Tas de sable en équilibre sur un réseau hétérogène

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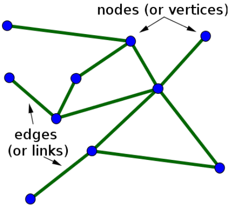
We shall consider the system of PDE for (u, v)

$$
\begin{array}{ll}
-D\,(v(x)\,\eta(x)\,Du(x)) = f(x) & \text{in } \mathcal{N} \setminus \partial\mathcal{N} \\
\eta(x)\,|Du(x)| \leq 1 & \text{in } \mathcal{N} \setminus \partial\mathcal{N} \\
\eta(x)\,|Du(x)| = 1 & \text{in } \{x \in \mathcal{N} \setminus \partial\mathcal{N} \,:\, v(x) \neq 0\} \\
u,\ v \geq 0 & \text{in } \mathcal{N}
\end{array}
$$

on a finite connected network $\mathcal{N}$



This is a joint work with F. Camilli and S. Cacace
("La Sapienza" - Università di Roma)

Mass transfer problem

The Euler-Lagrange equation for the Monge-Kantorovich's dual problem

$$\max_{u \in \mathrm{Lip}^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} u(f^+ - f^-)dx$$

is

$$\begin{aligned} -\mathrm{div}\,(a\,\nabla u) &= f^+ - f^- \\ a \geq 0\,, \quad |\nabla u| &\leq 1\,, \quad a(|\nabla u| - 1) = 0 \end{aligned}$$

where a is the Lagrange multiplier

See Evans-Gangbo, Mem. Am. Math. Soc. (1999)
Villani, Topics in OT, GMS AMS (2003),
Santambrogio, Progress in Non. Diff Eq. Appl. (2015)

A variational problem

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbf{1}_D(\nabla u) + g(u))$$

$D \subseteq \mathbb{R}^n$ convex and closed, $\mathbf{1}_D$ characteristic function of D

g strictly increasing

The Euler-Lagrange equation for the above variational problem is

$$\begin{aligned} &\mathrm{div}(\pi(x)) = g'(u(x)) \\ &\pi(x) \cdot \nabla u(x) = \max\{\pi(x) \cdot d; d \in D\} \end{aligned}$$

See Bianchini, DCDS (2007), and the references therein

Equilibrium configurations of sand-piles

$\rightsquigarrow$ Dry matter is poured vertically and continuously, by a constant in time source of density $f(x) \geq 0$, onto a "table" $\Omega \subset \mathbb{R}^2$

$\rightsquigarrow$ The evolution of the heap of sand is described mathematically through the functions

$u \geq 0$: height of the standing layer (matter that stay at rest)

$v \geq 0$: thickness of the rolling layer (matter moving down)

and assuming that :

- the slope of u can not exceed the critical "angle of repose"
  $\Rightarrow |\nabla u| \leq 1$ in Ω
- the flow of v follows the slope of $u \Rightarrow J_v = -v\nabla u$
- superfluous matter runs down at $\partial\Omega \quad \Rightarrow \quad u = 0$ on $\partial\Omega$

$\rightsquigarrow$ At the equilibrium :

- $-\mathrm{div}(v\nabla u) = f$
- the slope of u has to be maximal if $v > 0 \Rightarrow |\nabla u| = 1$ in $\{v > 0\}$

Equilibrium configurations of sand-piles

- Configuration corresponding to a point source f at $y \in \Omega$

$$u^f(x) = [\mathrm{dist}(y, \partial\Omega) - |x - y|]_+$$

- Union of all "cones"

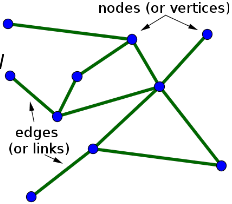
$$u^f(x) = \max_{y \in supp(f)} [\mathrm{dist}(y, \partial\Omega) - |x - y|]_+$$

- "$v(x)$ is determined by adding up all the matter coming down (from the source) along the *transport ray* between the *singular set* of $\mathrm{dist}_{\partial\Omega}$ and x "

Hadeler & Kuttler, Granular matter (1999);
Boutreux & de Gennes, J. Phys. (1996);
Cannarsa & Cardaliaguet, JEMS (2004);
Crasta & Malusa, Calc. Var. (2012);
and the references therein ....

The mathematical framework on the network $\mathcal{N}$

- The network $\mathcal{N}$ is composed of
  $\mathcal{E} := \{e_1, \dots, e_M\} = M$ curves (edges) in $\mathbb{R}^N$
  $\mathcal{V} := \{x_1, \dots, x_N\} = N$ vertices (nodes) in $\mathbb{R}^N$



- The vertices are endpoints of the edges
- The closed edges $\bar{e}_j$ are parametrized by

$$\pi_j : [0, \ell_j] \mapsto \mathbb{R}^n$$

The π_j induce an orientation on each e_j but the results do not depend on it

- The vertices are either transition vertices or boundary vertices.

In particular, if x_i is the endpoint of only one edge, then it is a boundary vertex

- $\mathrm{Inc}_i := \{j \in \{1, \dots, M\} \, : \, x_i \in \bar{e}_j\}$

The mathematical framework on the network $\mathcal{N}$

- Two type of functions to be considered

$$u : \mathcal{N} \to \mathbb{R} \quad \text{and} \quad u = (u_{\bar{e}_j})_{j=1,\ldots,M} \ , \quad u_{\bar{e}_j} : \bar{e}_j \to \mathbb{R}$$

- To each of them we associate the projections

$$u_j(t) := u(\pi_j(t)) \quad \text{and} \quad u_j(t) := u_{\bar{e}_j}(\pi_j(t)) \, , \qquad t \in [0, \ell_j]$$

- With $Du(x)$, $x \in \mathcal{N}$, we denote $(D_j u(x))_{j=1,\ldots,M}$, where

$$D_j u(x) = u_j'(\pi_j^{-1}(x)) \, , \qquad \text{if } x \in e_j$$

$$D_j u(x_i) = \begin{cases} \lim\limits_{h \to 0^+} (u_j(h) - u_j(0))/h \, , & \text{if } x_i = \pi_j(0) \\ \lim\limits_{h \to 0^+} (u_j(\ell_j - h) - u_j(\ell_j))/h \, , & \text{if } x_i = \pi_j(\ell_j) \end{cases}$$

The mathematical framework on the network $\mathcal{N}$

$$\mathrm{Dist}(x,y) := \inf_{\mathcal{P}(x,y)} \{ |\pi_{j_1}^{-1}(x) - \pi_{j_1}^{-1}(x_1)| + \sum_{i=2}^{n} \ell_{j_i} + |\pi_{j_{n+1}}^{-1}(y) - \pi_{j_{n+1}}^{-1}(x_n)| \}$$

$$\int_{\mathcal{N}} u(x)dx = \sum_{j=1}^{M} \int_0^{\ell_j} u_j(t)dt$$

$$C(\mathcal{N}) := \{ (u_j)_{j=1,\dots M} : u_j \in C([0,\ell_j]) \text{ and } u_j(x_i) = u_k(x_i) \text{ if } j,k \in \mathrm{Inc}_i \}$$

$$C^1(\mathcal{N}) := \{ u \in C(\mathcal{N}) : u_j \in C^1([0,\ell_j]) \}$$

$$L^\infty(\mathcal{N}) := \prod_{j=1}^{M} L^\infty(0,\ell_j) \quad \text{and} \quad W^{1,\infty}(\mathcal{N}) := \prod_{j=1}^{M} W^{1,\infty}(0,\ell_j)$$

A weighted distance function

We assume a non constant "angle of repose" η^{-1} s.t.

$$(\eta_j)_j \in \prod_{j=1}^{M} C([0,\ell_j]) \quad \text{and} \quad \min_{j=1,\dots,M} \{\eta_{\bar{e}_j}(x);\, x \in \bar{e}_j\} > 0$$

We define the weighted metric

$$\mathcal{D}(x,y) := \inf_{\mathcal{P}(x,y)} \Big\{ \Big| \int_{\pi_{j_1}^{-1}(x)}^{t_1} \frac{1}{\eta_{j_1}(s)} ds \Big| + \sum_{i=2}^{n} \Big| \int_{t_{i-1}}^{t_i} \frac{1}{\eta_{j_i}(s)} ds \Big| + \Big| \int_{t_n}^{\pi_{j_{n+1}}^{-1}(y)} \frac{1}{\eta_{j_{n+1}}(s)} ds \Big| \Big\}$$

and the weighted distance function

$$d_{\partial\mathcal{N}}(x) := \min_{y \in \partial\mathcal{N}} \mathcal{D}(x,y)\,, \quad x \in \mathcal{N}$$

We say that (u, v) is a weak solution of

$$\begin{array}{ll} -D\left(v(x)\,\eta(x)\,Du(x)\right) = f(x) & \text{in } \mathcal{N}\setminus\partial\mathcal{N} \\ \eta(x)\,|Du(x)| \le 1 & \text{in } \mathcal{N}\setminus\partial\mathcal{N} \\ \eta(x)\,|Du(x)| - 1 = 0 & \text{in } \{x\in\mathcal{N}\setminus\partial\mathcal{N} \,:\, v(x)\neq 0\} \\ u,\ v \ge 0 & \text{in } \mathcal{N} \end{array}$$

if

(i) $v \ge 0$ and s.t. $v_j \in C([0,\ell_j])$ for $j = 1,\dots,M$

(ii) $u \in (W^{1,\infty}\cap C)(\mathcal{N})$, $u \ge 0$, $\eta(x)\,|Du(x)| \le 1$ a.e. in $\mathcal{N}\setminus\partial\mathcal{N}$

(iii) u is a viscosity solution of the eikonal equation in $\{x\in\mathcal{N}\setminus\partial\mathcal{N} :\ v(x)\neq 0\}$

(iv) $\forall\ \psi \in (W^{1,\infty}\cap C)(\mathcal{N})$ s.t. $\psi_{|\partial\mathcal{N}} = 0$:
$$\int_{\mathcal{N}} v\,\eta\,Du\,D\psi dx = \int_{\mathcal{N}} f\psi\,dx$$

(v) $u_{|\partial\mathcal{N}} = 0$

(vi) (u, v) satisfies transmission conditions at the transition vertex

The transmission conditions for (u, v)

$\rightsquigarrow$ The transmission conditions has to be defined so that the conservation of the flux at each transition vertices x_i is satisfied

$$\sum_{j \in \mathrm{Inc}_i} v_j(x_i)\, \eta_j(x_i)\, D_j u(x_i) = 0 \qquad (CF)$$

$\rightsquigarrow$ The conservation of the flux itself is not sufficient to define $v_j(x_i)$ at x_i for all $j \in \mathrm{Inc}_i$ and to guarantee the uniqueness of the solution (u, v)

$\rightsquigarrow$ The choice of the transmission conditions is not unique. Our choice amounts to impose that "the quantity of the rolling layer v entering in x_i along the *incoming edges*, is distributed to the *outgoing edges*, so that to satisfy (CF)"

The transmission conditions for (u, v)

• Given $u \in W^{1,\infty}(\mathcal{N})$, if x_i transition vertex and $j \in \mathrm{Inc}_i$ are s.t. $D_j u(x_i)$ exists and is not zero, we set

$$\sigma_{ij}(u) := \mathrm{sgn}[D_j u(x_i)]$$

Moreover, we set : $\mathrm{Inc}_i^{\pm}(u) := \{j \in \mathrm{Inc}_i : \sigma_{ij}(u) = \pm 1\}$

• Given $C_{ij} > 0$ s.t. $\sum_{j \in \mathrm{Inc}_i^-(u)} C_{ij} = 1$, the transmission conditions at x_i are as following :

$\rightsquigarrow$ if $\sigma_{ij}(u)$ is not defined and $j \in \mathrm{Inc}_i$ then

$$v_j(\pi_j^{-1}(x_i)) = 0$$

$\rightsquigarrow$ if $\sigma_{ij}(u)$ is defined and $j \in \mathrm{Inc}_i^-(u)$, then

$$v_j(\pi_j^{-1}(x_i)) = C_{ij} \sum_{k \in \mathrm{Inc}_i^+(u)} v_k(\pi_k^{-1}(x_i))$$

The viscosity solution definition

(i) $u \in C(\mathcal{N})$ is a sub-solution if for any $\phi \in C^1(\mathcal{N})$ and any $x \in e_j$, $j = 1, \dots, M$, s.t. $(u - \phi)$ attains a local maximum at x, it holds : $|D_j\phi(x)| - \frac{1}{\eta_j(x)} \leq 0$

(ii) $u \in C(\mathcal{N})$ is a super-solution if :

- for any $\phi \in C^1(\mathcal{N})$ and any $x \in e_j$, $j = 1, \dots, M$, s.t. $(u - \phi)$ attains a local minimum at x, it holds :

$$|D_j\phi(x)| - \frac{1}{\eta_j(x)} \geq 0$$

- for any $\phi \in C^1(\mathcal{N})$ and any transition vertex x_i s.t. $(u - \phi)$ attains a local minimum at x_i, it holds :

$$\max_{j \in \mathrm{Inc}_i} \left\{ |D_j\phi(x_i)| - \frac{1}{\eta_j(x_i)} \right\} \geq 0$$

(iii) $u \in C(\mathcal{N})$ is a solution if it is both a sub- and a super-solution

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See Lions & Souganidis, Rendiconti Lincei (2016), and the references therein ....

The u-component of the solution

The weighted distance function to $\partial\mathcal{N}$

$$d_{\partial\mathcal{N}}(x) := \min_{y\in\partial\mathcal{N}} \mathcal{D}(x,y)\,, \quad x\in\mathcal{N}$$

is the good candidate to be the viscosity solution of the eikonal equation

$$|Du(x)| - \frac{1}{\eta(x)} = 0 \qquad (Ek)$$

Proposition

For any fixed $x_0\in\mathcal{N}$, the function $\mathcal{D}(x_0,\cdot)$ is a viscosity solution of (Ek) in $\mathcal{N}\setminus(\partial\mathcal{N}\cup\{x_0\})$ and $d_{\partial\mathcal{N}}$ is the unique viscosity solution of (Ek) in $\mathcal{N}$ with $d_{\partial\mathcal{N}}=0$ on $\partial\mathcal{N}$.

What about the v-component of the solution ?

(i) $v \geq 0$ and s.t. $v_j \in C([0, \ell_j])$ for $j = 1, \dots, M$

(ii) $u \in (W^{1,\infty} \cap C)(\mathcal{N})$, $u \geq 0$, $\eta(x)\,|Du(x)| \leq 1$ a.e. in $\mathcal{N} \setminus \partial\mathcal{N}$

(iii) u is a viscosity solution of the eikonal equation in $\{x \in \mathcal{N} \setminus \partial\mathcal{N} : \; v(x) \neq 0\}$

(iv) $\forall\, \psi \in (W^{1,\infty} \cap C)(\mathcal{N})$ s.t. $\psi_{|\partial\mathcal{N}} = 0$:
$$\int_{\mathcal{N}} v\, \eta\, Du\, D\psi dx = \int_{\mathcal{N}} f \psi\, dx$$

(v) $u_{|\partial\mathcal{N}} = 0$

(vi) (u, v) satisfies transmission conditions at each transition vertex

What about the v-component of the solution ?

$\rightsquigarrow$ Hadeler & Kuttler, Granular matter (1999) :
"$v(x)$ is determined by adding up all the matter coming down (from the source) along the *transport ray* between the *singular set* of $\mathrm{dist}(\cdot, \partial\Omega)$ and x "

$\rightsquigarrow$ We need to define the singular set of the distance function $d_{\partial\mathcal{N}}$

$\rightsquigarrow$ If $\Omega \subset \mathbb{R}^n$ is smooth, the singular set of the euclidian distance from $\partial\Omega$ is the set of points where this function is not differentiable. Its closure coincides with the set of points having multiple geodesics connecting them to $\partial\Omega$

$\rightsquigarrow$ In the case of the network $\mathcal{N}$, the singular set of $d_{\partial\mathcal{N}}$ is the set of points where $d_{\partial\mathcal{N}}$ attains a local maximum

Singular set of $d_{\partial\mathcal{N}}$

Proposition

Let

$$S_j(d_{\partial\mathcal{N}}) := \{t \in (0, \ell_j) \ : \ d_j \textit{ is not differentiable at } t\}\,, \quad j = 1, \dots, M$$

It holds :

(i) $d_{\partial\mathcal{N}}$ *does not attain a local minimum on* $\mathcal{N} \setminus \partial\mathcal{N}$

(ii) $d_{\partial\mathcal{N}}$ *attains a local maximum at* $x \in e_j$ *iff* $\pi_j^{-1}(x) \in S_j(d_{\partial\mathcal{N}})$

(iii) $\# S_j(d_{\partial\mathcal{N}}) \in \{0, 1\}$

Definition

We define the singular set of $d_{\partial\mathcal{N}}$ as

$$S(d_{\partial\mathcal{N}}) := \{\pi_j(S_j(d_{\partial\mathcal{N}}))\,;\, j = 1, \dots, M\} \cup \{x_i \in \mathcal{V} \ : \ d_{\partial\mathcal{N}} \text{ has a local maximum at } x_i\}$$

The v-component of the solution

For $d = d_{\partial\mathcal{N}}$ we set

$$T_j(d) := \{\pi_j^{-1}(x_i)\,;\, x_i \in \overline{e}_j \text{ s.t. } j \in \mathrm{Inc}_i^-(d)\}$$
$$\Sigma_j(d) := S_j(d) \cup T_j(d)$$

$$P_j(t) := t + \tau_j(t)\,\eta_j(t)\,d_j'(t)\,, \qquad t \in [0, \ell_j]$$
$$\tau_j(t) := \min\{s \geq 0 :\ t + s\,\eta_j(t)\,d_j'(t) \in \Sigma_j(d)\}$$

and we define v component-wise as

$$v_j^f(t) = \int_0^{\tau_j(t)} f_j\left(t + r\,\eta_j(t)\,d_j'(t)\right) dr$$
$$+ \Big(C_{ij} \sum_{k \in \mathrm{Inc}_i^+(d)} v_k^f(\pi_k^{-1}(x_i))\Big)\chi_{T_j(d)}(P_j(t))$$

The existence result

Theorem (Cacace, Camilli, C.)

The pair $(d_{\partial\mathcal{N}}, v^f)$ is a weak solution of the differential system. Moreover,

(i) $v^f = 0$ *over the singular set* $S(d_{\partial\mathcal{N}})$

(ii) $v^f \in W^{1,\infty}(\mathcal{N})$

(iii) $(d_{\partial\mathcal{N}}, v^f)$ *satisfies*

$$-D(v\,\eta\,Du) = f$$

pointwise on $\mathcal{E} \setminus \{\pi_j(S_j(d_{\partial\mathcal{N}})); j = 1, \ldots, M\}$

Proof.

Choose ad hoc test functions and use the fact that one set between $S_j(d_{\partial\mathcal{N}})$ and $T_j(d_{\partial\mathcal{N}})$ is a singleton and the other one is empty $\square$

What about uniqueness ?

$$X := \{u \in (W^{1,\infty} \cap C)(\mathcal{N}) \ : \ \eta(x)|Du(x)| \le 1 \text{ a.e. } x \in \mathcal{N}\}$$

$$X_0 := \{u \in X \ : \ u = 0 \text{ on } \partial\mathcal{N}\}$$

$$u^f(x) := \max_{y \in \mathrm{supp}(f)} [d_{\partial\mathcal{N}}(y) - \mathcal{D}(x,y)]_+ \ , \qquad x \in \mathcal{N} \, ,$$

$\rightsquigarrow$ $d_{\partial\mathcal{N}}$ is the maximal nonnegative element in X_0

$\rightsquigarrow$ u^f is the minimal one in the following sense

Lemma

(i) $0 \le u^f \le d_{\partial\mathcal{N}}$ *in* $\mathcal{N}$ *and* $u^f = d_{\partial\mathcal{N}}$ *on* $supp(f)$

(ii) $u^f \in X_0$

(iii) u^f *is the smallest nonnegative function among the nonnegative functions* $u \in X$ *such that* $u = d_{\partial\mathcal{N}}$ *on* $supp(f)$

(iv) $u^f = d_{\partial\mathcal{N}}$ *in* $supp(v^f) = \Pi_{j=1}^{M} \pi_j(supp(v_j^f))$

(v) $u^f = d_{\partial\mathcal{N}}$ *in* $\mathcal{N}$ *iff* $S(d_{\partial\mathcal{N}}) \subset supp(f)$

Proof.

(iv) To prove that $u^f = d_{\partial\mathcal{N}}$ in $\mathrm{supp}(v^f)$ we prove that for all $x_0 \in \mathrm{supp}(v^f)$ there exists $x_1 \in \mathrm{supp}(f)$ s.t.

$$d_{\partial\mathcal{N}}(x_0) = d_{\partial\mathcal{N}}(x_1) - \mathcal{D}(x_0, x_1)$$

i.e. any geodesic path from $\mathrm{supp}(v^f)$ to $\partial\mathcal{N}$ is contained into at least one geodesic path from $\mathrm{supp}(f)$ to $\partial\mathcal{N}$

(v) To prove that $u^f = d_{\partial\mathcal{N}}$ in $\mathcal{N}$ iff $S(d_{\partial\mathcal{N}}) \subset \mathrm{supp}(f)$ we use in particular the fact that any geodesic path from $x \in \mathcal{N} \setminus S(d_{\partial\mathcal{N}})$ to $\partial\mathcal{N}$ does not cross $S(d_{\partial\mathcal{N}})$

$\square$

As a consequence of the previous Lemma, all nonnegative functions $u \in X_0$ s.t. $u = d_{\partial \mathcal{N}}$ on $\mathrm{supp}(f)$, satisfy also

$$u = u^f = d_{\partial \mathcal{N}} \quad \text{on } \mathrm{supp}(v^f)$$

Therefore, these functions u are all good candidates to be the first component of a weak solution, with v^f the second component

Question : does (u, v^f) satisfies the transmission conditions at the transition verteces ? only in the verteces x_i s.t. $v^f(x_i) > 0$ since there

$$\sigma_{ij}(u^f) = \sigma_{ij}(d_{\partial \mathcal{N}}) \quad \text{and} \quad \mathrm{Inc}_i^{\pm}(u^f) = \mathrm{Inc}_i^{\pm}(d_{\partial \mathcal{N}})$$

The uniqueness result

Theorem (Cacace, Camilli, C.)

If (u, v) is a weak solution, then

(i) $u = d_{\partial\mathcal{N}} = u^f$ *on* $supp(v^f)$

(ii) $v = v^f$ *on* $\Pi_{i=1}^M \bar{e}_j$

(iii) *if* $S(d_{\partial\mathcal{N}}) \subset supp(f)$, *then* $(u, v) = (d_{\partial\mathcal{N}}, v^f)$ *on* $\mathcal{N} \times \Pi_{i=1}^M \bar{e}_j$

Corollary

If (u, v) is a weak solution, then for all transition vertex x_i s.t. $v(x_i) \neq 0$, the sets $\mathrm{Inc}_i^{\pm}(u)$ are not empty and satisfy

(i) $\{j \in \mathrm{Inc}_i^+(d_{\partial\mathcal{N}}) \,:\, v_j(x_i) > 0\} \subseteq \mathrm{Inc}_i^+(u) \subseteq \mathrm{Inc}_i^+(d_{\partial\mathcal{N}})$

(ii) $\mathrm{Inc}_i^-(u) = \mathrm{Inc}_i^-(d_{\partial\mathcal{N}})$

Two key tools for the proof of the uniqueness result :

- the following partition $(\mathcal{E}_m)_m$ of $\mathcal{E}$

$$\begin{aligned}
&\mathcal{E}_0 = \mathcal{E}_0' \cup \mathcal{E}_0'' \\
&\mathcal{E}_0' := \{e_j \in \mathcal{E} \,:\, S_j(d_{\partial\mathcal{N}}) \neq \emptyset\} \\
&\mathcal{E}_0'' := \{e_j \in \mathcal{E} \,:\, \text{one endpoint is a maximum point of } d_{\partial\mathcal{N}}\}
\end{aligned}$$

and

$$\mathcal{E}_m := \{e_j \in \mathcal{E} \,:\, \exists\ i \in \mathcal{I}_T \text{ and } e_k \in \mathcal{E}_{m-1} \text{ s.t. } j,k \in \mathrm{Inc}_i\}$$

- the transmission conditions

Numerical approximations of $(d_{\partial\mathcal{N}}, v^f)$ and tests

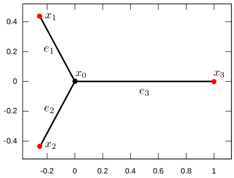
- To compute an approximation of $d_{\partial\mathcal{N}}$ we consider

$$\begin{cases} \max\limits_{y\in\mathcal{G}^h\,:\,y\sim x}\left(-\dfrac{u^h(y)-u^h(x)}{\mathrm{Dist}(x,y)}\right)-\dfrac{1}{\eta(x)}=0 & x\in\mathcal{G}^h\setminus\partial\mathcal{G}^h\\ u^h(x)=0 & x\in\partial\mathcal{G}^h \end{cases}$$

whose solution is

$$d^h(x)=\min\left(\sum_{m=0}^{n-1}\frac{1}{\eta(x_m)}\mathrm{Dist}\left(x_m,x_{m+1}\right)\right) \qquad x\in\mathcal{G}^h$$

- The formula for $(v_j^f)_{j=1,\dots,M}$ is approximated by a quadrature rule methode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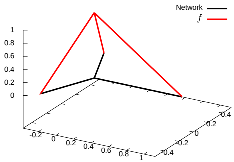
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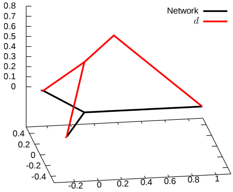
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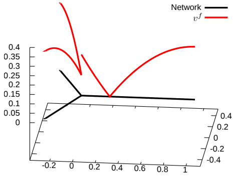
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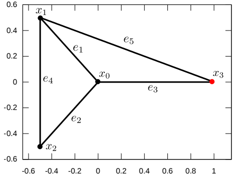
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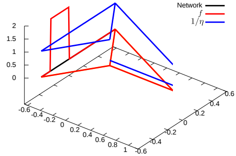
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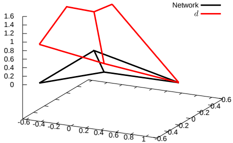
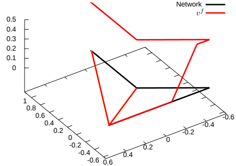
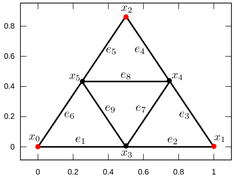
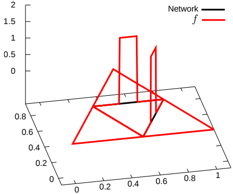
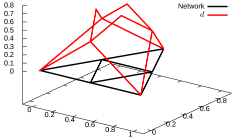
-0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1
-0.6 -0.4 -0.2 0 0.2 0.4 0.6

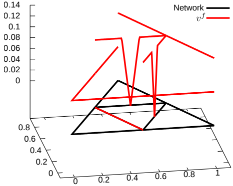

Network
d
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1
-0.6 -0.4 -0.2 0 0.2 0.4 0.6


Network
v^f
0.5
0.4
0.3
0.2
0.1
0
1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6
0.6 0.4 0.2 0 -0.2 -0.4 -0.6


x_0
x_1
x_2
x_3
x_4
x_5
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9


Network
f


Network
d


Network
v^f

The software SPNET

The software **SPINET** (for **S**and **P**iles on **NET**works) for the numerical approximations is due to Simone Cacace and it can be downloaded at
`http://www.dmmm.uniroma1.it/~fabio.camilli/spnet.html`.

The associated variational problem

$$\sup_{u \in X_0} \int_{\mathcal{N}} f(x)\, u(x)\, dx$$

or

$$\inf_{u \in X_0} \int_{\mathcal{N}} \left(\mathbf{1}_{[-1,1]}(\eta(x) Du(x)) - f(x)\, u(x)\, dx\right)$$

- $d_{\partial\mathcal{N}}$ is the maximal solution
- u^f is the minimal solution
- $u \in X_0$ is a solution iff $u^f \leq u \leq d_{\partial\mathcal{N}}$ in $\mathcal{N}$
- $u \in X_0$ is a solution iff there exists $(v_j)_j \in \Pi_{j=1}^M C([0, \ell_j])$ s.t.

$$\int_{\mathcal{N}} v\, \eta\, Du\, D\psi dx = \int_{\mathcal{N}} f \psi\, dx \qquad \forall\, \psi \in (W^{1,\infty} \cap C)(\mathcal{N})\,,\ \psi_{|\partial\mathcal{N}} = 0$$

Merci !