Collège de France Seminar – April 19, 2013

A fluid-structure model coupling the Navier-Stokes equations

and the Lamé system

M. Vanninathan, TIFR–CAM – Bangalore

joint work with Jean-Pierre Raymond, IMT – Toulouse

Plan of the talk

1. Brief review of known results

2. The model and the method of proof

- The fluid model in Eulerian variables
- The Lamé system in Lagrangian variables
- The flow associated with the velocity field
- The interface conditions
- The full nonlinear system

3. Analysis of the linearized coupled model

- Analysis of the interface condition
- Analysis of the Lamé system
- Analysis of the Stokes system
- The linearized coupled system

4. The Lispchitz estimates

5. The method of successive approximations

1. Known results

- N.S.E. + Rigid body (Takahashi '03, San Martin, Starovoitov, Tucsnak '02, Cumsille, Takahashi '09, ...)

- N.S.E. + Deformable body described by a system of O.D.E. (San Martin, Scheid, Takahashi, Tucsnak '08, Court '10)

- N.S.E. + Elasticity system with a damping (Boulakia '07)

- Elastic structure modeled by the Lamé system (Coutand, Shkoller '05, Kukavica, Tuffaha, '12)

- A damped beam or plate located at the fluid boundary (Beirao da Veiga '04, Chambolle, Desjardins, Esteban, Grandmont '05, R. '09, Lequeurre '10)

2. The model and the method of proof

The fluid model in Eulerian variables

$$\frac{\partial u}{\partial t} + (u \cdot \nabla_x)u - \mathrm{div}_x\, \sigma(u,p) = 0, \quad \mathrm{div}_x\, u = 0 \quad \text{in } \Omega_F(t), \text{ for } t > 0,$$

$$u(0) = u_0 \text{ in } \Omega_F = \Omega_F(0),$$

at the F-S interface $u(x,t) =$ velocity of the solid displacement,

where

$$\sigma(u,p) = \nu(\nabla_x u + (\nabla_x u)^T) - pI.$$

The Lamé system in Lagrangian variables

$$\frac{\partial^2 w}{\partial t^2} - \text{div}_y \sigma(w) = 0 \;\; \text{in } Q_S^T = \Omega_S \times (0, T),$$

$$w(\cdot, 0) = I \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \;\; \text{in } \Omega_S,$$

$$\sigma(w)n = \text{force exerted by the fluid},$$

where

$$\sigma(w) = \lambda \, \text{trace}\, \varepsilon(w) I + 2\mu \, \varepsilon(w) \quad \text{with} \quad \varepsilon(w) = \frac{1}{2}(\nabla_y w + (\nabla_y w)^T),$$

$\mu > 0$ and $\lambda + \mu > 0$.

The flow associated with the velocity field

The mapping $X(\cdot, t)$ from $\Omega_F(0) = \Omega_F$ to $\Omega_F(t)$ satisfies the differential equation

$$\frac{\partial X}{\partial t}(y,t) = u(X(y,t),t), \quad X(y,0) = y \quad \text{for all } y \in \Omega_F.$$

The equality of velocity is expressed by

$$u(X(y,t),t) = \frac{\partial w}{\partial t}(y,t) \quad \text{on } \Sigma_S^T.$$

The equality of forces reads as

$$\sigma(w)\tilde{n} = (\sigma(u,p)\circ X)\ \mathrm{cof}(\nabla_y X)\tilde{n} = (\sigma(u,p)\circ X)\ n \ \text{ on } \Sigma_S^T,$$

where $n = \mathrm{cof}(\nabla_y X)\tilde{n}$, $\tilde{n}$ is the unit normal to Γ_S exterior to Ω_F.

The Lagrangian formulation of the coupled system

We introduce

$$\tilde{u}(y,t) = u(X(y,t),t) \quad \text{and} \quad \tilde{p}(y,t) = p(X(y,t),t).$$

$X(\cdot,t)$ is a C^1-diffeomorphism from Ω_F into $\Omega_F(t) = X(\Omega_F,t)$ for all $t \in [0,T^*]$, where $T^* > 0$ only depends on the initial conditions u_0 and w_1.

We denote by $Y(\cdot,t)$ the inverse of $X(\cdot,t)$, that is the mapping from $\Omega_F(t)$ into Ω_F satisfying

$$Y(X(y,t),t) = y, \quad y \in \Omega_F \quad \text{and} \quad X(Y(x,t),t) = x, \quad x \in \Omega_F(t).$$

Let us notice that

$$\nabla u(x,t) = \nabla\tilde{u}(Y(x,t),t)J_Y(x,t), \quad x \in \Omega_F(t), \text{ for } t \in [0,T^*],$$

while

$$\nabla p(x,t) = J_Y(x,t)^T\nabla\tilde{p}(Y(x,t),t), \quad x \in \Omega_F(t), \text{ for } t \in [0,T^*],$$

where $J_Y(x,t) = (J_X(Y(x,t),t))^{-1}$.

The full nonlinear system

$$\frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + \nabla \tilde{p} = \mathcal{F}(\tilde{u}, \tilde{p}) \quad \text{in } Q_F^T,$$

$$\operatorname{div} \tilde{u} = \mathcal{G}(\tilde{u}) = \operatorname{div}(\mathbf{g}(\tilde{u})) \quad \text{in } Q_F^T,$$

$$\tilde{u}(0) = u_0 \;\; \text{in } \Omega_F, \quad \tilde{u} = 0 \quad \text{on } \Sigma_e^T,$$

$$\tilde{u} = \frac{\partial w}{\partial t} \quad \text{and} \quad \sigma(\tilde{u}, \tilde{p})\tilde{n} = \sigma(w)\tilde{n} + \mathcal{H}(\tilde{u}, \tilde{p}) \quad \text{on } \Sigma_S^T,$$

$$\frac{\partial^2 w}{\partial t^2} - \operatorname{div}\sigma(w) = 0 \;\; \text{in } Q_S^T,$$

$$w(0) = I \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \;\; \text{in } \Omega_S,$$

$$X(y,t) = y + \int_0^t \tilde{u}(y,s)ds, \quad \text{for all } y \in \Omega_F \text{ and all } t \in [0,T],$$

$$\Omega_F(t) = X(\Omega_F, t),$$

$$Y(X(y,t),t) = y, \quad y \in \Omega_F \quad \text{and} \quad X(Y(x,t),t) = x, \quad x \in \Omega_F(t).$$

The nonlinear terms are defined by

$$\mathcal{F}(\tilde{u},\tilde{p}) = \mathcal{F}_1(\tilde{u}) + \mathcal{F}_2(\tilde{u}) + \mathcal{F}_3(\tilde{u},\tilde{p}),$$

$$\mathcal{F}_1(\tilde{u}) = \nu \sum_{j,k} \frac{\partial^2 Y_k}{\partial x_j^2}(X(y,t),t)\frac{\partial \tilde{u}}{\partial y_k}(y,t),$$

$$\mathcal{F}_2(\tilde{u}) = \nu \sum_{i,j,k} \frac{\partial Y_i}{\partial x_j}\frac{\partial Y_k}{\partial x_j}(X(y,t),t)\frac{\partial^2 \tilde{u}}{\partial y_i \partial y_k}(y,t) - \nu\,\Delta \tilde{u},$$

$$\mathcal{F}_3(\tilde{u},\tilde{p}) = \left(I - J_Y^T\right)\nabla \tilde{p},$$

$$\mathcal{G}(\tilde{u}) = \nabla \tilde{u} : \left(I - J_Y^T\right),$$

$$\mathcal{H}(\tilde{u},\tilde{p}) = -\nu(\nabla\tilde{u}\,J_Y + J_Y^T(\nabla\tilde{u})^T)n + \nu(\nabla\tilde{u} + (\nabla\tilde{u})^T)\tilde{n} - \tilde{p}(\tilde{n} - \mathrm{cof}(\nabla X)\tilde{n}).$$

Let us notice that

$$\nabla \tilde{u} : \left(I - J_Y^T \right) = \mathcal{G}(\tilde{u}) = \operatorname{div}\left(\mathbf{g}(\tilde{u})\right) \quad \text{with} \quad \mathbf{g}(\tilde{u}) = (I - J_Y)\tilde{u},$$

because

$$\nabla \tilde{u} : J_{Y_{\tilde{u}}}^T = \operatorname{div}_x u = 0.$$

Otherwise, any regular vector field $\tilde{v}$, we have

$$\mathcal{G}(\tilde{v}) = \operatorname{div}_y \mathbf{g}(\tilde{v}) + \mathrm{j}(\tilde{v}),$$

where

$$\mathrm{j}(\tilde{v}) = -\nabla \left(\det J_{X_{\tilde{v}}}\right) \cdot J_{Y_{\tilde{v}}} \tilde{v} / \det\left(J_{X_{\tilde{v}}}\right).$$

3. The linearized model

The method of successive approximations is based on fine estimates for the following linearized system

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla q = f \quad \text{in } Q_F^T,$$

$$\operatorname{div} v = g = \operatorname{div} \mathbf{g} + \mathrm{j} \quad \text{in } Q_F^T,$$

$$v(0) = u_0 \ \text{ in } \Omega_F, \quad v = \frac{\partial w}{\partial t} \quad \text{on } \Sigma_S^T,$$

$$\sigma(v, q)n = \sigma(w)n + h \quad \text{on } \Sigma_S^T,$$

$$\frac{\partial^2 w}{\partial t^2} - \operatorname{div}\sigma(w) = 0 \ \text{ in } Q_S^T,$$

$$w(0) = I \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \ \text{ in } \Omega_S.$$

For simplicity we have replaced $\tilde{n}$ by n.

Maximal regularity results for the Stokes system

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla q = 0 \quad \text{in } Q_F^T,$$
$$\operatorname{div} v = 0 \quad \text{in } Q_F^T,$$
$$v(0) = u_0 \ \text{ in } \Omega_F, \qquad v = \frac{\partial w}{\partial t} \quad \text{on } \Sigma_S^T,$$
$$\sigma(v, q)n = \sigma(w)n + h \quad \text{on } \Sigma_S^T.$$

For Dirichlet boundary conditions, if $v|_{\Sigma_S^T} \in H^{\ell,\ell/2}(\Sigma_S^T)$, $\ell > 0$ then $v \in H^{\ell+1/2,\ell/2+1/4}(Q_F^T)$.

Conversely, if $v \in H^{\ell+1/2,\ell/2+1/4}(Q_F^T)$, then $v|_{\Sigma_S^T} \in H^{\ell,\ell/2}(\Sigma_S^T)$.

For Neumann boundary conditions, if $\sigma(v, q)n \in H^{\ell,\ell/2}(\Sigma_S^T)$, $\ell > 0$ then $v \in H^{\ell+3/2,\ell/2+3/4}(Q_F^T)$. Conversely, if $v \in H^{\ell+3/2,\ell/2+3/4}(Q_F^T)$, then $\sigma(v, q)n \in H^{\ell,\ell/2}(\Sigma_S^T)$.

Regularity results for the Lamé system

$$\frac{\partial^2 w}{\partial t^2} - \mathrm{div}\sigma(w) = F \quad \text{in } Q_S^T,$$

$$w = G = w_0|_{\Gamma_S} + \int_0^t v\, d\tau \qquad \text{on } \Sigma_S^T,$$

$$w(0) = w_0 \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \quad \text{in } \Omega_S.$$

If $w_0 \in H^2(\Omega_S)$, $w_1 \in H^1(\Omega_S)$, $F \in L^1(H^1) \cap W^{1,1}(L^2)$, $G \in H^2(\Sigma_S^T)$, $G|_{t=0} = w_0|_{\Gamma_S}$ and $\partial_t G|_{t=0} = w_1|_{\Gamma_S}$, then

$$w \in C([0,T]; H^2(\Omega_S)) \cap C^1([0,T]; H^1(\Omega_S)) \cap C^2([0,T]; L^2(\Omega_S))$$

and

$$\sigma(w)n \in H^1(\Sigma_S^T).$$

More generally if $G \in H^s(\Sigma_S^T)$, then $\sigma(w)n \in H^{s-1}(\Sigma_S^T)$.

Analysis of the interface conditions

$$w = G = w_0|_{\Gamma_S} + \int_0^t v\, d\tau \quad \text{and} \quad \sigma(v,q)n = \sigma(w)n + h \quad \text{on } \Sigma_S^T.$$

If $\sigma(w)n + h \in H^1(\Sigma_S^T)$, then $v \in H^{5/2,5/4}(Q_F^T)$, $v|_{\Sigma_S^T} \in H^{2,1}(\Sigma_S^T)$, and $w_0|_{\Gamma_S} + \int_0^t v\, d\tau \in H^2(\Sigma_S^T)$, which gives again $\sigma(w)n \in H^1(\Sigma_S^T)$.

Thus $v \in H^{5/2,5/4}(Q_F^T)$ and
$w \in C([0,T]; H^2(\Omega_S)) \cap C^1([0,T]; H^1(\Omega_S)) \cap C^2([0,T]; L^2(\Omega_S))$
with $\sigma(w)n \in H^1(\Sigma_S^T)$ seems to be a good candidate family of spaces for a fixed point method.
Unfortunately, for the nonlinear terms in 3D, we need

$$v \in H^{2+\ell,1+\ell/2}(Q_F^T), \quad \text{with } \ell > 1/2.$$

We redo the previous analysis

If $\sigma(w)n + h \in H^{1/2+\ell}(\Sigma_S^T)$, then $v \in H^{2+\ell,1+\ell/2}(Q_F^T)$,

$v|_{\Sigma_S^T} \in H^{3/2+\ell,3/4+\ell/2}(\Sigma_S^T)$, and

$$w_0|_{\Gamma_S} + \int_0^t v\, d\tau \in H^1(0,T;H^{3/2+\ell}(\Gamma_S)) \cap H^{7/4+\ell/2}(0,T;L^2(\Gamma_S)).$$

We would like to have $w_0|_{\Gamma_S} + \int_0^t v\, d\tau \in H^{3/2+\ell}(\Sigma_S^T)$, to recover $\sigma(w)n \in H^{1/2+\ell}(\Sigma_S^T)$.

But $7/4 + \ell/2 < 3/2 + \ell$ and we cannot get

$$w_0|_{\Gamma_S} + \int_0^t v\, d\tau \in H^{3/2+\ell}(\Sigma_S^T).$$

We have to prove new anisotropic hidden regularity for the Lamé system.

We prove that if

$$w_0|_{\Gamma_S} + \int_0^t v\, d\tau \in H^1(0,T;H^{3/2+\ell}(\Gamma_S)) \cap H^{7/4+\ell/2}(0,T;L^2(\Gamma_S)),$$

then

$$\sigma(w)n \in H^{5/8-\ell/4}(0,T;H^{1/2+\ell}(\Gamma_S)) \cap H^{3/4+\ell/2}(0,T;L^2(\Gamma_S)).$$

Analysis of the Stokes system

$$
\begin{aligned}
&\frac{\partial v}{\partial t} - \nu \Delta v + \nabla q = 0 \quad \text{in } Q_F^T, \\
&\operatorname{div} v = g = \operatorname{div} \mathbf{g} + \mathrm{j} \quad \text{in } Q_F^T, \\
&v(0) = u_0 \ \text{ in } \Omega_F, \\
&\sigma(v, q) n = \sigma(w) n + h \quad \text{on } \Sigma_S^T.
\end{aligned}
$$

To simplify we set $\mathrm{j} = 0$.

The difficult part is

$$\begin{aligned}
&\frac{\partial \tilde{v}}{\partial t} - \nu \Delta \tilde{v} + \nabla \tilde{q} = 0 \quad \text{in } Q_F^T, \\
&\operatorname{div} \tilde{v} = g = \operatorname{div} \mathbf{g} \quad \text{in } Q_F^T, \\
&\tilde{v}(0) = 0 \;\; \text{in } \Omega_F, \\
&\sigma(\tilde{v}, \tilde{q}) n = 0 \quad \text{on } \Sigma_S^T.
\end{aligned}$$

We have to use the decomposition $\tilde{v} = P\tilde{v} + (I - P)\tilde{v}$.

The regularity of $g \in L^2(0, T; H^{1+\ell}(\Omega_F)) \cap H^{\ell/2}(0, T; H^1(\Omega_F))$ is used to recover the best regularity with respect to the space variable $\tilde{v} \in L^2(0, T; H^{2+\ell}(\Omega_F))$,

while the regularity $\mathbf{g} \in H^{1+\ell/2}(0, T; L^2(\Omega_F))$ is used to recover the best regularity with respect to the time variable $\tilde{v} \in H^{1+\ell/2}(0, T; L^2(\Omega_F))$.

Regularity result for the Stokes system. If

$$f \in H^{\ell,\ell/2}(Q_F^T), \quad v_0 \in H^{1+\ell}(\Omega_F) \cap H^1_{\Gamma_e}(\Omega_F), \quad \mathrm{div}\, v_0 = 0,$$

$$h \in H^{1/2+\ell,1/4+\ell/2}(\Sigma_S^T), \quad \mathbf{g} \in H^{1+\ell/2}(0,T;L^2(\Omega_F)),$$

$$\mathbf{g} \in C([0,T];H^{1+\ell}(\Omega_F)), \quad \mathbf{g}|_{\Sigma_e^T} = 0, \quad \mathbf{g}(\cdot,0) = 0 \quad \text{in } \Omega_F,$$

$$g \in L^2(0,T;H^{1+\ell}(\Omega_F)) \cap H^{\ell/2}(0,T;H^1(\Omega_F)),$$

and if h, v_0 obey the compatibility conditions

$$2\nu\,(\varepsilon(v_0)n)\cdot\tau = h(0)\cdot\tau \quad \text{on } \Gamma_S,$$

then

$$\|Pv\|_{H^{2+\ell,1+\ell/2}(Q_F^T)} + \|\nabla q\|_{H^{\ell,\ell/2}(Q_F^T)} \le C(\|f\|_{H^{\ell,\ell/2}(Q_F^T)}$$

$$+\|g\|_{L^2(H^{1+\ell})\cap H^{\ell/2}(H^1)} + \|\mathbf{g}\|_{H^{1+\ell/2}(L^2)} + \|h\|_{H^{\ell,\ell/2}(\Sigma_S^T)} + \|v_0\|_{H^{1+\ell}(\Omega_F)}),$$

and

$$\|(I-P)v\|_{H^{2+\ell,1+\ell/2}(Q_F^T)} \le C(\|g\|_{L^2(0,T;H^{1+\ell}(\Omega_F))} + \|\mathbf{g}\|_{H^{1+\ell/2}(L^2)}).$$

In particular $v|_{\Sigma_S^T}$ belongs to $H^{3/2+\ell,3/4+\ell/2}(\Sigma_F^T)$ because $v \in H^{2+\ell,1+\ell/2}(Q_F^T)$.

The linearized coupled system

The Stokes system

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla q = f \quad \text{in } Q_F^T,$$

$$\operatorname{div} v = g = \operatorname{div} \mathbf{g} + \mathrm{j} \quad \text{in } Q_F^T,$$

$$v(0) = u_0 \ \text{ in } \Omega_F,$$

$$\sigma(v, q)n = \sigma(w)n + h = \zeta + h \quad \text{on } \Sigma_S^T,$$

The Lamé system

$$\frac{\partial^2 w}{\partial t^2} - \operatorname{div} \sigma(w) = 0 \ \text{ in } Q_S^T,$$

$$w = I + \int_0^t v(s)\, ds \quad \text{on } \Sigma_S^T,$$

$$w(0) = I \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \ \text{ in } \Omega_S.$$

For ζ given, we denote by (v_ζ, q_ζ) the solution to the Stokes equation. We denote by w_ζ the solution to the Lamé system with the Dirichlet boundary condition

$$w_\zeta = I + \int_0^t v_\zeta(s)\,ds \quad \text{on } \Sigma_S^T.$$

We show that the mapping

$$\zeta \longmapsto \sigma(w_\zeta)n$$

is a contraction in

$$\{\zeta \in H^{1/2+\ell,1/4+\ell/2}(\Sigma_S^T) \mid \zeta\cdot\tau\big|_{t=0} = 0 \text{ on } \Gamma_S \text{ for all vector } \tau \text{ tangent to } \Gamma_S\}.$$

4. Lipschitz estimates

We need to introduce

$$K_0 = \|u_0\|_{H^{1+\ell}(\Omega_F)} + \|w_0\|_{H^{3/2+\ell+\beta}(\Omega_S)} + \|w_0|_{\Gamma_S}\|_{H^{3/2+\ell}(\Gamma_S)}$$
$$+\|w_1\|_{H^{1/2+\ell+\beta}(\Omega_S)}, \quad 0 < \beta \leq 5/8 - \ell/4.$$

The solution $(\tilde{u}^0, \tilde{p}^0, w^0)$ to

$$\frac{\partial v}{\partial t} - \nu\Delta v + \nabla q = 0, \quad \operatorname{div} v = 0, \quad \text{in } Q_F^T,$$
$$v(0) = u_0 \ \text{ in } \Omega_F, \quad v = \frac{\partial w}{\partial t} \quad \text{on } \Sigma_S^T,$$
$$\sigma(v, q)n = \sigma(w)n \quad \text{on } \Sigma_S^T,$$
$$\frac{\partial^2 w}{\partial t^2} - \operatorname{div}\sigma(w) = 0 \ \text{ in } Q_S^T,$$
$$w(0) = I \quad \text{and} \quad \frac{\partial w}{\partial t}(\cdot, 0) = w_1 \ \text{ in } \Omega_S.$$

The estimate

$$\|\tilde{u}^0\|_{H^{2+\ell,1+\ell/2}_{\#}(Q_F^T)} + \|\nabla \tilde{p}^0\|_{H^{\ell,\ell/2}_{\#}(Q_F^T)} \leq C_1\, K_0,$$

We set

$$\widetilde{K}_0 = \|\mathcal{G}(\tilde{u}^0)\|_{L^2(0,T;H^{1+\ell}_{\#}(\Omega_F))\cap H^{\ell/2}(0,T;H^1_{\#}(\Omega_F))} + \|\mathbf{g}(\tilde{u}^0)\|_{H^{1+\ell/2}(0,T;L^2_{\#}(\Omega_F))}.$$

and $M_0 = 3C_1(K_0 + \widetilde{K}_0) + 3.$

The maximal time T^* depends only on M_0.

Lipschitz estimates

$$\|\mathcal{F}(\tilde{u},\tilde{p})\|_{F_T} \le K_{\mathcal{F}}T^{1-\ell}\chi(M_0),$$
$$\|\mathcal{F}(\tilde{u}^1,\tilde{p}^1)-\mathcal{F}(\tilde{u}^2,\tilde{p}^2)\|_{F_T}$$
$$\le K_{\mathcal{F}}T^{1-\ell}\chi(M_0)\left(\|\tilde{u}^1-\tilde{u}^2\|_{E_T}+\|\nabla\tilde{p}^1-\nabla\tilde{p}^2\|_{F_T}\right),$$

for all $\tilde{u}$, $\tilde{u}^1$ and $\tilde{u}^2$ bounded by M_0 in E_T, equal to u_0 at $t=0$, and all $\tilde{p}$, $\tilde{p}^1$ and $\tilde{p}^2$ bounded by M_0 in F_T, with

$$E_T = H^{2+\ell,1+\ell/2}(Q_F^T), \quad F_T = H^{\ell,\ell/2}(Q_F^T),$$

χ is a polynomial of high degree.

5. The method of successive approximations

We look for a solution $(\tilde{u}, \tilde{p}, w)$ to system the nonlinear system in the form

$$\tilde{u} = v + \tilde{u}^0, \quad \tilde{p} = q + \tilde{p}^0, \quad w = z + w^0.$$

Thus (v, q, z) obeys

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla q = \mathcal{F}(v + \tilde{u}^0, q + \tilde{p}^0) \quad \text{in } Q_F^T,$$

$$\operatorname{div} v = \mathcal{G}(v + \tilde{u}^0) = \operatorname{div}\left(\mathbf{g}(v + \tilde{u}^0)\right) + \mathbf{j}(v + \tilde{u}^0) \quad \text{in } Q_F^T,$$

$$v(0) = 0 \ \text{ in } \Omega_F, \quad v = 0 \quad \text{on } \Sigma_e^T,$$

$$v = \frac{\partial z}{\partial t} \quad \text{and} \quad \sigma(v, q)n = \sigma(z)n + \mathcal{H}(v + \tilde{u}^0, q + \tilde{p}^0) \quad \text{on } \Sigma_S^T,$$

$$\frac{\partial^2 z}{\partial t^2} - \operatorname{div}\sigma(z) = 0 \ \text{ in } Q_S^T,$$

$$z(0) = 0 \quad \text{and} \quad \frac{\partial z}{\partial t}(\cdot, 0) = 0 \ \text{ in } \Omega_S,$$

and X and Y are determined by

$$X(y,t) = y + \int_0^t (v + \tilde{u}^0)(y,s)ds, \quad \text{for all } y \in \Omega_F \text{ and all } t \in [0,T],$$

$$\Omega_F(t) = X(\Omega_F, t),$$

$$Y(X(y,t),t) = y, \quad y \in \Omega_F \quad \text{and} \quad X(Y(x,t),t) = x, \quad x \in \Omega_F(t).$$

We prove the existence of a fixed point to the above system, by the method of successive approximation.

We choose (v^k, q^k, z^k) in the nonlinear terms of the RHS, and we denote by $(v^{k+1}, q^{k+1}, z^{k+1})$ the corresponding solution.

At the first iterate, that is for $(v^0, q^0, z^0) = (0, 0, 0)$, we have

$$\mathcal{G}(\tilde{u}^0) = \operatorname{div}(\mathbf{g}(\tilde{u}^0)),$$

because $\nabla \tilde{u}^0 : J^T_{Y_{\tilde{u}^0}} = 0$.

For the other iterates, we have

$$\mathcal{G}(\tilde{u}^k) = \operatorname{div}(\mathbf{g}(v^k + \tilde{u}^0)) + \mathrm{j}(v^k + \tilde{u}^0).$$

This method is used with some gap in the literature on free boundary value problems.

Existence and uniqueness theorem

Assumptions.

$$u_0 \in H^{1+\ell}(\Omega_F), \quad u_0|_{\Gamma_e} = 0, \quad \mathrm{div}\, u_0 = 0, \quad w_1 \in H^{1/2+\ell+\beta}(\Omega_S),$$

with $\ell \in (1/2, 1)$ and $0 < \beta \leq \frac{5}{8} - \frac{\ell}{4}$, and

$$u_0|_{\Gamma_S} = w_1|_{\Gamma_S}, \quad 2\nu\,(\varepsilon(u_0)n)\cdot\tau = \sigma(w_0)n\cdot\tau = \sigma(I)n\cdot\tau = 0 \ \text{ on } \Gamma_S,$$

for any unit vector τ tangent to Γ_S.

Metric spaces.

$$E_{T,M_0,u_0} = \{\tilde{u} \in H^{2+\ell,1+\ell/2}(Q_F^T) \mid \tilde{u}(0) = u_0, \ \|\tilde{u}\|_{H^{2+\ell,1+\ell/2}(Q_F^T)} \leq M_0\},$$

$$P_{T,M_0} = \{p \in L^2(Q_F^T) \mid \|\nabla p\|_{H^{\ell,\ell/2}(Q_F^T)} \leq M_0\},$$

and

$$X_{T,1/2} = \{X \in H^1(0,T;H^{2+\ell}(\Omega_F)) \cap H^{2+\ell/2}(0,T;L^2(\Omega_F)) \mid \|\nabla_y X - I\|_{C\left(\overline{Q_F^T}\right)} \leq 1/2\}.$$

Conclusion. Then, there exists $T > 0$ such that the nonlinear system admits a unique solution $(\tilde{u}, \nabla\tilde{p}, w, X)$ in $E_{T,M_0,u_0} \times P_{T,M_0} \times \left(C^0([0,T];H^{7/4+\ell/2}(\Omega_S)) \cap C^1([0,T];H^{3/4+\ell/2}(\Omega_S))\right) \times X_{T,1/2}$.

Solution to the initial system.

If u_0 and w_1 obey the previous compatibility conditions. Assume that $(\tilde{u}, \nabla\tilde{p}, w, X) \in E_{T,M_0,u_0} \times P_{T,M_0} \times \left(C^0([0,T]; H^{7/4+\ell/2}(\Omega_S)) \cap C^1([0,T]; H^{3/4+\ell/2}(\Omega_S))\right) \times X_{T,1/2}$ is a solution to the nonlinear system in Lagrangian variables. Let us set

$$u(x,t) = \tilde{u}(Y(x,t),t), \quad p(x,t) = \tilde{p}(Y(x,t),t) \quad \text{for all } x \in \Omega_F(t), \; t \in [0,T].$$

Then (u, p, w) is a solution to system initial system in Eulerian-Lagrangian variables.

References

H. Beirao da Veiga, On the existence of strong solutions to a coupled fluid-structure evolution system, JMFM 6 (2004), 21-52.

A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont, Existence of weak solutions for unsteady fluid-plate interaction problem, J. Math. Fluid Mech. 7 (2005), 368-404.

M. Boulakia, S. Guerrero, Regular solutions of a problem coupling a compressible fluid and an elastic structure, 2010.

M. Boulakia, Existence of weak solutions for the three dimensional motion of an emastic structure in an incompressible fluid, J. Math. Fluid Mech. 9 (2007), 262-294.

C. Conca, J. San Martin, M. Tucsnak, Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid, CPDE 25 (2000), 1019-1042.

C. Grandmont, Existence of weak solutions for unsteady interaction of a viscous fluid with an elastic plate, SIMA, 2008.

J. Lequeurre, Existence of strong solutions to a fuid-structure system, SIAM J. Math. Anal., 43 (2011), 389–410.

I. Kukavica, A. Tuffaha, Solutions to a fluid-structure interaction free boundary problem, Discrete and Cont. Dyn. Systems, 32 (2012), 1355–1389.

D. Coutand and S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, Arch. Ration. Mech. Anal., 176 (2005), 25-102.

J.-P. Raymond, Feedback stabilization of a fuid - structure model, SIAM J. Control Optim., 48(2010), 5398–5443.

J-P. Raymond, *Stokes and Oseen equations with a nonhomogeneous divergence condition*. Discrete Contin. Dyn. Syst., Ser. B, 14 (2010), 1537-1564.

J. San Martín, J-F. Scheid, T. Takahashi, M. Tucsnak, *An initial and boundary value problem modeling of fish-like swimming,* Arch. Rational Mech. Anal. 188 (2008),429-455.

T. Takahashi, *Analysis of strong solutions for the equations modelling the motion of a rigid-fluid system in a bounded domain,* Adv. Differ. Equ. 8 (2003),1499-1532.

T. Takahashi, M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid*, J. Math. Fluid Mech. 6 (2004) 53-77.