From Saint-Venant to Navier-Stokes with variable density

Modeling, kinetic interpretation, simulations

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Joint work with

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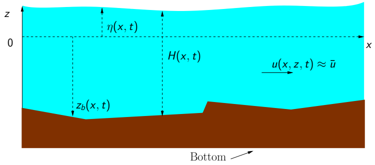
Shallow water systems

$$(NS)\begin{cases} \operatorname{div}\underline{\mathbf{u}} = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = G + \operatorname{div}\underline{\underline{\Sigma}}, \end{cases}$$

- small parameter $\varepsilon = \frac{H_0}{L_0}$, expansion in $\mathcal{O}(\varepsilon^2)$
- Saint-Venant 1872, Gerbeau 2001, Saleri 2004, Marche 2007

$$(SV)\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\mu\frac{\partial \bar{u}}{\partial x}\right) - \kappa(\bar{u}) \end{cases}$$

- Low complexity
- Many applications



NOTATIONS: free surface η, bottom z_b, water height $H = \eta - z_b$, velocities $\underline{\mathbf{u}} = (u, w)$, $\bar{u}(x,t) = \frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$

FROM SAINT-VENANT TO NAVIER-STOKES ?

Navier-Stokes vs. Saint-Venant

2D Navier-Stokes $\mathbf{u} = (u, w)$

$$\left\{\begin{array}{l}\operatorname{div} \underline{\mathbf{u}} = 0,\\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = G + \operatorname{div} \underline{\underline{\Sigma}},\end{array}\right.$$

Saint-Venant $H, \bar{u}$

$$\left\{\begin{array}{l}\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0,\\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} - \kappa(\bar{u})\end{array}\right.$$

Complexity of Navier-Stokes

- functional analysis
- numerical schemes
- geometrical meshes, ALE
- computational costs and use

Advantages of Saint-Venant

- fixed meshes
- reduced computational costs
- efficient numerical methods (FV, DG)

Derivation of the Saint-Venant system

(Navier-Stokes) $$\left\{\begin{array}{l}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,\\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\Sigma_{xx}+\frac{\partial}{\partial z}\Sigma_{xz},\\ \frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g+\frac{\partial}{\partial x}\Sigma_{zx}+\frac{\partial}{\partial z}\Sigma_{zz},\end{array}\right.$$

One main assumption $(\varepsilon = H_0/L_0 \ll 1)$

1. Hydrostatic

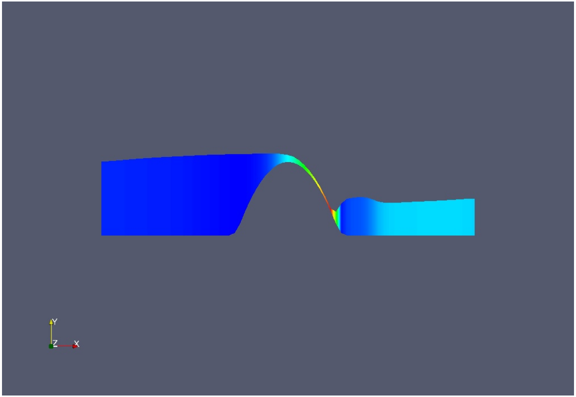
$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g \quad\Rightarrow\quad \frac{\partial p}{\partial z}=-g$$

2. Averaged velocity $\bar{u}(x,t)=\frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$

(St-Venant) $$\left\{\begin{array}{l}\frac{\partial H}{\partial t}+\frac{\partial (H\bar{u})}{\partial x}=0,\\ \frac{\partial (H\bar{u})}{\partial t}+\frac{\partial}{\partial x}\left(H\bar{u}^2+\frac{g}{2}H^2-4\mu H\frac{\partial \bar{u}}{\partial x}\right)=-gH\frac{\partial z_b}{\partial x}-\kappa(\bar{u})\end{array}\right.$$

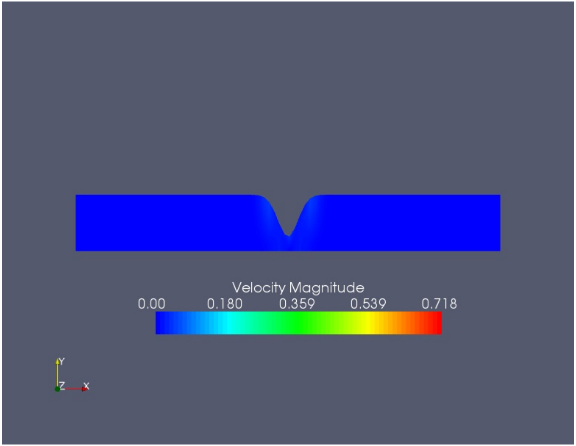
Three numerical examples

1. Hydraulic jump



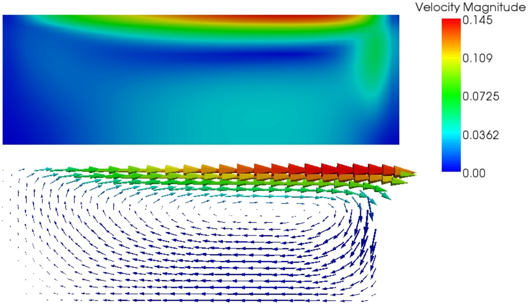
Three numerical examples

1. Waves propagation



Three numerical examples

1. Stratified velocities (wind effect)



Typical applications (Etang de Berre)

- Large scale : time (1 year) & space (155 km^2)
- Water with variable density (temperature & salinity)
- Hydrodynamics and biology

Outline

Classical Saint-Venant system

Proposition (Gerbeau-Perthame 2001)

The viscous Saint-Venant system with friction

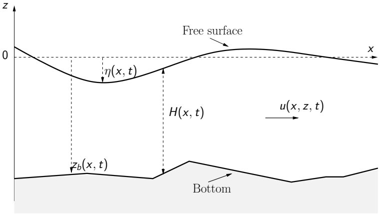
$$\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0,$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\mu\frac{\partial \bar{u}}{\partial x}\right) - \kappa(\bar{u})$$

results from an approximation in $O(\varepsilon^2)$ of the Navier-Stokes equations ($\varepsilon = H_0/L_0$).

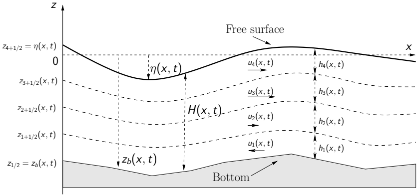
- Strictly hyperbolic
- Numerical schemes
  - finite volumes (relaxation schemes, cf. Bouchut, kinetic schemes, cf. Perthame , Simeoni)
  - discontinuous Galerkin (Ern)
- Not only free surface flows (blood in arteries, Quarteroni *et al.*, avalanches, Bouchut *et al.*)

Vertical discretization

Enrich the velocity field



$$\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0,$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\mu\frac{\partial \bar{u}}{\partial x}\right) - \kappa(\bar{u})$$



- Several attempts (Audusse, Pares)
- Valid for non-miscible fluids
- Hyperbolicity ?
- Key point :

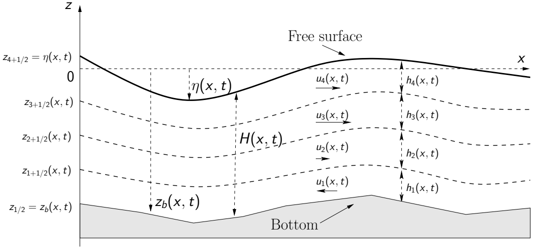
$$H = \sum_{\alpha=1}^{N} h_\alpha, \quad h_\alpha(x,t) = l_\alpha H(x,t)$$
$$u(x,z,t) = \sum_{\alpha=1}^{N} 1_{z\in h_\alpha}(z)\, u_\alpha(x,t)$$

Derivation of the (new) multilayer system

Starting point

$$(\text{Euler hydro}) \quad \left\{ \begin{array}{l} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial z} = -g \end{array} \right.$$

+ two kinematic boundary conditions

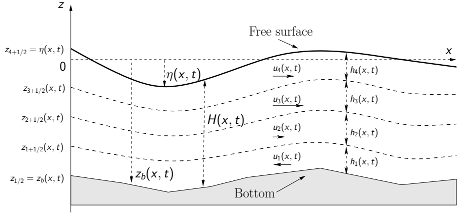


- Viscous terms neglected in the presentation
- Discretization in z of the velocities u, w
- Also done in 3D

Derivation of the (new) multilayer system

Starting point

$$\text{(Euler hydro)} \quad \left\{ \begin{array}{l} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial z} = -g \end{array} \right.$$



$$u(x,z,t) = \sum_{\alpha=1}^{N} 1_{z\in h_\alpha}(z) u_\alpha(x,t)$$

- Integration on each layer of the continuity equation

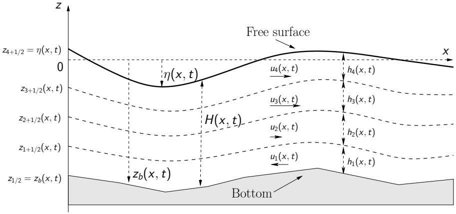
$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial h_\alpha u_\alpha}{\partial x} = G_{\alpha+1/2} - G_{\alpha-1/2}$$

$$G_{\alpha+1/2} = \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2}\frac{\partial z_{\alpha+1/2}}{\partial x} - w_{\alpha+1/2},$$
$$G_{1/2} = G_{N+1/2} = 0 \text{ (kinematic boundary conditions)}$$

Derivation of the (new) multilayer system

Starting point

$$\text{(Euler hydro)}\quad \left\{\begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial z} = -g \end{array}\right.$$



$$u(x,z,t) = \sum_{\alpha=1}^{N} 1_{z\in h_\alpha}(z)\, u_\alpha(x,t)$$

- Integration on each layer of the momentum equation

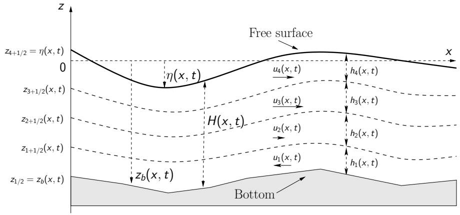
$$\frac{\partial (h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + g h_\alpha \left[\frac{h_\alpha}{2} + \sum_{j=\alpha+1}^{N} h_j \right] \right) = F_{\alpha+1/2} - F_{\alpha-1/2},$$

with $F_{\alpha+1/2} = u_{\alpha+1/2} G_{\alpha+1/2} + \left(\sum_{j=\alpha+1}^{N} g h_j \right) \frac{\partial z_{\alpha+1/2}}{\partial x},$

Derivation of the (new) multilayer system

Starting point

$$(\text{Euler hydro}) \quad \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial z} = -g \end{cases}$$



$$u(x,z,t) = \sum_{\alpha=1}^{N} 1_{z\in h_\alpha}(z)\, u_\alpha(x,t)$$

- Integration on each layer of the momentum equation $(\times u)$

$$\frac{\partial}{\partial t} E_\alpha + \frac{\partial}{\partial x}\left(u_\alpha\left(E_\alpha + g h_\alpha\left(\frac{h_\alpha}{2} + \sum_{j=\alpha+1}^{N} h_j\right)\right)\right) = E_{\alpha+1/2} - E_{\alpha-1/2}$$

with $E_\alpha = \frac{h_\alpha u_\alpha^2}{2} + \frac{g(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2)}{2}$

$E_{\alpha+1/2} = \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + \left(\sum_{j=\alpha+1}^{N} g h_j\right) \frac{\partial z_{\alpha+1/2}}{\partial x} u_{\alpha+1/2}$

Energy equality

For the multilayer system

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{N} E_\alpha\right) + \frac{\partial}{\partial x}\left(\sum_{\alpha=1}^{N} u_\alpha \left(E_\alpha + \frac{g}{2} h_\alpha f(\{h_j\}_{j\leq\alpha}) - 4\mu h_\alpha \frac{\partial u_\alpha}{\partial x}\right)\right) =$$

$$-\kappa(\bar{\mathbf{v}}, H)u_1^2 - \frac{\mu}{h_\alpha}\sum_{\alpha=1}^{N-1}(u_{\alpha+1/2} - u_{\alpha-1/2})^2 - 4\mu\sum_{\alpha=1}^{N} h_\alpha\left(\frac{\partial u_\alpha}{\partial x}\right)^2,$$

with $E_\alpha = \frac{h_\alpha u_\alpha^2}{2} + \frac{g(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2)}{2}$

For the 2D Navier-Stokes eq.

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\left(\frac{u^2}{2} + gz\right)dz + \frac{\partial}{\partial x}\left(u\left(\int_{z_b}^{\eta}\left(\frac{u^2}{2} + gz\right)dz + p\right)\right) = -\kappa u^2$$

Complete formulation

Proposition (Audusse, Bristeau, Perthame, JSM 2009)

The multilayer Saint-Venant system

$$\begin{cases} \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{N} \frac{\partial}{\partial x}(h_\alpha u_\alpha) = 0 \\ \frac{\partial (h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha f(\{h_j\}_{j\geq\alpha})\right) = F_{\alpha+1/2} - F_{\alpha-1/2} \\ \frac{\partial E_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(u_\alpha\left(E_\alpha + \frac{g}{2} h_\alpha H\right)\right) = E_{\alpha+1/2} - E_{\alpha-1/2} \end{cases}$$

results in an approximation in $\mathcal{O}(1/N)$ of the Euler (Navier-Stokes) equations ($h_\alpha = H/N$).

- Only one "global" continuity eq.
- Exchange terms $G_{\alpha+1/2}$, $F_{\alpha+1/2} = u_{\alpha+1/2} G_{\alpha+1/2} + P_{\alpha+1/2}$,
  $E_{\alpha+1/2} = \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + u_{\alpha+1/2} P_{\alpha+1/2}$
- If $G_{\alpha+1/2} = 0$, non-miscible fluids

The multilayer system is hyperbolic

The multilayer system reads

$$M(U)\dot{U} + F(U)\frac{\partial U}{\partial x} = 0,$$

with $U = (h_1, \ldots, h_N, u_1, \ldots, u_N)^T$

- For $N = 2$, eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfy
$$\lambda_1 \leq u_1 - \sqrt{gH} < \lambda_2 < u_2 + \sqrt{gH} \leq \lambda_3$$
- For $N > 2$, interlacing of the eigenvalues but only numerical proof (arrow matrices)
- Most finite volume schemes fail
- Velocity discretization (more complex)

$$u(x,z,t) = \sum_{\alpha=1}^{N} P_\alpha(z)\, u_\alpha(x,t), \quad \text{with} \quad deg(P_\alpha) = k$$

Kinetic interpretation (Lions, Perthame, Tadmor, Souganidis)

- Gibbs equilibrium $M(x,t,\xi) = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right)$ with $c = \sqrt{gH}$

where $\chi(\omega) = \chi(-\omega)$, supp$\,(\chi) \subset \Omega$, $\int_{\mathbb{R}} \chi(\omega) = \int_{\mathbb{R}} \omega^2 \chi(\omega) = 1$

Proposition (classical Saint-Venant)

The functions $(H, \bar{u}, E)(t,x)$ are strong solutions of the Saint-Venant system if and only if $M(t;x;\xi)$ is solution of the kinetic equation

$$(\mathcal{B}), \qquad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

where $Q(t,x,\xi)$ is a “collision term”.

- Proof : $\int_{\mathbb{R}}(1,\xi,|\xi|^2/2)(\mathcal{B})\; d\xi$
- See Audusse *et al.*

Kinetic interpretation for the multilayer system

- Gibbs equilibrium
  - $M_\alpha(x,t,\xi) = \frac{h_\alpha}{c_\alpha}\chi\left(\frac{\xi-u_\alpha}{c_\alpha}\right)$, with $c_\alpha = \sqrt{gf(\{h_j\}_{j\geq\alpha})}$
  - $N_{\alpha+1/2}(x,t,\xi) = G_{\alpha+1/2}\,\delta\left(\xi - u_{\alpha+1/2}\right)$,

Proposition (Audusse, Bristeau, Perthame, JSM 2009)

The functions $(h_\alpha, u_\alpha, E_\alpha)(t,x)$ are strong solutions of the multilayer Saint-Venant system if and only if the set $\{M_j(t;x;\xi)\}_{j=1}^N$ is solution of the kinetic equations

$$\frac{\partial M_\alpha}{\partial t} + \xi\frac{\partial M_\alpha}{\partial x} - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha(x,t,\xi)$$

- Pressure terms omitted
  - more complex kinetic interpretation
  - hydrostatic reconstruction (Audusse *et al.*)

2^{nd} order finite volume

Properties

- Positive scheme (CFL=1 but more complex)
- 2^{nd} order scheme (space)
- Well balanced (hyd. reconst., cf. Audusse *et al.* 2004, 2005)
- (Discrete entropy)
- Heun scheme (Runge-Kutta, 2^{nd} order in time)

$$H_j^{n+1} = H_j^n - \sigma_j^n \sum_{\alpha=1}^{N} \left(\mathcal{F}_{\alpha,j+1/2}^n - \mathcal{F}_{\alpha,j-1/2}^n \right),$$

and for each layer α

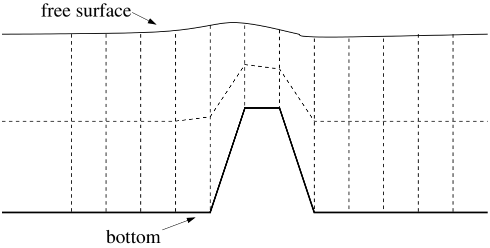
$$Q_{\alpha,j}^{n+1} = Q_{\alpha,j}^n - \sigma_j^n \Big(\mathcal{Q}_{\alpha,j+1/2}^n - \mathcal{Q}_{\alpha,j-1/2}^n + N_{\alpha+1/2}^{n+1/2} - N_{\alpha-1/2}^{n+1/2} \Big)$$

upwinding of $u_{\alpha+1/2}$ and $u_{\alpha-1/2}$

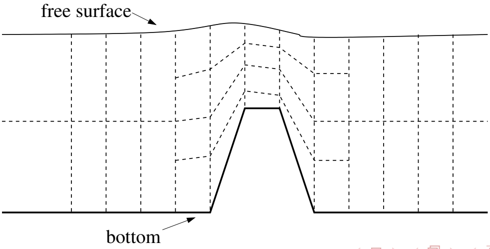
- Numerical cost $\mathcal{O}(N \times 1)$

Mesh considerations

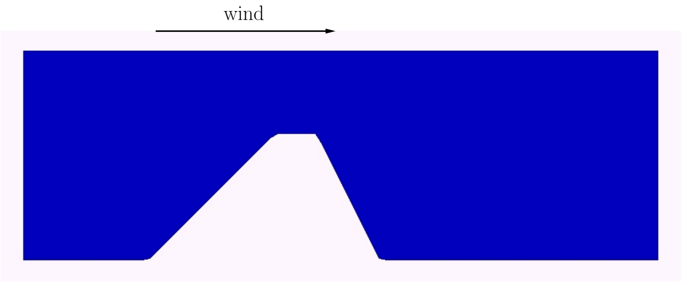
- This version

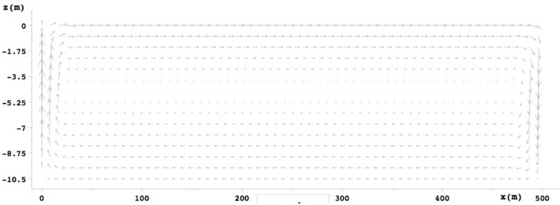


- Improvement



Wind effects (stratified velocities)



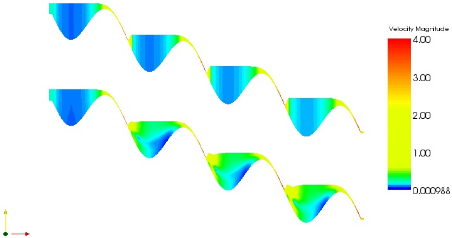


TELEMAC3D (3D hydrostatic Navier-Stokes)

Courtesy of A. Decoene

Simulations

- With 1 and 5 layers
- $Q_{in} = 2.749.10^{-2}\ m^2.s^{-1}$, $\mu = 0.001$, Strickler 40.0
- Rain 0.00687 $m.s^{-1}$



Transport of a passive pollutant

- 2D Euler (Navier-Stokes) with pollutant

$$\begin{cases} \operatorname{div} \underline{u} = 0 \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)(\underline{\mathbf{u}}) + \nabla p = G \\ \dot{T} + \operatorname{div}(T\underline{\mathbf{u}}) = \frac{\partial}{\partial z}\left(\mu_T \frac{\partial T}{\partial z}\right) \end{cases}$$

- Discretized by layer

$$\begin{cases} \frac{\partial}{\partial t}\sum_{\alpha=1}^{N} h_\alpha + \frac{\partial}{\partial x}\sum_{\alpha=1}^{N} h_\alpha u_\alpha = 0, \\ \frac{\partial (h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + g h_\alpha \left[\frac{h_\alpha}{2} + \sum_{j=\alpha+1}^{N} h_j\right]\right) = F_{\alpha+1/2} - F_{\alpha-1/2}, \\ \frac{\partial (h_\alpha T_\alpha)}{\partial t} + \frac{\partial}{\partial x}(h_\alpha u_\alpha T_\alpha) = T_{\alpha+1/2} G_{\alpha+1/2} - T_{\alpha-1/2} G_{\alpha-1/2}, \end{cases}$$

with $G_{\alpha+1/2} = \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2}\frac{\partial z_{\alpha+1/2}}{\partial x} - w_{\alpha+1/2}$

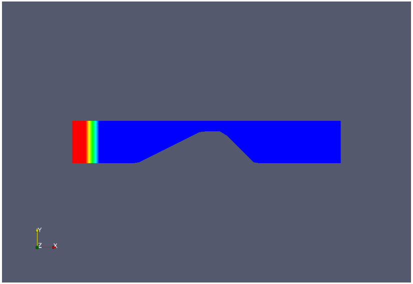
and $F_{\alpha+1/2} = u_{\alpha+1/2} G_{\alpha+1/2} + \left(\sum_{j=\alpha+1}^{N} g h_j\right)\frac{\partial z_{\alpha+1/2}}{\partial x}$,

Transport of a passive pollutant

- 2D Euler (Navier-Stokes) with pollutant

$$\begin{cases} \operatorname{div} \underline{u} = 0 \\ \underline{\dot{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)(\underline{\mathbf{u}}) + \nabla p = G \\ \dot{T} + \operatorname{div}(T\underline{\mathbf{u}}) = \frac{\partial}{\partial z}\left(\mu_T \frac{\partial T}{\partial z}\right) \end{cases}$$

- Also kinetic interpretation
- Same numerical scheme as previously



Pollutant in 3D

Multilayer with variable density

3D Euler/Navier-Stokes (hydrostatic) with variable density

$$\begin{cases} \dot{\rho} + \mathrm{div}\,(\rho\mathbf{u}) = 0, \\ \dot{\overline{\rho\mathbf{u}}} + (\mathbf{u}.\nabla)(\rho\mathbf{u}) + \nabla p = \rho G, \\ \dot{\overline{\rho T_j}} + \mathrm{div}\,(\rho T_j \mathbf{u}) = \frac{\partial}{\partial z}\left(\mu_T \frac{\partial T_j}{\partial z}\right), \quad j = 1, \dots, p \\ \rho = \rho(\{T_j\}_{j=1}^p) = \rho(H, \{T_j\}_{j=i}^p) \end{cases}$$

- Discretized by layer

$$\begin{cases} \frac{\partial}{\partial t} \sum_{\alpha=1}^N (\rho_\alpha h_\alpha) + \sum_{\alpha=1}^N \frac{\partial}{\partial x}(\rho_\alpha h_\alpha u_\alpha) = 0, \\ \frac{\partial(\rho_\alpha h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(\rho_\alpha h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha f(\{\rho_j h_j\}_{j\geq\alpha})\right) = \\ \qquad + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} + Sc.\ Terms, \\ \frac{\partial(\rho_\alpha h_\alpha T_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(\rho_\alpha h_\alpha u_\alpha T_\alpha\right) \\ \qquad + T_{\alpha+1/2} G_{\alpha+1/2} - T_{\alpha-1/2} G_{\alpha-1/2}, \end{cases}$$

Multilayer with variable density

- Gibbs equilibrium
  - $M_\alpha(x,t,\xi) = \frac{\rho_\alpha h_\alpha}{c_\alpha}\chi\left(\frac{\xi - u_\alpha}{c_\alpha}\right)$, with $c_\alpha = \sqrt{gf(\{\rho_j h_j\}_{j\geq\alpha})}$
  - $N_{\alpha+1/2}(x,t,\xi) = G_{\alpha+1/2}\,\delta\left(\xi - u_{\alpha+1/2}\right)$,

Proposition

The functions $(\rho_\alpha, h_\alpha, u_\alpha, T_\alpha, E_\alpha)(t,x)$ are strong solutions of the multilayer Saint-Venant system with variable density if and only if the set $\{M_j(t;x;\xi)\}_{j=1}^N$ is solution of the kinetic equations

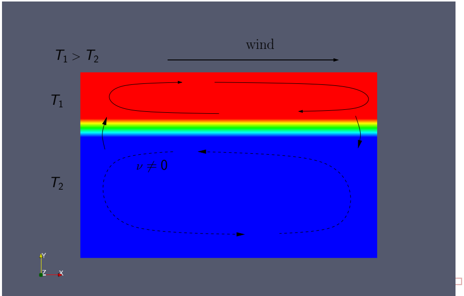
$$\frac{\partial M_\alpha}{\partial t} + \xi\frac{\partial M_\alpha}{\partial x} - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha(x,t,\xi)$$

- Numerical scheme
  - maximum principle for T, positivity of T, ...
  - no more explicit, more complex and costly

Multilayer with variable density

3D compressible Euler/Navier-Stokes (hydrostatic)

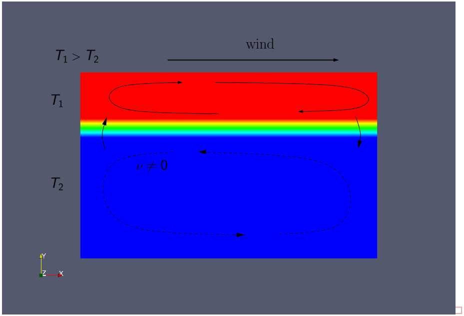
$$\begin{cases} \dot{\rho} + \mathrm{div}\,(\rho\underline{\mathbf{u}}) = 0, \\ \dot{\overline{\rho\underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho\underline{\mathbf{u}}) + \nabla p = \rho G, \\ \dot{\overline{\rho T_j}} + \mathrm{div}\,(\rho T_j\underline{\mathbf{u}}) = 0, \quad j = 1,\ldots,p \\ \rho = \rho(\{T_j\}_{j=1}^{p}) = \rho(H, \{T_j\}_{j=i}^{p}) \end{cases}$$



Multilayer with variable density

3D hydrostatic Euler/Navier-Stokes (constant density)

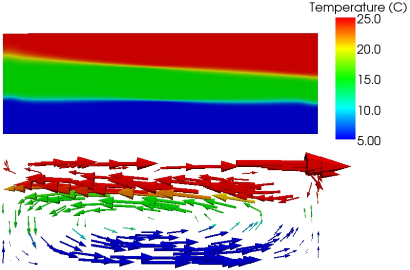
$$\left\{\begin{array}{l} \mathrm{div}\,(\underline{\mathbf{u}}) = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)(\underline{\mathbf{u}}) + \nabla p = G, \\ \dot{\overline{T_j}} + \mathrm{div}\,(T_j\underline{\mathbf{u}}) = 0, \quad j = 1, \dots, p \end{array}\right.$$



Multilayer with variable density

3D compressible hydrostatic Navier-Stokes

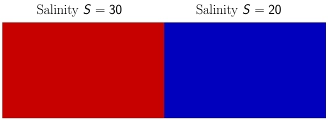
$$\left\{\begin{array}{l}
\dot{\rho} + \mathrm{div}\,(\rho\underline{\mathbf{u}}) = 0,\\
\dot{\overline{\rho\underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho\underline{\mathbf{u}}) + \nabla p = \rho G,\\
\dot{\overline{\rho T_j}} + \mathrm{div}\,(\rho T_j\underline{\mathbf{u}}) = 0, \quad j = 1, \ldots, p\\
\rho = \rho(\{T_j\}_{j=i}^{p}) = \rho({\color{red}H}, \{T_j\}_{j=i}^{p})
\end{array}\right.$$



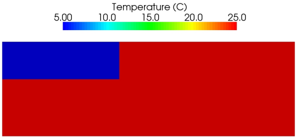
Archimedes' principle

- Limitations for hydrostatic systems

  - Re-ordering



  - Behaviour (hot/cold water)



$$\left\{\begin{array}{l} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ {\color{red}\frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho w^2}{\partial z}} + \frac{\partial p}{\partial z} = -\rho g \end{array}\right.$$

- Static equilibrium stable if $\frac{\partial \rho}{\partial z} \geq 0$

Gravity waves

Non-hydrostatic

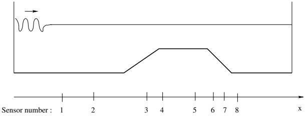
$$u(x,z,t) = \bar{u}(x,t) + \mathcal{O}(\varepsilon), \qquad \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -g$$

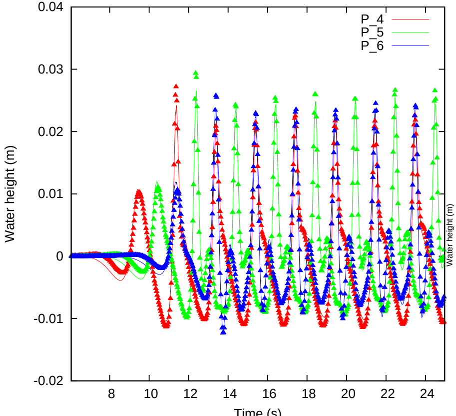
$$\left\{ \begin{array}{l} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) + \mathcal{D}(H)\frac{\partial^3 \bar{u}}{\partial^2 x \partial t} = -H\frac{\partial z_b}{\partial x} - \kappa(\bar{u}) \end{array} \right.$$

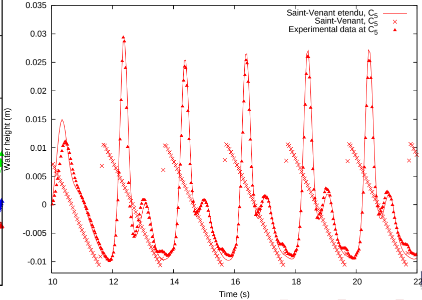
- Peregrine 1957, BBM 1972, Bristeau-JSM 2008
- Kinetic type interpretation & scheme
- Implementation in Mascaret (©EDF-MEEDDAT)
- Other approach : irrotational velocity field (Bona, Saut, Lannes)
  - often ill-posed problems

Experimental validation

- Dinguemans experiments
- Animation
- Comparisons (sensors 4, 5 and 6)







General outlook

- From Saint-Venant to Navier-Stokes
  - multilayer system (vertical dicretization)
  - variable density
  - kinetic interpretation, associated numerical scheme
- Mathematical works (compressible case)
- Convective terms (to be added)

$$\frac{\partial w}{\partial t} + \color{red}{\frac{\partial uw}{\partial x}} \color{red}{+} \color{red}{\frac{\partial w^2}{\partial z}} + \frac{\partial p}{\partial z} = -g$$

  - Boltzmann type equations but with reaction terms
- Exprerimental validations needed
- Biology/hydrodynamics coupling
- Inverse problems (data assimilation) using kinetic type interpretations

Comparison (flat bottom)

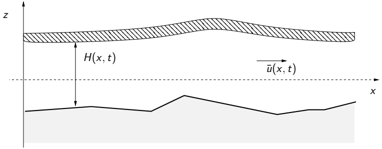
- Audusse formulation ($h_1 + h_2 = H$)

$$\left\{\begin{array}{l} \frac{\partial h_1}{\partial t} + \frac{\partial (h_1 u_1)}{\partial x} = 0, \quad \frac{\partial h_2}{\partial t} + \frac{\partial (h_2 u_2)}{\partial x} = 0, \\ \frac{\partial (h_1 u_1)}{\partial t} + \frac{\partial}{\partial x}\left(h_1 u_1^2 + g h_1 (h_1 + h_2)\right) = \frac{g(h_1+h_2)^2}{2} \frac{\partial}{\partial x} \frac{h_1}{h_1+h_2}, \\ \frac{\partial (h_2 u_2)}{\partial t} + \frac{\partial}{\partial x}\left(h_2 u_2^2 + g h_2 (h_1 + h_2)\right) = \frac{g(h_1+h_2)^2}{2} \frac{\partial}{\partial x} \frac{h_2}{h_1+h_2}, \end{array}\right.$$

- New formulation ($h_1 = \alpha H,\ h_2 = (1-\alpha)H$ and $\alpha \in (0,1)$)

$$\left\{\begin{array}{l} \frac{\partial H}{\partial t} + \frac{\partial (h_1 u_1)}{\partial x} + \frac{\partial (h_2 u_2)}{\partial x} = 0, \\ \frac{\partial (h_1 u_1)}{\partial t} + \frac{\partial}{\partial x}\left(h_1 u_1^2 + \frac{g}{2}(2h_2^2 + h_1^2)\right) = u_{3/2}\left(\frac{\partial h_1}{\partial x} + \frac{\partial (h_1 u_1)}{\partial x}\right), \\ \frac{\partial (h_2 u_2)}{\partial t} + \frac{\partial}{\partial x}\left(h_2 u_2^2 + \frac{g}{2} h_2^2\right) = -u_{3/2}\left(\frac{\partial h_1}{\partial x} + \frac{\partial (h_1 u_1)}{\partial x}\right), \end{array}\right.$$

Flows without free surface



- Flow with deformable tip

$$\frac{\partial H}{\partial t} + \frac{\partial H\bar{u}}{\partial x} = 0$$
$$\frac{\partial H\bar{u}}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2 + Hp_S\right) = -gH\frac{\partial z_b}{\partial x} - p_S\frac{\partial H}{\partial x}$$

- Kinetic interpretation
  - Gibbs equilibrium

$$M(x,t,\xi) = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right) \quad \text{with} \quad c = \sqrt{gH + p_S}$$

  - Solution of

$$(\mathcal{B}), \qquad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

Source terms - two writtings

Incompressible case i.e. $p = g(\eta - z)$

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial p}{\partial x} dx = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} g\frac{\partial \eta}{\partial x} dx = \frac{\partial}{\partial x}\left(\frac{g}{2} h_\alpha H\right) + gh_\alpha \frac{\partial z_b}{\partial x}$$

Compressible case i.e. $p = \int_z^\eta \rho g\ dz$

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial p}{\partial x} dx = \frac{\partial}{\partial x}\left(\int_z^\eta \rho g\ dz\right) - \frac{\partial z_{\alpha+1/2}}{\partial x} p_{\alpha+1/2} + \frac{\partial z_{\alpha-1/2}}{\partial x} p_{\alpha-1/2}$$

- Extension of Audusse, Bouchut, Klein, Perthame 2005
- Well balanced, 2^{nd} order in space ... rather tricky