Corpora and Fluids

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Outline:

1. Equilibrium
2. Kinetics
3. Non-Equilibrium Dynamics
4. Stochastics

Complex Fluids Models

- Landau Equilibrium models: order parameter (Director = Oseen, Zöcher, Frank, Ericksen, Leslie. Tensor = de Gennes.)

- Onsager Equilibrium models: (pdf of state), free energy derived from physics

- Passive Kinetic models: Doi, FENE and variants (pdf of state) effects of shear on dilute suspensions of rigid or extensible corpora => linear Fokker-Planck

- Tensorial models: (conformation tensors): closure of certain kinetic models, e.g. Oldroyd B

- Active Kinetic Models: (pdf) Onsager-Smoluchowski: Nonlinear Fokker-Planck, stochastic models

Applications

- Nanoscale self-assembly
- Microfluidics
- Biomaterials
- Gels and Foams
- Soft Lattices, Jamming

Major Problems

- Derivation of Micro-Macro Effect
- Dissipation of Energy: Complex Fluids “Onsager” conjecture
- Transitions: from isotropic to order (nematic, smectic)
- Modeling of interactions in the correct moduli space.
- Existence Theory

Nonlinear Stochastic System (in the sense of McKean): drift deterministic, but computed via functionals of the SDE driven by it.

Equilibrium

- M configuration space of corpora = metric space.
- $d\mu(p)$: “volume element” = Borel probability.
- $f(p)d\mu(p)$ probability density of corpora p.
- $K(p,q)$ interaction kernel: real symmetric, bounded below, Lipschitz.

- Mean field interaction potential $U = -\mathcal{K}f$

$$(\mathcal{K}f)(p) = \int_M K(p,q)f(q)d\mu(q)$$

- Free energy:

$$\mathcal{E} = \int_M (f \log f - \frac{1}{2} f\mathcal{K}f)d\mu$$

$$\boxed{\frac{\delta \mathcal{E}}{\delta f} = 0}$$

Onsager equation

$$\log f(m) = \int_M K(m,p) f(p) d\mu(p) - \log Z$$

Z a normalizing constant.

$$\boxed{f = Z^{-1} e^{\mathcal{K} f}}$$

- Nonlinear, nonlocal.

Example: Maier-Saupe potential

$M = \mathbb{S}^n, \quad d\mu =$ area.

$$\boxed{\mathcal{K}f(p) = b \int\limits_{\mathbb{S}^n} \left((p \cdot q)^2 - \frac{1}{n} \right) f(q) d\mu}$$

- b: inverse temperature, or concentration. $b \to \infty$: transition to nematics.

Dimension Reduction, Maier-Saupe

For any real, $n \times n$ symmetric, traceless matrix S and positive b:

$$S \mapsto Z(S)$$

$$Z(S) = \int_{\mathbb{S}^{n-1}} e^{b(S^{ij} m_i m_j)} d\mu.$$

$$\psi_S(m) = (Z(S))^{-1} e^{b(S^{ij} m_i m_j)}$$

$$\sigma(S)_{ij} = \int_{\mathbb{S}^{n-1}} \left(m_i m_j - \frac{\delta_{ij}}{n} \right) \psi_S(m) d\mu.$$

Theorem 1 *Onsager's equation with Maier-Saupe potential is equivalent to*

$$\sigma(S) = S.$$

• $O(n)$ Rotation invariance.

Theorem 2 *Let $n = 2$. Maier-Saupe potential. Let $N(b)$ denote the number of distinct steady solutions modulo the $O(2)$ conjugacy. Then, if $b \leq 4$ then $N(b) = 1$. If $b > 4$ then $N(b) = 2$. The non-trivial steady state converges, as $b \to \infty$, to a delta function concentrated on the unit circle.*

Onsager Equation, Maier-Saupe $n = 3$.

$$S^{ij} = \lambda_i \delta_{ij}$$

$\lambda_i \in [-\frac{1}{3}, \frac{2}{3}]$,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Let

$$v_1 = \frac{1}{2}(\lambda_1 + \lambda_2), \quad v_2 = \frac{1}{2}(\lambda_1 - \lambda_2).$$

$$\begin{cases} y_1(p) = 1 - 3p^2 \\ y_2(p,t) = (1 - p^2)\cos t \end{cases}$$

for $(p, t) \in K = [-1, 1] \times [0, 2\pi]$.

$$y = y(p,t) = (y_1(p), y_2(p,t)), \;\; v = (v_1, v_2).$$

Theorem 3 *Let*

$$Z_2(v) = \int\limits_K e^{bv\cdot y(p,t)} dpdt$$

$$\mathcal{F}(v) = \log(Z_2(v)) - b\left(3v_1^2 + v_2^2\right).$$

Onsager's equation: critical points of $\mathcal{F}$, $v \in [-\frac{1}{3}, \frac{2}{3}] \times [0, \frac{1}{2}]$, i.e.:

$$\begin{cases} 6v_1 = [y_1](v) \\ 2v_2 = [y_2](v) \end{cases}$$

where, for any $\phi : K \to \mathbb{R}$,

$$[\phi](v) = (Z_2(v))^{-1} \int_K \phi(p,t) e^{bv\cdot y(p,t)} dpdt$$

•If $0 < b < 1/2$ the function $\mathcal{F}$ is strictly concave and has a unique critical point at $v = 0$. The corresponding unique steady state is the uniform distribution.

•If $b \geq 8$ then $v = 0$ is an isolated critical point. Consequently, no bifurcations from the uniform distribution occur for $b \geq 8$.

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m)\psi_{S,b}(m)dm.$$

•**Isotropic:** $\lambda_1 = \lambda_2 = \lambda_3 = 0.$

$$\lim_{b\to\infty}[\phi] = \frac{1}{4\pi}\int_{\mathbb{S}^2} \phi(p)dp$$

•**Oblate:** $\lambda_1 = \frac{1}{6},\ \lambda_2 = \frac{1}{6},\ \lambda_3 = -\frac{1}{3}.$

$$\lim_{b\to\infty}[\phi] = \frac{1}{2\pi}\int_0^{2\pi} \phi(\cos\varphi, \sin\varphi, 0)d\varphi$$

•**Prolate:**$\lambda_1 = \frac{2}{3}$, $\lambda_2 = -\frac{1}{3}$, $\lambda_3 = -\frac{1}{3}$.

$$\boxed{\lim_{b \to \infty} [\phi] = \phi(e_1).}$$

•axisymmetry, two lambdas are equal. Finitely many solutions at finite b. (Fatkulin-Slastikov, Luo-Zhang-Zhang, Zhou-Wang-Forest-Wang, Liu-Zhang-Zhang).

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{K}(p_1, q_1, p_2, q_2, \dots) = K_1(p_1, q_1) + \dots K_N(p_N, q_N)$$

$$\widetilde{\mathcal{K}} f = \sum_{j=1}^{N} \mathcal{K}_j f, \quad \text{with } \mathcal{K}_j f(p_j) = \int_{\widetilde{M}} K_j(p_j, q_j) f(q_1, \dots q_N) d\mu$$

Onsager Equation $\widetilde{f} = \widetilde{Z}^{-1} e^{\widetilde{K} \widetilde{f}}$

$$\widetilde{Z} = \Pi_{j=1}^{N} Z_j, \quad \text{with } Z_j = \int_{M_j} e^{\mathcal{K}_j f_j} d\mu_j, \quad f_j = (Z_j)^{-1} e^{\mathcal{K}_j f_j}$$

$$\widetilde{f}(p_1, \dots p_N) = f_1(p_1) f(p_2) \dots f_N(p_N) \quad \text{product measure}$$

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$\mathcal{K}f(p_1, p_2) = -b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$$\|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$$

Dimension reduction: Onsager's equation $f = Z^{-1} e^{\mathcal{K}f}$ reduces to

$$z = [\sin \theta](z)$$

with

$$\begin{cases} [\phi](z, \gamma) = \int_0^{2\pi} \phi(\theta) g(\theta) d\theta \\ g(\theta) = Z^{-1} e^{-b \sin^2(\theta) + 2bz \sin \theta} \\ Z = \int_0^{2\pi} e^{-b \sin^2(\theta) + 2bz \sin \theta} d\theta \end{cases}$$

The solution is $f(\theta_1, \theta_2) = g(\theta_1 - \theta_2)$. Let

$$u(\theta, z) = \sin\theta - z,$$

and let

$$[u](b, z) = \frac{\int_0^{2\pi} u(\theta, z) e^{-bu^2(\theta, z)} d\theta}{\int_0^{2\pi} e^{-bu^2(\theta, z)} d\theta}.$$

The Onsager equation is equivalent to

$$[u](b, z) = 0.$$

This determines z, which in turn determines g, f.

$z = 0$ always a solution. It yields

$$f_0(p_1, p_2) = Z^{-1} e^{-b\sin^2(p_1 - p_2)}.$$

As $b \to \infty$ this tends to $\delta((p_1 - p_2)\text{mod}\pi)$.

Consider

$$\lambda(z,\tau) = b^{\frac{1}{2}} \int_0^{2\pi} e^{-b(\sin\theta - z)^2} d\theta$$

with $\tau = b^{-1}$. Note

$$[u] = \frac{1}{2b} \frac{\partial_z \lambda}{\lambda}$$

and

$$\partial_\tau \lambda = \frac{1}{4} \partial_z^2 \lambda$$

$$\lim_{\tau \to 0} \lambda(z,\tau) = 2\sqrt{\pi} \frac{1}{\sqrt{1-z^2}}$$

Increasing. But things are subtle, clearly $\frac{\partial \lambda}{\partial z}(1,\tau) < 0$! In fact, phase transition at positive τ

$$\partial_z \lambda(z_b, \tau) = 0$$

and limit $\lim_{\tau\to 0} z_b = 1$, and consequently

$$\lim_{b\to\infty} f(p_1 - p_2) = \delta\left(\left(p_1 - p_2 - \frac{\pi}{2}\right) \mathrm{mod}\pi\right)$$

Packing, Jamming. $V(r)$ nonnegative, nonincreasing, compactly supported. $p = (x_1, \ldots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$. Packing energy:

$$F(p) = \sum_{i<j} V(|x_i - x_j|).$$

$\widetilde{M} = \Omega \times \cdots \times \Omega \cap \{F \le F_0\}$.

$$(\mathcal{K}f)(p) = -\int_{\widetilde{M}} |F(p) - F(q)|^2 f(q) dq$$

General Onsager Equation

• Conjecture: General configuration space M, generic potential. The zero temperature limit is concentrated on a single corpus

Partition function

$$Z(f,b) = \int_M e^{b\mathcal{K}f} d\mu$$

Define, for $\phi : M \to \mathbb{R}$,

$$[\phi](f,b) = (Z(f,b))^{-1} \int_M \phi(m) e^{b\mathcal{K}f} d\mu.$$

$$K(m,p) = \sum_{j=1}^{\infty} \mu_j \phi_j(m)\phi_j(p)$$

ϕ_j real, complete, orthonormal in $L^2(M)$,

$$\mathcal{K}\phi_j = \mu_j \phi_j$$

Expand f:

$$v_j(f) = \int_M f(p)\phi_j(p) d\mu.$$

Onsager's equation

$$f = Z^{-1} e^{b\mathcal{K}f}$$

is equivalent to the system

$$v_j(f) = [\phi_j](f,b).$$

Onsager solution is a critical point of the free energy

$$\mathcal{F}(v,b) = \log Z(v,b) - b\sum_{j=1}^{\infty}\mu_j \frac{v_j^2}{2}$$

Differentiation: For any function $\phi(p)$

$$\frac{\partial[\phi]}{\partial v_i} = b\mu_i \left\{[\phi\phi_i] - [\phi][\phi_i]\right\}$$

Therefore the Hessian $\frac{\partial^2 \mathcal{F}}{\partial v_i \partial v_j}$ is

$$\mathcal{H}_{ij} = b^2\mu_i\mu_j[\xi_i\xi_j] - b\mu_i\delta_{ij}$$

with $\xi_j = \phi_j - [\phi_j]$. For b small the isotropic state $v = 0$ is stable.

$$\lim_{b\to\infty}[\phi](v,b) = \phi(p(v))$$

Onsager equation on metric spaces

M compact metric space, d distance, μ Borel probability measure on M, uniform in the sense that there exist $0 < k < 1$, $c > 0$

$$(A) \qquad \mu(B(p,r)) \geq c e^{-r^{-k}}$$

for all $p \in M$, and all r sufficiently small. (e.g.: Riemannian).

$$\boxed{U(p) = -\left(\mathcal{K} f\right)(p) = -\int_M u(d(p,q)) f(q) d\mu(q)}$$

Assume

$$(B) \qquad \begin{cases} 0 \leq u(d) \\ |u(d) - u(t)| \leq L|d-t| \end{cases}$$

Theorem 4 *Let M be a compact metric space with distance d. Let μ be a Borel probability measure on M that satisfies (A). Let u satisfy (B). Then:*

(I) For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f>0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

$$g(x) = (Z(b))^{-1} e^{-bU(x)}$$

with

$$Z(b) = \int_M e^{-bU(x)} d\mu(x)$$

and

$$U(x) = \int_M u(d(x,y))g(y)d\mu(y)$$

The function g is normalized $\int g d\mu = 1$, strictly positive and Lipschitz continuous.

(II) Let $b_n \to \infty$ and let $d\nu_n = g_n d\mu$ be a sequence of solutions of Onsager equations corresponding to b_n. By passing to a subsequence we may assume that the sequence converges weakly to a probability measure $\nu = \lim_n \nu_n$. There exists a non-negative Lipschitz continuous function $U_\infty(x)$ on M such that ν is concentrated on the set

$$\Sigma = \{x \in M \mid U_\infty(x) = \min_{y \in M} U_\infty(y)\}$$

Thus, for any ϕ continuous, supported in the open set $M \setminus \Sigma$,

$$\lim_{n \to \infty} \int_M \phi(x) g_n(x) d\mu = 0$$

Moduli spaces of corpora: n-gons, model interactions, e.g. Gromov-Hausdorff distance.

Kinetics

M compact connected Riemannian manifold with metric g.

$$\partial_t f = \mathrm{div}_g \left(f \nabla_g \left(\frac{\delta \mathcal{E}}{\delta f} \right) \right)$$

$$\frac{\delta \mathcal{E}}{\delta f} = \log f - \mathcal{K} f$$

$$\frac{d\mathcal{E}}{dt} = - \int_M f \, |\nabla_g (\log f - \mathcal{K} f)|^2 \, dp$$

Gradient system, steady solutions = Onsager equation.

$$\partial_t f = \Delta_g f - \mathrm{div}_g (f \nabla_g (\mathcal{K} f))$$

Lyapunov functional:

$$\frac{d}{dt}\mathcal{E} = -\int_M f \, |\nabla_g(\log f - b\mathcal{K}f)|^2 \, dp$$

Example $n = 2$, Maier-Saupe, Fourier representation.

$$\frac{dv_j}{dt} = -4j^2 v_j + bjv_1(v_{j-1} - v_{j+1})$$

The potential is determining. $n = 2, 3$: Inertial Manifolds (Vukadinovic).

Transport: Smoluchowski (Nonlinear Fokker-Planck) Equation

$$\boxed{\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Gf) = \frac{1}{\tau}\Delta_g f}$$

$$G = \frac{1}{\tau}\nabla_g \mathcal{K} f + W,$$

The $(0,1)$ tensor field W is:

$$W(x,m,t) =$$
$$= \left(\sum_{i,j=1}^{3} c_\alpha^{ij}(m)\frac{\partial u_i}{\partial x_j}(x,t)\right)_{\alpha=1,\ldots,d.}$$

Example, rod-like particles:

$$W(x,m,t) = (\nabla_x u(x,t))m - ((\nabla_x u(x,t))m \cdot m)m.$$

Macro-Micro Effect: from first principles, in principle...

Dynamics: Navier Stokes Equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma$$
$$\nabla \cdot u = 0$$

The tensor $\sigma_{ij}(x, t)$: added stress.

Sufficient for regularity, if σ smooth

$$\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty$$

Amplification factor of tracers

$$\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty$$

2D, Bounded stress

Theorem 5 *Let* $\sigma \in L^\infty(dtdx)$*. Let* $u_0 \in L^2(dx)$*. There exists a unique weak solution of the forced 2D NS eqns, with*

$$\boxed{u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))}$$

Moreover,

$$\boxed{\int_0^T \|\nabla u\|_{L^q(dx)}^{\frac{q}{q-1}} dt < \infty, \quad \forall\, q \geq 2}$$

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^{p} dt < \infty, \quad \forall\, p < 2.}$$

Open questions

$$\boxed{\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad \text{No ?}}$$

$$\boxed{\int_0^T \|u\|^2_{L^\infty(dx)} dt < \infty \quad \text{No ?}}$$

$$\boxed{\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty \quad \text{No ?}}$$

Partial regularity?

Navier-Stokes with nearly singular forces

$$\partial_t u + u \cdot \nabla_x u - \nu \Delta_x u + \nabla_x p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

Theorem 6 *Let u be a solution of the $2D$ Navier-Stokes system with divergence-free initial data $u_0 \in W^{1,2}(\mathbb{R}^2) \cap W^{1,r}(\mathbb{R}^2)$. Let $T > 0$ and let the forces $\nabla \cdot \sigma$ obey*

$$\sigma \in L^1(0,T; L^\infty(\mathbb{R}^2)) \cap L^2(0,T; L^2(\mathbb{R}^2))$$
$$\nabla \cdot \sigma \in L^1(0,T; L^r(\mathbb{R}^2)) \cap L^2(0,T; L^2(\mathbb{R}^2))$$

with $r > 2$.

$$\|\sigma\|_{L^\infty} \sim K, \quad \|\nabla \cdot \sigma\|_{L^r} \sim B$$

Then

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt \leq K \log_*(B)$$

and also

$$\frac{1}{M}\sum_{q=1}^{M}\int_0^T \|\Delta_q \nabla u(t)\|_{L^\infty} dt \leq K$$

with K *depending on* T*, norms of* σ *and the initial velocity, but not on gradients of* σ *nor* M*, and* B *depending on norms of the spatial gradients of* σ*.*

$$\boxed{u = \sum_{q=-1}^{\infty} \Delta_q(u)}$$

Micro-Macro Effect

$$\boxed{\sigma_{ij}(x) = -\epsilon \int_M \left(\text{div}_g c^{ij} + c^{ij} \cdot \nabla_g \mathcal{K} f(x,m)\right) f(x,m) dm \quad *}$$

Micro-Macro Effect: derived from Energetics

- $f = Z^{-1} e^{\mathcal{K} f} \Rightarrow \sigma = 0$

- $\mathcal{K} = 0,\ W = (\nabla_x u)m - m((\nabla_x u)m \cdot m) \Rightarrow \sigma = \epsilon \int (3n \otimes n - \mathbf{1}) dm$

Theorem 7 *For the coupled 3DNS + Nonlinear Fokker-Planck system, with macro-micro effect given in *,*

$$E(t) = \frac{1}{2}\int |u|^2 dx +$$
$$+\epsilon \int \left\{ f \log f - \frac{1}{2}(\mathcal{K}f) f \right\} dx dm.$$

is nondecreasing on solutions: If (u, f) is a smooth solution then

$$\frac{dE}{dt} = -\nu \int |\nabla_x u|^2 dx -$$
$$-\frac{\epsilon}{\tau} \int \int_M f \left| \nabla_g \left(\log f - \mathcal{K}f \right) \right|^2 dm dx.$$

If the smooth solution is time independent, then $u = 0$ and f solves the Onsager equation

$$f = Z^{-1} e^{\mathcal{K}f}.$$

Time dependent Stokes and Nonlinear Fokker-Planck in 3D

$$\boxed{\begin{array}{c}\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) + \frac{1}{\tau}\mathrm{div}_g(f\nabla_g(\mathcal{K}f)) = \epsilon\Delta_g f\\ \partial_t u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0.\end{array}}$$

Theorem 8 *Assume u_0 is divergence-free and belongs to $W^{2,r}(\mathbb{T}^3)$, $r > 3$, assume that f_0 is positive, normalized, and $f_0 \in L^\infty(dx;\mathcal{C}(M)) \cap \nabla_x f_0 \in L^r(dx; H^{-s}(M))$, $s \le \frac{d}{2} + 1$. Then the solution exists for all time and*

$$\|u\|_{L^p[(0,T);W^{2,r}(dx)]} < \infty,$$
$$\|\nabla_x f\|_{L^\infty[(0,T);L^r(dx;H^{-s}(M))]} < \infty$$

for any $p > \frac{2r}{r-3}$, $T > 0$ $\tau \le \infty$, $\epsilon \ge 0$.

Global existence, NSE and Nonlinear Fokker-Planck 2D

Theorem 9 *(C-Masmoudi) Let* $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$ *be divergence-free, and* $f_0 \in W^{1,r}(H^{-s}(M))$, *with* $r > 2$, $\alpha > 1$, $s \le \frac{d}{2}+1$ *and* $f_0 \ge 0$, $\int_M f_0 dm \in (L^1 \cap L^\infty)(\mathbb{R}^2)$. *Then the coupled NS and nonlinear Fokker-Planck system in 2D has a global solution* $u \in L^\infty_{loc}(W^{1,r}) \cap L^2_{loc}(W^{2,r})$ *and* $f \in L^\infty_{loc}(W^{1,r}(H^{-s}))$. *Moreover, for* $T > T_0 > 0$, *we have* $u \in L^\infty((T_0,T); W^{2-0,r})$.

No a priori bound.

$$\boxed{\sup_k \lambda_k^{\alpha-\frac{1}{k}\int_0^t \|\nabla_x S_{k-1}(u(s))\|_{L^\infty} ds} \|\Delta_k(u)(t)\|_{L^p}}$$

C-Fefferman-Titi-Zarnescu: a priori bounds if f is driven by a time average of u

Stochastic Lagrangian Representation: Navier-Stokes

Theorem 10 *(Iyer) Let W be an n-dimensional Wiener process. Let $k \geq 1$ and assume $u_0 \in C^{k+1,\alpha}$ is a deterministic divergence-free vector field. Let (u, X) solve the stochastic system*

$$\begin{cases} dX = u\,dt + \sqrt{2\nu}\,dW, \\ A = X^{-1}, \\ u = \mathbb{E}\mathbb{P}\left\{\left(\nabla^T A\right)(u_0 \circ A)\right\} \end{cases}$$

Then u solves the deterministic incompressible NSE:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0,$$

$$\nabla \cdot u = 0$$

•When $\nu = 0$, all is deterministic, and we recover the Eulerian-Lagrangian deterministic representation based on the Weber formula.

Remarks

•$A = X^{-1}$ is the spatial inverse (“back-to-labels”). It exists, and it is as smooth as X. Both are stochastic.

•**Forced NSE**

$$\begin{cases} dX = udt + \sqrt{2\nu}dW, \\ A = X^{-1} \\ u = \mathbb{EP}\left\{(\nabla^T A)\left[u_0 + \int_0^t (\nabla^t X) f(X_s, s) ds\right] \circ A(t)\right\} \end{cases}$$

represents

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0.$$

•Representations for Lans-alpha, Burgers. No direct representation for Leray regularization.

Local Existence for the Stochastic System, Remarkable Formulae

Theorem 11 *Let* $u_0 \in C^{k+1,\alpha}$ *be divergence-free. There exists a* $T > 0$ *depending on the norm of* u_0*, but independent of viscosity, so that a solution* (u, X) *of the stochastic system exists on* $[0, T]$*. Moreover,* $\|u\|_{C^{k+1,\alpha}} \leq U$ *for* $t \in [0, T]$ *with* U *dependent on the norm of the initial data and* T*.*

Theorem 12 *Let* $\omega = \nabla \times u$*,* $\omega_0 = \nabla \times u_0$*. Then*

$$\omega = \mathbb{E}\left\{((\nabla X)\omega_0) \circ A\right\}.$$

In two dimensions,

$$\omega = \mathbb{E}\left[\omega_0 \circ A\right].$$

For forced systems in $n = 2, 3$, replace in the formulae above ω_0 by

$$\xi_t = \omega_0 + \int_0^t (\nabla X_s)^{-1} g(X_s, s) ds$$

with $g = \nabla \times f$.

●**Circulation is conserved.**

Let

$$\widetilde{u} = \mathbb{P}\left\{(\nabla^t A)(u_0 \circ A)\right\}$$

This is a stochastic incompressible velocity, with initial data u_0 and

$$u = \mathbb{E}\widetilde{u}$$

$$\oint_{X(\gamma)} \widetilde{u} \cdot dr = \oint_{\gamma} u_0 \cdot dr.$$

Stochastic Lagrangian Transport

- The "back-to-labels" process obeys

$$\boxed{dA_t + [u \cdot \nabla A - \nu \Delta A]\, dt + \sqrt{2\nu}\nabla A dW = 0}$$

For any smooth function $\phi(a,t)$, $v(x,t) = \phi(A(x,t),t)$ obeys

$$\boxed{dv_t + [u \cdot \nabla v - \nu \Delta v]\, dt + \sqrt{2\nu}\nabla v dW = \partial_t \phi \circ A}$$

- Cancellation, chain rule as if it were a first order PDE, due to the joint quadratic variation.

- Valid if u is smooth, not necessarily divergence-free.

Stochastically Passive Scalars

$$d\theta_t + [u \cdot \nabla\theta - \nu\Delta\theta]\, dt + \sqrt{2\nu}\nabla\theta dW = 0$$

- $\theta_1,\ \theta_2,$ sps $\Rightarrow \theta_1\theta_2$sps
- with viscosity, inviscid invariants become stochastically passive

Stochastic Representation for Linear Fokker-Planck coupled with Navier-Stokes

Let

$$m = M(a, \alpha, t)$$

solve

$$dM = (u(X,t) + G(X,M,t))dt + \sqrt{2\kappa}dW$$

with

$$M(a, \alpha, 0) = \alpha.$$

Let

$$(A(x,t), R(x,m,t)) = (X(a,t), M(a,\alpha,t))^{-1}$$

It exists and a.s. for all t

$$A(X(a,t),t) = a, \quad R(X(a,t), M(a,\alpha,t)) = \alpha.$$

Then

$$\boxed{f(x,m,t) = f_0(A(x,t), R(x,m,t))\,\mathrm{det}\left(\nabla_m R\right)(x,m,t)}$$

solves

$$\boxed{\begin{array}{c} df + (u\cdot\nabla_x f + \mathrm{div}_g(Gf) - \kappa\Delta_g f - \nu\Delta_x f)dt = \\ -\sqrt{2\kappa}\nabla_g f \cdot dW - \sqrt{2\nu}\nabla_x f \cdot dW = 0 \end{array}}$$

and so

$$\overline{f} = \mathbb{E} f$$

solves

$$\boxed{\partial_t \overline{f} + u \cdot \nabla_x \overline{f} + \mathrm{div}_g(G\overline{f}) = \kappa \Delta_g \overline{f} + \nu \Delta_x \overline{f}.}$$

• $\nu \geq 0$, $\kappa \geq 0$.

• Modifications needed for manifolds.

Nonlinear Fokker-Planck and hybrid stochastic-deterministic (not closed) models: open.

Future work

●Traveling and standing waves in physical space, connecting solutions of Onsager's equation.

●Onsager equation on moduli spaces of n-gons.

●Hybrid stochastic-deterministic models for interacting corpora coupled to fluids and their relationship to deterministic models.

●Invariant measures for hybrid stochastic-deterministic models of interacting corpora.

●Partial regularity theory for NLFP-NS systems.