On the motion of rigid bodies in a perfect incompressible fluid

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Outline

I. Motion of a 2d rigid body in a perfect incompressible fluid. The Cauchy problem.

II. The particle limit ($\varepsilon \to 0$).

III. The mean-field limit ($N \to +\infty$).

IV. The gyroscopic limit ($m \to 0$).

- $\varepsilon > 0$: rigid body diameter,
- $N \in \mathbb{N}^*$: number of particles,
- m : individual mass of the particles.

Part I.

Motion of a 2d rigid body in a perfect incompressible fluid. The Cauchy problem.

Position of the rigid body

We consider the motion of a solid body occupying, at time t, the domain

$$\mathcal{S}(t) = \tau(t)\mathcal{S}_0,$$

where

- $\mathcal{S}_0 \subset \mathbb{R}^2$ is a closed, bounded, connected and simply connected regular domain, which denotes the initial position of the solid.
- $\tau(t) \in SE(2)$ can be decomposed into

  $$\tau(t) \cdot x = h(t) + Q(t)(x - h(0)),$$

  where

  - $h(t)$ is the position of the center of mass of $\mathcal{S}(t)$, and we will assume that $h(0) = 0$.
  - $Q(t)$ is the rotation matrix :

    $$Q(t) := \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

    and satisfies $Q(0) = Id$, that is $\theta(0) = 0$.

Velocity of the rigid body

The body velocity is given by

$$u_{\mathcal{S}}(t,x) := \ell(t) + r(t)(x - h(t))^{\perp},$$

where

- $\ell(t) := h'(t)$ is the velocity of the center of mass,
- $r(t) := \theta'(t)$ is the angular velocity of the body, and
- the notation $x^{\perp}$ stands for $x^{\perp} = (-x_2, x_1)$, when $x = (x_1, x_2)$.

A solid in a perfect incompressible fluid

- We consider the motion of a solid body immersed in a perfect incompressible fluid occupying

$$\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t).$$

  Hence we consider the incompressible Euler equation in the fluid domain :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0 \quad \text{for} \quad x \in \mathcal{F}(t)$$
$$\text{div } u = 0 \quad \text{for} \quad x \in \mathcal{F}(t).$$

- On the interface between the fluid and the solid there holds :

$$u \cdot n = u_{\mathcal{S}} \cdot n \quad \text{on } \partial\mathcal{S}(t).$$

Newton's law

- The solid motion is given by Newton's law.
- It evolves under the influence of the fluid pressure on its surface :

$$mh''(t) = \int_{\partial \mathcal{S}(t)} p\, n\, ds \quad \text{and} \quad \mathcal{J}\, r'(t) = \int_{\partial \mathcal{S}(t)} p\, (x - h(t))^{\perp} \cdot n\, ds,$$

where $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the moment of inertia of the body.

Initial data

We prescribe the initial velocities :

- $u|_{t=0} = u_0$ is
  - in the Hölder space
    $$C^{1,\lambda}(\mathcal{F}_0; \mathbb{R}^2),$$
    where $\lambda \in (0, 1)$,
  - in the space
    $$L^2(\mathcal{F}_0; \mathbb{R}^2) \oplus \mathbb{R}\chi H_0,$$
    where
    $$H_0(x) := \frac{x^\perp}{2\pi|x|^2} \text{ and } \chi(x) := \frac{|x|^2}{1+|x|^2},$$
  - and satisfies div $u_0 = 0$ in $\mathcal{F}_0$,

- $(\ell(0), r(0)) = (\ell_0, r_0)$ is in $\mathbb{R}^2 \times \mathbb{R}$,

  with the compatibility condition at the interface :
  $$u_0 \cdot n = u_{\mathcal{S}_0} \cdot n \text{ on } \partial\mathcal{S}_0, \text{ with } u_{\mathcal{S}_0}(x) := \ell_0 + r_0 x^\perp.$$

Global-in-time existence and uniqueness of classical solutions

Theorem

There exists a unique classical solution

$$(\ell, r, u) \in C^1([0,+\infty);\mathbb{R}^2) \times C^1([0,+\infty);\mathbb{R}) \times C_w([0,+\infty); C^{1,\lambda}(\mathcal{F}(t))).$$

References : Ortega-Rosier-Takahashi (2007) in the case of finite energy, Glass-S (2012) for the general case.

Why should we bother with infinite energy ?

The fluid part of the system can also be written thanks to the vorticity :

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega u) = 0 \text{ in } \mathcal{F}(t), \\ \omega_{|t=0} = \omega_0, \end{cases}$$

and

$$\begin{cases} \operatorname{curl} u = \omega \text{ in } \mathcal{F}(t), \\ \operatorname{div} u = 0 \text{ in } \mathcal{F}(t), \\ u \cdot n = u_{\mathcal{S}} \cdot n \text{ on } \partial\mathcal{S}(t), \\ \lim_{|x| \to +\infty} u(t,x) = 0, \\ \int_{\partial\mathcal{S}(t)} u(t,x) \cdot \tau \, ds = \int_{\partial\mathcal{S}_0} u_0(x) \cdot \tau \, ds \text{ (Kelvin's law).} \end{cases}$$

Why should we bother with infinite energy ?

- In particular the unique regular vector field H such that

$$\begin{cases} \operatorname{curl} H = 0 \text{ in } \mathcal{F}_0, \\ \operatorname{div} H = 0 \text{ in } \mathcal{F}_0, \\ H \cdot n = 0 \text{ on } \partial\mathcal{S}_0, \\ \lim_{|x|\to+\infty} H(x) = 0, \\ \int_{\partial\mathcal{S}_0} H \cdot \tau \, ds = 1. \end{cases}$$

  behaves like H_0 at infinity, and therefore is not in $L^2(\mathcal{F}_0)$.

- Still it is a steady (irrotationnal) solution of the Euler incompressible equations in $\mathcal{F}_0$.
- For fluid velocities which are potential in $\mathcal{F}$, stationary and constant, say equal to u_∞, at infinity, D'Alembert's paradox states that the fluid does not influence the dynamics of the solid.
- If there is a circulation $\gamma \neq 0$, then the fluid acts on the solid with the Kutta-Joukowski force $F = -\gamma u_\infty^\perp$.

The Kutta-Joukowski force

- In an irrotationnal flow the calculation of the Kutta-Joukowski force relies on the following lemma :

Lemma (Blasius' lemma)

Let $\mathcal{C}$ be a smooth Jordan curve, $f := (f_1, f_2)$ and $g := (g_1, g_2)$ two smooth tangent vector fields on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} (f \cdot g) n \, ds = i \left(\int_{\mathcal{C}} (f_1 - i f_2)(g_1 - i g_2) \, dz \right)^*,$$

$$\int_{\mathcal{C}} (f \cdot g)(x^{\perp} \cdot n) \, ds = \mathfrak{Re} \left(\int_{\mathcal{C}} z (f_1 - i f_2)(g_1 - i g_2) \, dz \right).$$

where $(\cdot)^*$ denotes the complex conjugation.

- and on Cauchy's Residue Theorem, using that the Laurent series of H starts as follows :

$$(H_1 - i H_2)(z) = \frac{1}{2 i \pi z} + \mathcal{O}(\frac{1}{z^2}) \text{ as } z \to \infty.$$

Renormalized energy

- The vector field
$$\hat{u} := u - (\alpha + \gamma) H(t),$$
where
$$\alpha := \int_{\mathcal{F}(t)} \omega(t,x)\,dx = \int_{\mathcal{F}_0} \omega(0,x)\,dx.$$
is in $L^2(\mathcal{F}(t))$.
- Moreover
$$\mathcal{H} := \frac{1}{2}\left[m\ell^2 + \mathcal{J} r^2 + \int_{\mathcal{F}(t)} \hat{u}^2 + 2(\gamma+\alpha)\hat{u} \cdot H(t) \right]$$
is conserved.
- The "standard" energy would be
$$\mathcal{E} := \frac{1}{2}\left[m\ell^2 + \mathcal{J} r^2 + \int_{\mathcal{F}(t)} \left(\hat{u} + (\gamma+\alpha) H(t)\right)^2 \right],$$
but this is infinite in general.

Smoothness of the body motion

Theorem (Glass-S-Takahashi (2012), Glass-S (2012).)

Assume furthermore that $\partial \mathcal{S}_0$ is analytic.

- *Then the motion of the rigid body is analytic in time, that is*

$$(\ell, r) \in C^\omega([0,\infty);\mathbb{R}^2) \times C^\omega([0,\infty);\mathbb{R})$$

- *Moreover it depends smoothly on the initial data.*

References

This theorem extends some results about the smoothness of the trajectories of the fluid particles :

- Chemin (1991, 1992) : perfect incompressible fluid filling the full space, trajectories for classical solutions are C^∞.
- Serfati (1992) : perfect incompressible fluid filling the full space, trajectories for classical solutions are C^ω.
- Gamblin (1994) : perfect incompressible fluid filling the full plane, trajectories for Yudovich solutions are Gevrey 3.
- Kato (2000) : perfect incompressible fluid filling a C^∞ bounded domain, trajectories for classical solutions are C^∞.

A few ingredients used in the proof

The proof uses

- a decomposition into two parts of the pressure which encodes the added mass effect,
- a precise study of the commutation of the iterated material derivatives D^k, where
$$D := \partial_t + u \cdot \nabla$$
with an equivalent formulation of the problem, in particular with the div-curl systems satisfied by the two parts of the pressure.

Kirchoff's potentials.

- One introduces Kirchoff's potentials $\Phi_1(t), \Phi_2(t), \Phi_3(t)$:

$$\Delta \Phi_i = 0 \ \text{ in } \ \mathcal{F}(t),$$
$$\partial_n \Phi_i = \begin{cases} n_i \ \ (\text{ if } i = 1, 2), \\ (x - h(t))^{\perp} \cdot n \ \ (\text{ if } i = 3), \end{cases} \quad \text{on } \ \partial \mathcal{S}(t).$$

- The solid equations become

$$\begin{bmatrix} m \, \mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} \begin{bmatrix} \ell \\ r \end{bmatrix}' = \begin{bmatrix} \displaystyle\int_{\partial \mathcal{S}(t)} p \, n \, ds \\ \displaystyle\int_{\partial \mathcal{S}(t)} p \, (x - h(t))^{\perp} \cdot n \, ds \end{bmatrix}$$
$$= \left[\int_{\partial \mathcal{S}(t)} p \partial_n \Phi_i \, dx \right]_{i=1,2,3}$$
$$= \left[\int_{\mathcal{F}(t)} \nabla p \cdot \nabla \Phi_i \, dx \right]_{i=1,2,3}.$$

Decomposition of the pressure

The pressure decomposes as follows :

$$\nabla p = -\begin{bmatrix} \ell \\ r \end{bmatrix}' \cdot \left[\nabla \Phi_i\right]_{i=1,2,3} - \nabla \mu,$$

where

$$\begin{aligned} -\Delta \mu &= \mathrm{tr}\left\{\nabla u \cdot \nabla u\right\} \quad \text{for} \quad x \in \mathcal{F}(t), \\ \frac{\partial \mu}{\partial n} &= \sigma, \quad \text{for } x \in \partial \mathcal{S}(t), \\ \mu(t,x) &\to 0 \quad \text{as} \quad x \to \infty, \end{aligned}$$

where

$$\sigma := \nabla^2 \rho \left\{u - u_{\mathcal{S}}, u - u_{\mathcal{S}}\right\} - n \cdot \left(r \left(2u - u_{\mathcal{S}} - \ell\right)^{\perp} \right),$$

with

$$\rho(t,x) := \mathrm{dist}(x, \partial \mathcal{S}(t)).$$

Added mass effect

We end up with this new equation for the solid :

$$\mathcal{M} \begin{bmatrix} \ell \\ r \end{bmatrix}' = \left[\int_{\mathcal{F}(t)} \nabla \mu \cdot \nabla \Phi_i \, dx \right]_{i=1,2,3},$$

where

$$\mathcal{M} := \begin{bmatrix} m \, \mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} + \underbrace{\left[\int_{\mathcal{F}(t)} \nabla \Phi_i \cdot \nabla \Phi_j \; dx \right]_{i,j=1,2,3}}_{=:\mathcal{M}_2}.$$

The matrix $\mathcal{M}_2$ is a matrix of added inertia, expressing how the fluid opposes the movement of the solid. It is positive as a Gram matrix.

Weak solutions

There exists some results of global existence of weak solutions.

- Glass-S. (2011) : weak solutions with $\omega := \operatorname{curl} u \in L^\infty_c$, for which uniqueness holds true and the motion of the rigid body is Gevrey 3, if the boundary is analytic.
  This corresponds to solutions “à la Yudovich”.
- Glass-Lacave-S. (2011) : weak solutions with $\omega := \operatorname{curl} u \in L^p_c$, where $p > 2$. These solutions satisfy renormalization properties and the corresponding velocity u is continuous.
  This corresponds to solutions “à la Di Perna-Lions”.

Remark

One can also prove the existence of even weaker solutions, for $\omega \in L^p_c$ with $p > 1$, or for finite-energy weak solutions for ω bounded Radon measure with symmetry (solutions “à la Delort"). cf. Glass-S. (2012), Xin-Wang (2012), S. (2012).

Part II.

The particle limit ($\varepsilon \to 0$).

The problem of a small body

- Question. What can be said if the size ε of the solid goes to zero, so that the body shrinks to a point ?
- For $\varepsilon \in (0, 1)$, we define

$$\mathcal{S}_0^\varepsilon := \varepsilon \mathcal{S}_0, \quad \mathcal{F}_0^\varepsilon := \mathbb{R}^2 \setminus \mathcal{S}_0^\varepsilon.$$

- We will be interested in the following particular regime of a massive point in the limit :

$$m_\varepsilon = m \text{ and } \mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J},$$

where m and $\mathcal{J}$ are some fixed positive constants.

The problem of a small body, continued

- Let

$$\omega_0 \in L^p_c(\mathbb{R}^2), \text{ with } p > 2, \quad \gamma \in \mathbb{R}, \quad (\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}.$$

- For $\varepsilon \in (0,1)$, we define u_0^ε satisfying

$$\begin{cases} \operatorname{curl} u_0^\varepsilon = \omega_0 \ \text{ in } \ \mathcal{F}_0^\varepsilon, \\ \operatorname{div} \ u_0^\varepsilon = 0 \ \text{ in } \ \mathcal{F}_0^\varepsilon, \\ u_0^\varepsilon \cdot n = (\ell_0 + r_0 x^\perp) \cdot n \ \text{ on } \ \partial \mathcal{S}_0^\varepsilon, \\ \lim_{|x| \to +\infty} u_0^\varepsilon = 0, \\ \int_{\partial \mathcal{S}_0^\varepsilon} u_0^\varepsilon \cdot \tau \, ds = \gamma. \end{cases}$$

- What can be said about a sequence of global weak solutions

$$(\ell^\varepsilon, r^\varepsilon, u^\varepsilon)$$

associated to these data ?

A brief recall of some notations

- Recall that

$$h^{\varepsilon}(t) := \int_0^t r^{\varepsilon}(s)ds, \quad \theta^{\varepsilon}(t) := \int_0^t r^{\varepsilon}(s)ds,$$

$$\text{and } H_0(x) := \frac{x^{\perp}}{2\pi|x|^2}$$

- We will also use the Biot-Savart operator $K[\cdot]$ which is the convolution with H_0, and maps to a reasonable scalar function ω the vector field $K[\omega]$ solution of

$$\begin{cases} \operatorname{curl} K[\omega] = \omega \ \text{ in } \ \mathbb{R}^2, \\ \operatorname{div} K[\omega] = 0 \ \text{ in } \ \mathbb{R}^2, \\ \lim_{|x|\to+\infty} K[\omega](x) = 0. \end{cases}$$

Main result

Theorem (Glass-Lacave-S. 12'). Up to a subsequence,

- $(h^\varepsilon, \varepsilon\theta^\varepsilon) \xrightarrow{w*} (h, 0)$ in $W^{2,\infty}(0, T; \mathbb{R}^2 \times \mathbb{R})$,
- $\omega^\varepsilon \xrightarrow{w} \omega$ in $C^0([0, T]; L^p(\mathbb{R}^2))$,
- one has

$$\partial_t \omega + \operatorname{div}(\omega u) = 0, \quad u = K[\omega + \gamma\delta_{h(t)}],$$
$$mh''(t) = \gamma\Big(h'(t) - \tilde{u}(t, h(t))\Big)^\perp, \quad \tilde{u} = K[\omega],$$

with

$$(\omega|_{t=0}, h(0), h'(0)) = (\omega_0, 0, \ell_0).$$

Comparison of the limit system

Our limit system : Euler + massive point vortex

$$\partial_t \omega + \operatorname{div}(\omega u) = 0, \quad u = K[\omega + \gamma \delta_{h(t)}],$$
$$m h''(t) = \gamma \Big(h'(t) - \tilde{u}(t, h(t)) \Big)^{\perp}, \quad \tilde{u} = K[\omega],$$

Euler + (massless) point vortex, see Marchioro-Pulvirenti :

$$\partial_t \omega + \operatorname{div}(\omega u) = 0, \quad u = K[\omega + \gamma \delta_{h(t)}],$$
$$h'(t) = \tilde{u}(t, h(t)), \quad \tilde{u} = K[\omega].$$

Uniform (in ε) a priori estimates

- Using
  - the renormalized energy
  - and the conservations of $\|\omega\|_{L^p}$,

  one obtains that, for $T > 0$, the quantities

  $$|\ell^\varepsilon|, \quad \varepsilon|r^\varepsilon|, \quad \|u^\varepsilon - \gamma H^\varepsilon\|_\infty, \quad \text{and} \quad \text{diam}(\text{Supp}(\omega^\varepsilon))$$

  are bounded on $[0, T]$ independently of ε.
- Let us point pout that H^ε is of order $\mathcal{O}(1/\varepsilon)$ on $\partial\mathcal{S}_0^\varepsilon$ ....
- When $\varepsilon \to 0$, the added inertia is negligible with respect to the body inertia.
- One uses a potential approximation of the velocity on the solid's boundary, satisfying the interface condition.

Description of the shrinking body's behaviour

The solid equations become

$$\begin{bmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} \begin{bmatrix} h^\varepsilon \\ \varepsilon\theta^\varepsilon \end{bmatrix}'' = \gamma \begin{bmatrix} ((h^\varepsilon)' - u^\varepsilon(t, h^\varepsilon))^\perp \\ 0 \end{bmatrix} + \gamma \underbrace{\begin{bmatrix} -(\varepsilon(\theta^\varepsilon)')Q^\varepsilon(t)\alpha \\ \alpha \cdot Q^\varepsilon(t)^*((h^\varepsilon)' - u^\varepsilon) \end{bmatrix}}_{\text{converges weak-}* \text{ to 0 in } W^{1,\infty}} + o(1),$$

where

- $u^\varepsilon = K[\omega^\varepsilon]$,
- ω^ε is extended by 0 inside $\mathcal{S}_0^\varepsilon$,
- and $\alpha \in \mathbb{R}^2$ depends only on the geometry.

Part III.

The mean-field limit ($N \to \infty$).

N pointwise massive particles in a perfect incompressible fluid

Let us now generalize the previous system to the case of N pointwise particles of mass m_i, of circulation γ_i and of position $h_i(t)$, moving into a perfect and incompressible planar fluid :

$$\partial_t \omega + \operatorname{div}_x(\omega u) = 0, \quad u(t,x) = K[\omega + \sum_{j=1}^{N} \gamma_j \delta_{h_j(t)}],$$

$$m_i h_i''(t) = \gamma_i \Big(h_i'(t) - \tilde{u}_i(t, h_i(t)) \Big)^{\perp}, \quad \tilde{u}_i = K[\omega + \sum_{j \neq i} \gamma_j \delta_{h_j(t)}],$$

$$\omega|_{t=0} = \omega_0, \;\; h_i(0) = h_{i,0}, \;\; h_i'(0) = h_{i,1}.$$

The mean-field limit

- We want to study the mean-field limit of the previous system, that is the limit system obtained by the empirical measure

$$f_N(t) := \frac{1}{N}\sum_{i=1}^{N} \delta_{(h_i(t),h_i'(t))}$$

when N goes to infinity, with an appropriate scaling of the amplitudes.

- We therefore consider now the solutions of

$$\partial_t \omega + \operatorname{div}_x(\omega u) = 0, \quad u(t,x) = K[\omega + \frac{1}{N}\sum_{j=1}^{N} \delta_{h_j(t)}],$$

$$h_i''(t) = \Big(h_i'(t) - \tilde{u}_i(t, h_i(t))\Big)^{\perp}, \quad \tilde{u}_i = K[\omega + \frac{1}{N}\sum_{j\neq i} \delta_{h_j(t)}],$$

$$\omega|_{t=0} = \omega_0,\ h_i(0) = h_{i,0},\ h_i'(0) = h_{i,1}.$$

An Euler-Vlasov system

In the case of several massive vortices, in the mean-field regime, one obtains :

$$\partial_t \omega + \operatorname{div}_x(\omega u) = 0,$$
$$\partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot [f(\xi - u)^\perp] = 0,$$
$$u := K[\omega + \rho] \text{ and } \rho := \int_{\mathbb{R}^2} f d\xi.$$

Comparison of different sprays models

- Our model : Euler-Vlasov in 2d, coupled by a gyroscopic force.
- Spherical particles in a 3d potential flow. No gyroscopic force, but thicker spray with some added mass effect.
  cf. Russo-Smerecka, Herrero-Lucquin-Perthame, Jabin-Perthame.
- Vlasov-Stokes in 3d, coupled by the Brinkman drag force.
  cf. Jabin-Perthame, Desvillettes-Golse-Ricci.

The Cauchy problem for the Euler-Vlasov system

For this system, one can prove (Moussa-S. 12') :

- a well-posedness result "à la Dobrushin" in the space of Radon measures when the Biot-Savart kernel H_0 is regularized into a Lipshitz kernel.
- the existence of weak solutions, for $\omega_0 \in (L^{4/3} \cap L^1)(\mathbb{R}^2)$, $f_0 \in (L^\infty \cap L^1)(\mathbb{R}^2 \times \mathbb{R}^2)$ such that the kinetic energy of the dispersed phase is finite :

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x,\xi)|\xi|^2 \mathrm{d}x \, \mathrm{d}\xi < +\infty,$$

- the uniqueness of solution for solutions "à la Loeper", with the main assumption that $\rho \in L^\infty((0,T) \times \mathbb{R}^2)$,
- the persistence, globally in time, of regularity, "à la Degond".

Part IV.

The gyroscopic limit ($m \to 0$).

- We investigate the behavior, when the individual mass m of the particles converges to 0, of the system :

$$\partial_t \omega^m + \operatorname{div}_x(\omega^m u^m) = 0,$$
$$\partial_t f^m + \operatorname{div}_x(f^m \xi) + \frac{1}{m}\operatorname{div}_\xi(f^m(\xi - u^m)^\perp) = 0,$$
$$u^m = K[\omega^m + \rho^m] \text{ and } \rho^m := \int_{\mathbb{R}^2} f^m d\xi.$$

- One may guess that in the limit $m \to 0^+$ the density of particles becomes monokinetic with a velocity $\xi = u$ so that

$$j^m := \int_{\mathbb{R}^2} f^m \xi d\xi \to \rho u, \text{ where } \rho := \lim \rho^m \text{ and } u = \lim u^m.$$

- Therefore the equations would degenerate into :

$$\partial_t \omega + \operatorname{div}_x(\omega u) = 0, \quad \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \quad u = K[\omega + \rho],$$

thus yielding the incompressible Euler equation with vorticity $\omega + \rho$.

Theorem (Moussa-S. 12')

Let be given

- $u_0 \in L^2(\mathcal{F}_0; \mathbb{R}^2) \oplus \mathbb{R}\chi H_0$,
- *some smooth compactly supported functions* $(\omega_0^m, f_0^m)_m$ *such that*

$$(\omega_0^m, \rho_0^m)_m \text{ is bounded in } L^2(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$$

$$m \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\xi|^2 f_0^m(x,\xi) dx d\xi + \int_{\mathbb{R}^2} |u_0^m - u_0|^2 dx \to 0, \text{ when } m \to 0^+,$$

  where $u_0^m := K[\omega_0^m + \rho_0^m]$.
- *the corresponding smooth solutions* $(\omega^m, f^m)_m$ *of the Euler-Vlasov equations.*

Then, up to an extraction, $(u^m)_m$ *converges in*

$$L^\infty((0,T); L^2(\mathcal{F}_0; \mathbb{R}^2) \oplus \mathbb{R}\chi H_0 - w)$$

to a dissipative solution of the incompressible Euler equation with initial condition u_0.

Idea of the proof

- Let

$$\alpha := \int_{\mathbb{R}^2} \big(\omega^m(t,x) + \rho^m(t,x)\big)\,dx = \int_{\mathbb{R}^2} \big(\omega^m(0,x) + \rho^m(0,x)\big)\,dx.$$

- Consider a smooth (in time/space) vector field v such as

$$v(t) \in L^2(\mathcal{F}_0;\mathbb{R}^2) \oplus \alpha\chi H_0$$

and $\operatorname{curl} v(t)$ is compactly supported, for all t.

- Let us denote

$$2\mathcal{H}_v^m(t) := m \int_{\mathbb{R}^2\times\mathbb{R}^2} |\xi - v(t,x)|^2 f^m(t,x,\xi)\,dx\,d\xi + \int_{\mathbb{R}^2} |u^m(t,x) - v(t,x)|^2\,dx.$$

- Observe that the modulated energy $\mathcal{H}_v^m(t)$ is the sum of two nonnegative finite terms.
- The proof relies on the dynamics of $\mathcal{H}_v^m(t)$.

Open questions

- We considered here successively the particle limit $\varepsilon \to 0$, the mean-field limit $N \to +\infty$ and finally the gyroscopic limit $m \to 0$.
- Is that possible to proceed in a different order ? To consider correlated limits in order to cover a larger range of parameters ?
- Control issues ?
- Does there remain something of this with some viscosity ?

Thank you for your attention !