0) = 0$ for $\hat{\cdot}$ the Fourier transform.
- We enforce a "weak form" of $\hat{\eta}_0(0) = 0$ by requiring the Riesz potential $\mathcal{I}_\lambda \eta_0 \in L^2$ for some $\lambda \in (0,1)$, where

$$\mathcal{I}_\lambda f(x) = \int_{\mathbb{R}^2} \hat{f}(\xi) \left|\xi\right|^{-\lambda} e^{2\pi i x \cdot \xi} d\xi.$$

- Analytic utility = controls low frequency Fourier modes = something like a Poincaré inequality that we get in the periodic case from the zero-average condition. Essential use in interpolation estimates in a priori estimates.

Outline

Sketch of principal difficulties, pt. 1

Usual nonlinear energy method runs into some problems:

- Domain moves, so applying derivatives breaks the boundary conditions. Solution: introduce a flattened "geometric" coordinate system that fixes the domain to $\Omega = \{x \in \mathbb{R}^3 \mid -b < x_3 < 0\}$ (not Lagrangian coordinates).
- The dissipation always fails to control the energy by a $1/2$ derivative gap for η. This prevents us from deducing exponential decay from the energy evolution equation. Solution: introduce two tiers of energies / dissipations, one with high regularity and one with low regularity. Use an interpolation argument to compensate for the $1/2$ derivative gap in the low energy. This leads to algebraic decay of the low-regularity energy.

Sketch of principal difficulties, pt. 2

- The nonlinearity that appears in the high-regularity energy estimates involves more derivatives of the free surface, η, than can be controlled by the high-level energy and dissipation, which breaks the usual energy method. Solution: estimate the highest derivatives of η using the kinematic transport equation.
- Highest derivatives of η grow in time, so it's impossible to close the usual energy method estimates. Solution: use the decay of the low-regularity energy to balance this growth.

Note: in this scheme the existence of global-in-time solutions is predicated on their decay.

Outline

Two tiers

We define two tiers of energies and dissipations using the natural energy / dissipation structure described earlier. Let $N \geq 3$ be an integer.

- $\mathcal{E}_H$ and $\mathcal{D}_H$ – high derivatives: $2N$ temporal, $4N$ spatial
- $\mathcal{E}_L$ and $\mathcal{D}_L$ – low derivatives: $N+2$ temporal, $2N+4$ spatial
- Parabolic scaling dictates the relation between temporal and spatial derivative counts.
- "Low" is roughly half of "high" with extra $+2$ to help in Sobolev embeddings.

We get

$$\mathcal{E}_H(t) + \int_0^t \mathcal{D}_H(s)ds \lesssim \mathcal{E}_H(0) + \int_0^t \mathcal{N}_H(s)ds$$

$$\partial_t \mathcal{E}_L(t) + \mathcal{D}_L(t) \lesssim \mathcal{N}_L(t)$$

for some nonlinearities $\mathcal{N}_L$ and $\mathcal{N}_H$.

Absorbing

Suppose we can estimate the nonlinearities in terms of the dissipations (and data):

$$\int_0^t \mathcal{N}_H(s)ds \lesssim \varepsilon \int_0^t \mathcal{D}_H(s)ds + \mathcal{F}_H(0)$$

$$\mathcal{N}_L(t) \lesssim \varepsilon \mathcal{D}_L(t)$$

for $\varepsilon > 0$ small and $\mathcal{F}_H(0)$ some norms of the data at $t = 0$. Then we can absorb the nonlinear terms into the LHS:

$$\mathcal{E}_H(t) + \int_0^t \mathcal{D}_H(s)ds \lesssim \mathcal{E}_H(0) + \mathcal{F}_H(0) := \mathcal{C}_0$$

$$\partial_t \mathcal{E}_L(t) + C\mathcal{D}_L(t) \leq 0.$$

High-level bounds imply low-level decay

It is not true that $\mathcal{E}_L \lesssim \mathcal{D}_L$ (1/2 derivative gap persists). However, we can now interpolate and use the high-level bound:

$$\mathcal{E}_L \lesssim \mathcal{E}_H^{\theta} \mathcal{D}_L^{1-\theta} \lesssim \mathcal{C}_0^{\theta} \mathcal{D}_L^{1-\theta}$$

for $\theta \in (0,1)$ small (determined by N and λ). Then for $1/(1-\theta) = 1 + 1/r$ we have

$$\begin{aligned} \partial_t \mathcal{E}_L(t) + C\mathcal{D}_L(t) \le 0 &\Rightarrow \partial_t \mathcal{E}_L(t) + C(\mathcal{C}_0)(\mathcal{E}_L(t))^{1+1/r} \le 0 \\ &\Rightarrow \mathcal{E}_L(t) \lesssim \mathcal{C}_0/(1+t)^r, \end{aligned}$$

and so we get algebraic decay. Note that the decay rate r is determined by $1-\theta$, which is ultimately determined by N and λ. Only by taking $\lambda \in (0,1)$ can we get $r = 2 + \delta$ for some $\delta > 0$.

Estimates of the nonlinearities

Now we need to justify the estimates of the nonlinearities $\mathcal{N}_H$ and $\mathcal{N}_L$.

- Problem 1: $\mathcal{N}_L$ involves more derivatives of η than can be controlled by $\mathcal{E}_L$ or $\mathcal{D}_L$. Solution: interpolate with $\mathcal{E}_H$. We get

$$\mathcal{N}_L \lesssim \mathcal{E}_H^q \mathcal{D}_L \text{ for some } q > 0$$
$$\Rightarrow \mathcal{N}_L \lesssim \varepsilon \mathcal{D}_L \text{ if } \mathcal{E}_H \text{ is small enough.}$$

- Problem 2: $\mathcal{N}_H$ involves more derivatives of η $(4N + 1/2)$ than can be controlled by $\mathcal{E}_H$ $(4N)$ or $\mathcal{D}_H$ $(4N - 1/2)$. We can't interpolate now. Solution: use the kinematic transport equation

$$\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3$$
$$\Rightarrow \partial_t \eta \approx u_3|_\Sigma \in H^{4N+1/2} \text{ since } \|u_3|_\Sigma\|^2_{4N+1/2} \lesssim \|u\|^2_{4N+1} \lesssim \mathcal{D}_H.$$

Transport estimate

Define $\mathcal{F}_H = \|\eta\|^2_{4N+1/2}$. Then we use a transport estimate for η (Danchin, '05):

$$\sup_{0\le s\le t} \mathcal{F}_H(s) \le C \exp\left(C \int_0^t \sqrt{\mathcal{E}_L(s)} ds \right) \left[\mathcal{F}_H(0) + t \int_0^t \mathcal{D}_H(s) ds \right].$$

- The RHS can grow exponentially in time unless $\mathcal{E}_L$ decays like $1/(1+t)^{2+\delta}$. Even if $\mathcal{E}_L$ decays this fast, the RHS still grows linearly in time.

Estimate of $\mathcal{N}_H$

In order to balance the growth of $\mathcal{F}_H$, we have to identify a special structure in the estimate of $\mathcal{N}_H$: it always appears in a product $\mathcal{F}_h \mathcal{E}_L$, so we can use the decay of $\mathcal{E}_L$ to balance the growth of $\mathcal{F}_h$. Fortunately, this structure is there:

$$\int_0^t \mathcal{N}_H(s) ds \lesssim \int_0^t \mathcal{E}_H(s)^q \mathcal{D}_H(s) ds + \int_0^t \sqrt{\mathcal{D}_H(s)\mathcal{E}_L(s)\mathcal{F}_H(s)} ds$$

for some $q > 0$.

Decay at the low level implies bounds at the high level

Since $\mathcal{E}_L(t)$ decays like $1/(1+t)^{2+\delta}$ we can get

$$\int_0^t \mathcal{N}_H(s)ds \lesssim \mathcal{F}_H(0) + \varepsilon \int_0^t \mathcal{D}_H(s)ds.$$

We then deduce that

$$\mathcal{E}_H(t) + \int_0^t \mathcal{D}_H(s)ds \lesssim \mathcal{E}_H(0) + \mathcal{F}_H(0) = \mathcal{C}_0$$

and

$$\frac{\mathcal{F}_H(t)}{1+t} \lesssim \mathcal{C}_0.$$

Summary of a priori estimates

- We will build a “total energy” that couples the bounds at the high order to the decay at the low order and the growth of $\mathcal{F}_H$.
- Low order decay estimate $\Rightarrow$ high order bounds in terms of data.
- High order bounds $\Rightarrow$ low order decay estimate in terms of data.
- Decay and high bounds $\Rightarrow$ linear growth estimate for $\mathcal{F}_H$ in terms of data.

Outline

1 Introduction
- Formulation of the problem
- History and motivation

2 Main results
- Overview
- Discussion of Beale's non-decay theorem

3 Sketch of the a priori estimates
- Difficulties
- Two-tier nonlinear energy method
- Particulars

Two tiers of energies (rough definition)

We define energies and dissipations for $n = 2N$ and $n = N + 2$:

$$\mathcal{E}_n = \|\mathcal{I}_\lambda u\|^2_{H^0(\Omega)} + \sum_{j=0}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j}(\Omega)} + \sum_{j=0}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j-1}(\Omega)} + \|\mathcal{I}_\lambda \eta\|^2_{H^0(\Sigma)} + \sum_{j=0}^{n} \left\|\partial_t^j \eta\right\|^2_{H^{2n-2j}(\Sigma)}$$

$$\mathcal{D}_n = \|\mathcal{I}_\lambda u\|^2_{H^1(\Omega)} + \sum_{j=0}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega)} + \|\nabla p\|^2_{H^{2n-1}(\Omega)} + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j}(\Omega)} + \|D\eta\|^2_{H^{2n-3/2}(\Sigma)} + \|\partial_t \eta\|^2_{H^{2n-1/2}(\Sigma)} + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|^2_{H^{2n-2j+5/2}(\Sigma)}$$

Total energy norm

- In our interpolation estimates we need $N \geq 5$, and for the infinite problem nothing improves for larger N, so we choose $N = 5$: ($2N = 10$ temporal, $4N = 20$ spatial), ($N + 2 = 7$ temporal, $2N + 4 = 14$ spatial).
- Let's now call $\mathcal{E}_H = \mathcal{E}_{2N} = \mathcal{E}_{10}$, $\mathcal{E}_L = \mathcal{E}_{N+2} = \mathcal{E}_7$, $\mathcal{F}_H = \mathcal{F}_{10}$, etc.
- We combine the high and low terms into the total energy we use for our GWP result:

$$\mathcal{G}_{10}(t) := \sup_{0 \leq s \leq t} \mathcal{E}_{10}(s) + \int_0^t \mathcal{D}_{10}(s) dr + \sup_{0 \leq s \leq t} (1+s)^{2+\lambda} \mathcal{E}_7(s) + \sup_{0 \leq s \leq t} \frac{\mathcal{F}_{10}(s)}{(1+s)}.$$

- Bounds on $\mathcal{G}_{10}(t)$ couple the boundedness of high-order norms to the decay of low-order norms.

Interpolation remark

To close the estimates we need the interpolation estimate:

$$\mathcal{E}_7 \lesssim \mathcal{E}_{10}^{\theta} \mathcal{D}_7^{1-\theta}.$$

An example estimate:

$$\left\| D^2 \eta \right\|_0^2 \lesssim \left(\left\| \mathcal{I}_\lambda \eta \right\|_0^2 \right)^{\theta} \left(\left\| D^3 \eta \right\|_0^2 \right)^{1-\theta}$$

with $\theta = 1/(3+\lambda) \Rightarrow 1/(1-\theta) = 1 + 1/(2+\lambda) \Rightarrow r = 2 + \lambda.$

- Power improves with use of $\mathcal{I}_\lambda$: $\lambda > 0$ is necessary for $r > 2$.
- Proof of full estimate is fairly involved: multi-step bootstrap interpolation (using proper definitions of $N+2$ energies, which involve “minimal derivative counts”)

Theorem 1 – A priori estimates

The two-tier nonlinear energy method then works as described before, and we get:

Theorem

Let (u, p, η) be a solution on $(0, T)$. Then there exists a $\delta > 0$ so that if $\mathcal{G}_{10}(T) \leq \delta$, then

$$\mathcal{G}_{10}(T) \leq C\left(\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0)\right)$$

for a constant $C = C(\lambda, \mu, g, b)$.

Theorem 2 – GWP+decay (using LWP)

Theorem

Fix $\lambda \in (0,1)$. Then there exists a $\kappa > 0$ so that if $\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0) \leq \kappa$, then there exists a unique global-in-time solution satisfying

$$\mathcal{G}_{10}(\infty) \leq C\left(\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0)\right) \leq C\kappa$$

for a constant $C = C(\lambda, \mu, g, b)$. Moreover,

$$\sup_{0 \leq t}\left[(1+t)^{2+\lambda}\left\|u(t)\right\|^2_{C^2(\Omega)} + (1+t)^{1+\lambda}\left\|u(t)\right\|^2_{H^2(\Omega)}\right] \leq C\kappa,$$

$$\sup_{0 \leq t}\left[(1+t)^{1+\lambda}\left\|\eta(t)\right\|^2_{L^\infty(\Sigma)} + \sum_{j=0}^{1}(1+t)^{j+\lambda}\left\|D^j\eta(t)\right\|^2_{H^0(\Sigma)}\right] \leq C\kappa.$$

Thanks!

Thank you for your attention!