

The MBR social welfare criterion meets Rawls' view of intergenerational equity

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Abstract

In this paper we further explore the properties of the Mixed Bentham-Rawls (MBR) criterion for intergenerational justice that was introduced in Alvarez-Cuadrado and Long (2009). In a simple economy with a natural resource, cast in discrete time, we study the problem to find the best exploitation policy under the MBR criterion. We find out sufficient conditions for optimality, that complements the necessary first order conditions offered in Alvarez-Cuadrado and Long (2009). Then, we retain a specified version of the model in which we compute the optimal path and shed light on the consequences of using this criterion from the point of view of equity between generations. Interestingly enough, the best program under the MBR criterion shows a pattern of exploitation of the natural resource that is in line with recent proposals made by philosophers to go beyond the rawlsian criterion.

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1 Introduction

How could society best use natural resources overtime while ensuring a fair share of the benefits to all the successive generations? A traditional approach to this question is to formulate a criterion - also called an intertemporal social welfare function (ISWF) - that meets various normative conditions. One can then order alternative exploitation paths and single out the best of them, according to the chosen ISWF. Among the most well known ISWFs, one finds: the discounted utilitarian criterion, the golden rule, Ramsey's criterion, the average utilitarian criterion, the maximin (Rawlsian) criterion. Each has its advantages but none of them achieves the Holy Grail of this literature, *i.e.* no ISWF provides a complete ordering that satisfies both a concern for efficiency and the requirement that all generations should be treated equally (or anonymously). Optimists hold that implicit criteria can be constructed that meet those two requirements (Svensson, 1980). This, however, is an existence result that offers no explicit ISWF, therefore no practical way to identify fair and efficient management paths. Fleurbaey and Michel (2003) conjectured that "no explicit complete continuous ordering will ever be found that satisfies efficiency and weak anonymity"¹.

Those developments suggest we should now look for criteria that are less demanding about either efficiency or equity, or both. This new perspective finds a starting point in Chichilnisky's contribution (1996), who suggested to adopt a mixed criterion, which can be presented roughly as a convex combination of the discounted utilitarian criterion and the golden rule. Her ISWF achieves efficiency; besides it avoids dictatorship of the present generation and dictatorship of the future generations as well (see also Chichilnisky 1997, 2009). But of course other mixed criteria could be contemplated as well, and their respective merits be assessed. Alvarez-Cuadrado and Long (2009) began the analysis of an ISWF constructed as a convex combination of the discounted utilitarian and maximin criteria, which they called MBR (for mixed Bentham-Rawls). This ISWF avoids the dictatorships of the present and the future. Besides it also escapes the dictatorship of the least advantaged generations (see Alvarez-Cuadrado and Long, 2009).

In this paper we further explore the properties of the MBR criterion. More precisely, in a simple economy with a natural resource, cast in discrete time (Section 2), we study the problem to maximize the MBR criterion. In Section 3 we find out sufficient conditions for optimality, that complements the necessary first order conditions offered in Alvarez-Cuadrado and Long (2009). Then, in Section 4 we retain a specified version of the model in which we compute the optimal path and compare it with optimal paths that obtains under other ISWF, namely the discounted utilitarian criterion, the golden rule criterion, the maximin (Rawlsian) criterion. We are then in position to shed light on the consequences of using these criteria from the point of view of equity between generations. Interestingly enough, the best program under the MBR criterion

¹We have recently been informed of a manuscript by Lauwers (2007) that confirms this conjecture. See also Zame (2007).

shows a pattern of exploitation of the natural resource that is in line with recent proposals made by philosophers to go beyond the rawlsian criterion.

2 A natural resource economy

The economy has infinitely many successive generations. Let c_t be the consumption by generation t of a natural resource whose stock is x_t . This resource evolves according to:

$$x_{t+1} = x_t + f(x_t, c_t) . \quad (1)$$

When consuming c_t generation t enjoys a life-time utility level, or under a more adequate interpretation a standard of living, $U_t \equiv U(c_t)$. The utility function is continuous, non decreasing, concave and admits an upper and a lower bound. Without loss of generality, the numbers U_t can be normalized so that $0 \leq U_t \leq 1$. To any admissible path $\{c_t, x_t\}_{t=0}^{\infty}$, the MBR criterion associates the following value:

$$\theta \inf \{U(c_0), U(c_1), \dots\} + (1 - \theta) \sum_{t=0}^{\infty} \beta^t U(c_t) , \quad 0 < \theta < 1 .$$

3 Necessary and sufficient conditions for optimality

Consider first the finite horizon version of the problem, with $T < \infty$ periods: given the initial stock $x_0 > 0$, and a time path c_0, c_1, \dots, c_T of extractions, producing a path x_1, x_2, \dots, x_T for the stock, let $\underline{c} \equiv \inf \{c_0, c_1, \dots\}$ and $\underline{U} \equiv U(\underline{c})$. The corresponding social welfare under the MBR criterion is:

$$W^{mbr} = \theta \underline{U} + (1 - \theta) \sum_{t=0}^T \beta^t U(c_t)$$

The problem is to find \underline{U} and the path of extractions in order to maximize W^{mbr} subject to eq (1) and

$$c_t - \underline{c} \geq 0 \text{ for } t = 0, 1, \dots, T$$

For the time being, assume that x_{T+1} is fixed.

This kind of maximization exercise is unlike usual dynamic problems where standard optimization tools can be used. The necessary conditions for optimality have been derived in Alvarez-Cuadrado and Long (2009), from Hestenes' theorem. Let the trajectory $\{c_t^*, x_t^*, \lambda_t^*, \omega_t^*, \underline{c}^*\}_{t=0}^T$ denotes a solution to the MBR problem.

Definition 1 Let's call **Transformed Lagragian** the following function

$$\mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*) \equiv (1 - \theta)\beta^t U(c_t) + \lambda_t^* f(x_t, c_t) + \omega_t^* (c_t - \underline{c}) .$$

Then the following system of necessary conditions is verified at $(c_t, x_t, \underline{c}) = (c_t^*, x_t^*, \underline{c}^*)$, for $t = 0, 1, \dots, T$:

$$\frac{\partial \mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*)}{\partial c_t} = (1 - \theta)\beta^t U'(c_t) + \omega_t^* + \lambda_t^* \frac{\partial f}{\partial c_t} = 0, \quad (2)$$

$$\lambda_t^* - \lambda_{t-1}^* = -\frac{\partial \mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*)}{\partial x_t} = -\lambda_t^* \frac{\partial f}{\partial x_t}, \quad (3)$$

$$x_{t+1} = x_t + f(x_t, c_t),$$

$$\theta U'(\underline{c}) - \sum_{t=0}^T \frac{\partial \mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*)}{\partial \underline{c}} = 0. \quad (4)$$

In applications one finds several candidate programs that solves the necessary conditions. Some of them may correspond to minima instead of maxima. Further conditions are needed to help discard the former, and are offered in the following theorem.

Theorem 2 (*Sufficient conditions*) Let $(\{c_t^*, x_t^*, \lambda_t^*, \omega_t^*\}, \underline{c}^*)$ denote a solution to the necessary conditions. Let $\{c'_t, x'_t\}$ be any feasible path. If the following conditions hold:

- (a) $\mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*)$ is concave in $(c_t, x_t, \underline{c})$ and
 - (b) $\lambda_T^*(x'_{T+1} - x_{T+1}^*) \geq 0$,
- then $\{c'_t, x'_t\}$ is the optimal path.

Proof: Appendix

For the infinite version of the problem, the necessary conditions are unchanged. Regarding the sufficient conditions, we replace (b) of the theorem by the condition

$$\lim_{t \rightarrow \infty} \lambda_t x_t^* = 0 \text{ and } \lim_{t \rightarrow \infty} \lambda_t = 0$$

Since $x'_{T+1} \geq 0$, this implies

$$\lim_{T \rightarrow \infty} \lambda_T^*(x'_{T+1} - x_{T+1}^*) \geq 0$$

4 Implications in a simple example

We now investigate the consequences of using the MBR criterion in a simple example, where further analytical results can be obtained. Let the transition be specified as follows:

$$x_{t+1} = (x_t - c_t)^\alpha, \quad \alpha \in (0, 1).$$

And let the life-time utility from consumption be a linear function:

$$u(c_t) = c_t.$$

Appendices B,C,D and E compute the discounted utilitarian path, the golden rule, the maximin solution and the MBR path. The steady state of the first one gives the so-called *modified golden rule*:

$$x^{mgr} := (\alpha\beta)^{\frac{\alpha}{1-\alpha}},$$

while the golden rule stock is:

$$x^{gr} := \alpha^{\frac{\alpha}{1-\alpha}}.$$

Turning back to the MBR criterion, it is worth noting that this example satisfies the sufficient conditions given in the previous section. Hence the solutions to the necessary conditions, if there exists any, are optimal (and interior).

First, we can prove that poorest generations are the first ones or the last ones (intermediate generations cannot be the poorest).

Proposition 3 *Calling $\underline{c} = c_t$ the lowest consumption level of an optimal policy under the MBR criterion, then the following pattern:*

$$\underline{c} < c_{t-1}, \quad \underline{c} < c_{t+1}, \quad \text{for some } t \geq 1,$$

cannot occur.

Proof. Appendix F. ■

We will now see that the poorest generations are in the tail (resp. at the beginning) when the economy starts above (below) the modified golden rule.

4.1 When the economy starts "rich" (above the modified golden rule: $x_0 \geq x^{mgr}$)

Let

$$\begin{aligned} \hat{A} &\equiv \frac{\theta(1-\beta) + \beta(1-\theta)}{\theta(1-\beta) + (1-\theta)} = \frac{\theta + \beta - 2\beta\theta}{1 - \beta\theta}, \\ x_H &\equiv \left[\alpha \hat{A} \right]^{\alpha/(1-\alpha)}. \end{aligned}$$

Proposition 4 *If the initial condition x_0 satisfies*

$$x^{mgr} \leq x_0 \leq x_H$$

then $x_t = x_0$ for all $t = 0, 1, 2, \dots$ is an optimal program for the MBR criterion.

Proof. Appendix G. ■

Remark 5 Clearly, as $\theta \rightarrow 1$, $x_H \rightarrow x^{gr}$, and as $\theta \rightarrow 0$, $x_H \rightarrow x^{mgr} \equiv (\alpha\beta)^{\alpha/(1-\alpha)}$.

Proposition 6 (ONE DOWNWARD JUMP) If $x_0 > x_H$ then the policy made of one jump downward to x_H and then stay there for ever is an optimal solution.

Proof. Appendix H. ■

4.2 When the economy starts "poor" (below the modified golden rule: $x_0 < x^{mgr}$)

Unlike the solution dictated by the maximin criterion, the MBR ISWF does not require the economy to remain trapped in its initial poverty.

Proposition 7 Starting from $x_0 < x^{mgr}$, it is never optimal to put $x_t = x_0$ for all t .

Proof. Appendix I. ■

Better still, the economy should take off to reach, in a finite time, the modified golden rule as can be deduced from the two following propositions.

Proposition 8 Consumption paths cannot be non-monotone. If $c_t > \underline{c} \forall t > h$ then $x_{t+1} = x^{mgr} \forall t > h$, and $c_{t+1} = c_\infty$ for $\forall t > h$

Proof. Appendix J. ■

Proposition 9 If $c_t > \underline{c} \forall t \geq T$ then $x_t = x^{mgr} \forall t \geq T + 1$.

Proof. Appendix K. ■

Hence development is made of two phases. But its exact pattern, in particular the number of poorest generations in the take off phase, depends on the initial condition.

Given proposition 8, we are interested in studying consumptions of the form

$$\underline{c} = c_0 = \dots = c_{n-1} < c_n < c_\infty = \dots c_\infty. \quad (5)$$

We define:

$$x_1 = f_1(\underline{c}) = (x_0 - \underline{c})^\alpha, \quad x_n = f_n(\underline{c}) = (f_{n-1}(\underline{c}) - \underline{c})^\alpha, \forall n \geq 2.$$

We have the following properties for functions $f_n(\underline{c})$ and $c_n = c_n(\underline{c})$:

- For consumptions of the form (5),

$$x_{n+1} = x^{mgr} = (f_n(\underline{c}) - c_n)^\alpha \quad (6)$$

then

$$c_n(\underline{c}) = f_n(\underline{c}) - (x^{mgr})^{1/\alpha}. \quad (7)$$

- We now show by induction that $f'_n(\underline{c}) < 0$ and $f''_n(\underline{c}) < 0$. In fact:

$$f'_1(\underline{c}) = -\alpha(x_0 - \underline{c})^{\alpha-1} < 0, \quad f''_1(\underline{c}) = \alpha(\alpha-1)(x_0 - \underline{c})^{\alpha-2} < 0,$$

Now, if $f'_{n-1}(\underline{c}) < 0$ and $f''_{n-1}(\underline{c}) < 0$, then

$$f'_n(\underline{c}) = \alpha(f_n(\underline{c}))^{(\alpha-1)/\alpha}(f'_{n-1}(\underline{c}) - 1) < 0,$$

and

$$\begin{aligned} f''_n(\underline{c}) &= \alpha \frac{\alpha-1}{\alpha} (f_n(\underline{c}))^{((\alpha-1)/\alpha)-1} [f'_n(\underline{c})] (f'_{n-1}(\underline{c}) - 1) + \alpha (f_n(\underline{c}))^{(\alpha-1)/\alpha} f''_{n-1}(\underline{c}) \\ &= (\alpha)^2 \left(\frac{\alpha-1}{\alpha}\right) (f_n(\underline{c}))^{((\alpha-1)/\alpha)-1} (f'_{n-1}(\underline{c}) - 1)^\alpha + \alpha (f_n(\underline{c}))^{(\alpha-1)/\alpha} f''_{n-1}(\underline{c}) < 0 \end{aligned}$$

Now we consider the following problem, that corresponds to the optimization of the MBR criterion w.r.t. \underline{c} of a consumption given by (5):

$$\max_{\underline{c}} \left\{ [\theta + (1-\theta)(1+\beta+\dots+\beta^{n-1})] \underline{c} + (1-\theta)\beta^n c_n(\underline{c}) \right\} := F(\underline{c}) \quad (8)$$

where

$$x_1 = (x_0 - \underline{c})^\alpha, x_2 = (x_1 - \underline{c})^\alpha, \dots, x_n = (x_{n-1} - \underline{c})^\alpha, x^{mgr} = (x_n - c_n)^\alpha, \quad x_0 \text{ given.} \quad (9)$$

We must also impose the constraint that

$$c_n(\underline{c}) - \underline{c} \geq 0 \quad (10)$$

Define

$$K_n = \theta + (1-\theta)(1+\beta+\dots+\beta^{n-1}).$$

Let $\mu \geq 0$ be the Lagrange multiplier associated with the constraint (10). The first order condition of problem (8) is

$$F'(\underline{c}) + \mu [f'_n(\underline{c}) - 1] \leq 0 \quad (= 0 \text{ if } \underline{c} > 0)$$

i.e., if $\underline{c} > 0$,

$$K_n + (1-\theta)\beta^n f'_n(\underline{c}) + \mu [f'_n(\underline{c}) - 1] = 0 \quad (11)$$

where

$$\mu \geq 0, c_n(\underline{c}) - \underline{c} \geq 0, \mu [c_n(\underline{c}) - \underline{c}] = 0$$

We can also see that $F(\underline{c})$ is a concave function. In fact

$$F''(\underline{c}) = f''_n(\underline{c}) < 0.$$

Note also that

$$F'(0) = K_n - (1-\theta)(\beta\alpha)^n (x_0)^{n(\alpha-1)} \text{ which is } > 0 \text{ if } x_0 \text{ is not too small}$$

Note that an unique solution of problem (8) exists when $0 \leq \underline{c} \leq c_n(\underline{c})$. But what is the relation between problem (8) and an interior optimal solution of our MBR problem? The answer is given in the following proposition.

Proposition 10 *First order conditions of the MBR problem for a consumption path given by (5) are reduced to the existence of $0 < \underline{c} < c_n(\underline{c})$ that verifies the first order condition of problem (8), that is condition (11). In others words, if there exists initial conditions x_0 where (11) is verified with $0 < \underline{c} < c_n(\underline{c}) < c_\infty$, there exists initial conditions where the path given by (5) is an optimal solution of the MBR problem.*

Proof. First note that the following equivalences hold (equivalences related with equation (11) when $\mu = 0$):

$$\begin{aligned}
& K_n - (1 - \theta)\beta^n \alpha^n x_n^{(\alpha-1)/\alpha} \dots x_1^{(\alpha-1)/\alpha} - (1 - \theta)\beta^n \alpha^{n-1} x_n^{(\alpha-1)/\alpha} \dots x_2^{(\alpha-1)/\alpha} - \dots \\
& \quad - (1 - \theta)\beta^n \alpha x_n^{(\alpha-1)/\alpha} = \\
& = K_n - (1 - \theta)\beta^n \alpha x_n^{(\alpha-1)/\alpha} \left[1 + \alpha x_{n-1}^{(\alpha-1)/\alpha} \left[1 + \alpha x_{n-2}^{(\alpha-1)/\alpha} \dots \left[1 + \alpha x_1^{(\alpha-1)/\alpha} \right] \right] \right] = \\
& = K_n - (1 - \theta)\beta^n \alpha x_n^{(\alpha-1)/\alpha} \left[-f'_{n-1}(\underline{c}) + 1 \right] = K_n + (1 - \theta)\beta^n f'_n(\underline{c}). \quad (12)
\end{aligned}$$

A consumption of the form (5) verifies (9) and (7).

Suppose, for given x_0 , we have found an integer n and a real number $\underline{c} > 0$ such that condition (11) is satisfied with $\mu = 0$. Then we can compute $x_1, \dots, x_n, c_n(\underline{c})$. Suppose that $c_n(\underline{c}) > \underline{c}$. Then we can construct the multipliers λ_t and ω_t recursively as follows. First, since $x_{n+1} = x^{mgr}$, and $c_n > \underline{c}$, λ_n must satisfy

$$(1 - \theta)\beta^n + 0 - \lambda_n \alpha (x^{mgr})^{(\alpha-1)/\alpha} = 0$$

Next,

$$\lambda_{n-1} = \lambda_n \alpha (x^{mgr})^{(\alpha-1)/\alpha} \Rightarrow \lambda_{n-1} = (1 - \theta)\beta^n$$

Then it follows that

$$\begin{aligned}
\omega_{n-1} & = \lambda_{n-1} \alpha (x_n)^{(\alpha-1)/\alpha} - (1 - \theta)\beta^{n-1} = \\
& = (1 - \theta)\beta^{n-1} \left[(\beta \alpha) (x_n)^{(\alpha-1)/\alpha} - 1 \right] > 0 \\
\lambda_{n-2} & = \lambda_{n-1} \alpha x_n^{(\alpha-1)/\alpha} = (1 - \theta)\beta^{n-1} \alpha x_n^{(\alpha-1)/\alpha} \\
\omega_{n-2} & = \lambda_{n-2} \alpha x_{n-1}^{(\alpha-1)/\alpha} - (1 - \theta)\beta^{n-2} = \\
& = (1 - \theta)\beta^{n-2} \left[(\beta \alpha) x_{n-1}^{(\alpha-1)/\alpha} - 1 \right] > 0
\end{aligned}$$

etc.

Now we must verify to see whether the constructed $\omega_0, \dots, \omega_{n-1}$ satisfy the condition

$$\omega_0 + \omega_1 + \dots + \omega_{n-1} = \theta$$

Taking into account (12), the definition of K_n and the fact that \underline{c} verifies (11) with $\mu = 0$:

$$\begin{aligned}
& \omega_0 + \omega_1 + \dots + \omega_{n-1} = \\
& -(1 - \theta)(1 + \beta + \dots + \beta^{n-1}) + (1 - \theta)\beta^n \alpha^n x_n^{(\alpha-1)/\alpha} \dots x_1^{(\alpha-1)/\alpha} - \\
& (1 - \theta)\beta^n \alpha^{n-1} x_n^{(\alpha-1)/\alpha} \dots x_2^{(\alpha-1)/\alpha} - \dots - (1 - \theta)\beta^n \alpha x_n^{(\alpha-1)/\alpha} = \\
& = \theta - K_n - (1 - \theta)\beta^n f'_n(\underline{c}) = \theta.
\end{aligned}$$

Then we have proven that \underline{c} giving by (11) with $\mu = 0$ and the defined values of ω_i and λ_i verify the first order conditions of the MBR problem.

Reciprocally, first order conditions of the MBR problem are:

$$\begin{aligned}
& \omega_0 + \dots + \omega_{n-1} = \theta. \\
& (1 - \theta)\beta^i + \omega_i - \lambda_i \alpha x_{i+1}^{(\alpha-1)/\alpha} = 0, \quad i = 0 \dots n - 1. \quad (13) \\
& \lambda_{i-1} = \lambda_i \alpha x_{i+1}^{(\alpha-1)/\alpha}, \quad \forall i.
\end{aligned}$$

From these two last equations we can compute all values of λ_i , $i = 0 \dots n - 1$. Summing from $i = 0$ to $n - 1$ equations (13) we obtain

$$\begin{aligned}
& K_n - (1 - \theta)\beta^n \alpha^n x_n^{(\alpha-1)/\alpha} \dots x_1^{(\alpha-1)/\alpha} - (1 - \theta)\beta^n \alpha^{n-1} x_n^{(\alpha-1)/\alpha} \dots x_2^{(\alpha-1)/\alpha} - \dots \\
& - (1 - \theta)\beta^n \alpha x_n^{(\alpha-1)/\alpha} = 0,
\end{aligned}$$

from (12) this is equivalent to the first order condition of problem (8). ■

5 Conclusion

This paper further explore the properties of the MBR intertemporal welfare criterion proposed by Alvarez-Cuadrado and Long (2009). We find out sufficient conditions for optimality for the general formulation of the problem. Besides, for a simple natural resource economy we compute the optimal path and shed light on the consequences of using this criteria from the point of view of equity between generations.

Interestingly enough, the best exploitation program under the MBR criterion shows a pattern of exploitation of the natural resource that is in line with recent proposals made by philosophers to go beyond the Rawlsian criterion. Those scholars, including Rawls, argue that the maximin does not result in an acceptable normative prescription for poor economies, since its focus on equality implies no saving, hence no growth, and does not permit to reach the threshold at which basic liberties can be realized. They advocate an expansion in two stages, with an accumulation phase followed by a cruise phase. In the cruise phase, equality is justified again. This pattern is exactly to one obtained under the MBR intergenerational welfare criterion.

Appendix

A Derivation of the sufficient conditions

Let

$$V^* \equiv \theta U(\underline{c}^*) + (1 - \theta) \sum_{t=0}^T \beta^t U(c_t^*)$$

and

$$V \equiv \theta U(\underline{c}') + (1 - \theta) \sum_{t=0}^T \beta^t U(c_t')$$

Then

$$V^* = \theta U(\underline{c}^*) + \sum_{t=0}^T \{ \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \lambda_t^*(x_{t+1}^* - x_t^*) \}$$

because $\omega_t^*(c_t^* - \underline{c}^*) = 0$. And

$$V = \theta U(\underline{c}') + \sum_{t=0}^T \{ \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) - \lambda_t^*(x'_{t+1} - x'_t) - \omega_t^*(c_t' - \underline{c}') \}$$

Then, since $\omega_t^* \geq 0$ and $(c_t' - \underline{c}') \geq 0$,

$$V \leq \theta U(\underline{c}') + \sum_{t=0}^T \{ \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) - \lambda_t^*(x'_{t+1} - x'_t) \}$$

i.e.

$$-V \geq -\theta U(\underline{c}') - \sum_{t=0}^T \{ \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) - \lambda_t^*(x'_{t+1} - x'_t) \}$$

Thus

$$\begin{aligned} V^* - V &\geq \theta [U(\underline{c}^*) - U(\underline{c}')] + \sum_{t=0}^T \{ \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) \} \\ &\quad + \sum_{t=0}^T \lambda_t^*(x'_{t+1} - x'_t) - \sum_{t=0}^T \lambda_t^*(x_{t+1}^* - x_t^*) \end{aligned}$$

Now, from the assumption that $\mathcal{L}_t(c_t, x_t, \underline{c}, \lambda_t^*, \omega_t^*)$ is concave in $(c_t, x_t, \underline{c})$,

$$\begin{aligned} &\mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) \geq \\ &(c_t^* - c_t') \frac{\partial \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*)}{\partial c_t^*} + (x_t^* - x_t') \frac{\partial \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*)}{\partial x_t^*} + \end{aligned}$$

$$(\underline{c}^* - \underline{c}') \frac{\partial \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*)}{\partial \underline{c}^*}$$

Thus,

$$\begin{aligned} & \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) \geq \\ & (x_t^* - x_t') \lambda_t^* \frac{\partial f}{\partial x_t^*} + (\underline{c}^* - \underline{c}') \omega_t^* \text{ for } t = 0 \end{aligned}$$

But $x_0^* = x_0' = x_0$ given, so

$$\mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) \geq (\underline{c}^* - \underline{c}') \omega_t^* \text{ for } t = 0$$

And

$$\begin{aligned} & \mathcal{L}_t(c_t^*, x_t^*, \underline{c}^*, \lambda_t^*, \omega_t^*) - \mathcal{L}_t(c_t', x_t', \underline{c}', \lambda_t^*, \omega_t^*) \geq \\ & (x_t^* - x_t') (\lambda_{t-1}^* - \lambda_t^*) + (\underline{c}^* - \underline{c}') \omega_t^* \text{ for } t = 1, 2, \dots, T \end{aligned}$$

Hence

$$\begin{aligned} V^* - V & \geq \theta [U(\underline{c}^*) - U(\underline{c}')] + \sum_{t=0}^T (\underline{c}^* - \underline{c}') \omega_t^* + \\ & x_1^* \lambda_0^* - x_1^* \lambda_1^* + x_2^* \lambda_1^* - x_2^* \lambda_2^* + \dots + x_T^* \lambda_{T-1}^* - x_T^* \lambda_T^* \\ & - x_1' \lambda_0^* + x_1' \lambda_1^* + \dots - x_T' \lambda_{T-1}^* + x_T' \lambda_T^* + \sum_{t=0}^T \lambda_t^* (x_{t+1}' - x_t') - \sum_{t=0}^T \lambda_t^* (x_{t+1}^* - x_t^*) = \\ & \theta [U(\underline{c}^*) - U(\underline{c}')] + \left\{ \sum_{t=0}^T (\underline{c}^* - \underline{c}') \omega_t^* \right\} + \lambda_T^* (x_{T+1}' - x_{T+1}^*) = \\ & = \lambda_T^* (x_{T+1}' - x_{T+1}^*) \geq 0 \end{aligned}$$

(using eq (??), and condition (b)).

This completes the proof.

B The discounted utilitarian criterion

If $\beta \in]0, 1[$ stands for the planner's discount factor, the discounted utilitarian problem reads as:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t, \quad (14)$$

subject to the constraint

$$x_{t+1} = (x_t - c_t)^\alpha, \quad x_0 \text{ given.}$$

Using the maximum principle and asking for an interior solution, we obtain (as in the continuous case) only the stationary steady state (the so-called modified golden rule) given by

$$x^m := (\alpha \beta)^{\frac{\alpha}{1-\alpha}}.$$

To avoid this turnpike issue we propose to solve the same problem but with a finite horizon T and take the limit of the solution when T goes to infinity. Now, the problem is:

$$\max_{\{c_t\}_0^T} \sum_{t=0}^T \beta^t c_t + \beta^{T+1} x_{T+1}, \quad (15)$$

subject to the constraint

$$x_{t+1} = (x_t - c_t)^\alpha, \quad t = 0 \dots T, \quad x_0 \text{ given.}$$

Using dynamic programming, it is easy to check that problem (15) has the following optimal solution c_t^{*T} for all $t = 0, \dots, T$:

$$c_t^{*T} = \begin{cases} 0 & \text{if } x_t < \bar{x} \\ x_t - \bar{x} & \text{if } x_t \geq \bar{x}, \end{cases}$$

where $\bar{x} = (\alpha\beta)^{\frac{1}{1-\alpha}}$. The optimal consumption is zero when the stock is strictly below the steady state; otherwise it is optimal to consume the amount exactly necessary to settle the economy to its steady state. As we can see, this solution does not depend on T so, taking the limit when T goes to infinity, the optimal solution for our problem (14) is again:

$$c_t^* = \begin{cases} 0 & \text{if } x_t < \bar{x} \\ x_t - \bar{x} & \text{if } x_t \geq \bar{x}. \end{cases}$$

As far as the stock of the resource is concerned, the optimal path is, if $x_0 \geq \bar{x}$:

$$x_t^* = x^\infty, \quad \forall t \geq 1,$$

and if $x_0 < \bar{x}$

$$x_t^* = \begin{cases} (x_0)^{t\alpha} & t = 0 \dots t_0, \\ x^\infty & t > t_0, \end{cases}$$

where t_0 is such that $(x_0)^{t_0\alpha} \geq x^\infty$.

C The golden rule criterion

The problem is to find the highest stationary utility level. Formally:

$$\max_c c \quad (16)$$

such that

$$x = (x - c)^\alpha.$$

The solution to this program is

$$x^{GR} := (\alpha)^{\frac{\alpha}{1-\alpha}}, \quad c^{GR} := x^{GR} - (x^{GR})^{1/\alpha}.$$

D The maximin (Rawls) criterion

Under this criterion, the goal is to achieve the best utility level for the less advantaged generation. Or:

$$\max_{\bar{c}} \bar{c} \quad (17)$$

such that

$$x_{t+1} = (x_t - c_t)^\alpha, \quad x_0 \text{ given}, \quad x_t \geq 0, \quad c_t \geq \bar{c}.$$

We can remark that if there exists a date t_0 such that $c_{t_0} = 0$, then $\bar{c} = 0$ and that if there exists $c_{t_0} = x_{t_0}$, then $c_{t_0+1} = 0$ and $\bar{c} = 0$. So we can deduce that the optimal solution is an interior solution.

Appendix ?? gives the derivation of the solution which is:

- if $x_0 \geq x^{GR}$, the solution is the constant consumption path $c = c^{GR}$, which navigates the stock to a steady state value $x^H \geq x^{GR}$,
- if $x_0 < x^{GR}$ then the solution of (??) is $c = x_0 - x_0^{1/\alpha}$, which implies $x_t = x_0$ for all t .

E The mixed Bentham-Rawls (MBR) criterion

Inspired by the above analysis, the mixed Bentham-Rawls problem reads as:

$$\max_{\bar{c}, c_t} \left[\theta \bar{c} + (1 - \theta) \sum_{t=0}^{\infty} \beta^t c_t \right], \quad (18)$$

such that,

$$x_{t+1} = (x_t - c_t)^\alpha, \quad x_0, \text{ given}, \quad x_t \geq 0, \quad c_t \geq \bar{c} \geq 0.$$

Let the Lagrangian be

$$L = \theta \bar{c} + (1 - \theta) \sum_{t=0}^{\infty} \beta^t c_t + \sum_{t=0}^{\infty} \omega_t (c_t - \bar{c}) + \sum_{t=0}^{\infty} \lambda_t ((x_t - c_t)^\alpha - x_{t+1});$$

The first order conditions are:

$$\frac{\partial L}{\partial c_t} = (1 - \theta) \beta^t + \omega_t - \lambda_t \alpha (x_t - c_t)^{\alpha-1} = 0, \quad t = 0, 1, \dots \quad (19)$$

$$\frac{\partial L}{\partial x_t} = -\lambda_{t-1} + \lambda_t \alpha (x_t - c_t)^{\alpha-1} = 0, \quad t = 1, 2, \dots, \quad (20)$$

$$\frac{\partial L}{\partial \bar{c}} = \theta - \sum_{t=0}^{\infty} \omega_t = 0.$$

$$\omega_t \geq 0, \omega_t (c_t - \bar{c}) = 0, c_t - \bar{c} \geq 0 \quad (21)$$

F Proof of Proposition 3

For this policy we have

$$\omega_{t-1} = \omega_{t+1} = 0, \quad \omega_t > 0,$$

$$x_t = (x_{t-1} - c_{t-1})^\alpha, \quad x_{t+1} = (x_t - \underline{c})^\alpha, \quad x_{t+2} = (x_{t+1} - c_{t+1})^\alpha.$$

And from first order conditions

$$(1 - \theta)\beta^{t-1} = \lambda_{t-1}\alpha(x_t)^{\frac{\alpha-1}{\alpha}} \quad (22)$$

$$(1 - \theta)\beta^t + \omega_t = \lambda_{t-1} \quad (23)$$

$$(1 - \theta)\beta^{t+1} = \lambda_t \quad (24)$$

$$\lambda_{t-1} = \lambda_t\alpha(x_{t+1})^{\frac{\alpha-1}{\alpha}} \quad (25)$$

$$\lambda_t = \lambda_{t+1}\alpha(x_{t+2})^{\frac{\alpha-1}{\alpha}} \quad (26)$$

From (23), (24) and (25) we have

$$(1 - \theta)\beta^t + \omega_t = (1 - \theta)\beta^{t+1}\alpha(x_{t+1})^{\frac{\alpha-1}{\alpha}}$$

that gives

$$\omega_t = (1 - \theta)\beta^t \left[\alpha\beta(x_{t+1})^{\frac{\alpha-1}{\alpha}} - 1 \right] > 0 \quad \iff \quad x_{t+1} < x^{mgr}.$$

From (22), (24) and (25) we have

$$(1 - \theta)\beta^{t-1} = (1 - \theta)\beta^{t+1}\alpha(x_{t+1})^{\frac{\alpha-1}{\alpha}}\alpha(x_t)^{\frac{\alpha-1}{\alpha}} \quad \iff \quad 1 = (\alpha\beta)^2(x_{t+1}x_t)^{\frac{\alpha-1}{\alpha}}$$

this last equation gives

$$x_t > x^{mgr}.$$

From (24) and (26) we have

$$(1 - \theta)\beta^{t+1} = \lambda_{t+1}\alpha(x_{t+2})^{\frac{\alpha-1}{\alpha}}$$

and we can deduce that, if $\omega_{t+2} = 0$, then $x_{t+2} = x^{mgr}$ and if $\omega_{t+2} > 0$, then $x_{t+2} > x^{mgr}$, so

$$x_{t+2} \geq x^{mgr}.$$

We can conclude that

$$x^{mgr} \leq x_{t+2} = (x_{t+1} - c_{t+1})^\alpha = [(x_t - \underline{c})^\alpha - c_{t+1}]^\alpha <$$

$$(x^{mgr} - c_{t+1})^\alpha < (x_t - c_{t+1})^\alpha < (x_t - \underline{c})^\alpha = x_{t+1},$$

that is a contradiction.

G Proof of Proposition 4

We want to show that there exist values $\omega_t \geq 0$ for all $t = 0, 1, 2, \dots$ and λ_t such that, for all $x_0 \in [x^{mgr}, x_H]$, we have

$$(1 - \theta)\beta^t + \omega_t - \lambda_t \alpha (x_0 - c_0)^{\alpha-1} = 0, \quad t = 0, 1, \dots \quad (27)$$

$$-\lambda_{t-1} + \lambda_t \alpha (x_0 - c_0)^{\alpha-1} = 0, \quad t = 1, 2, \dots, \quad (28)$$

$$\theta - \sum_{t=0}^{\infty} \omega_t = 0 \quad (29)$$

Write

$$A^{-1} = \alpha(x_0 - c_0)^{\alpha-1} = \alpha \left[x_0^{1/\alpha} \right]^{\alpha-1}$$

ie

$$A = \frac{1}{\alpha} x_0^{(1-\alpha)/\alpha}$$

Eq (28) becomes

$$\lambda_t = A\lambda_{t-1} \text{ where } t = 1, 2, \dots$$

Therefore

$$\lambda_t = A^t \lambda_0 \text{ for } t = 1, 2, \dots$$

And

$$(1 - \theta) + \omega_0 - \lambda_0 A^{-1} = 0$$

$$(1 - \theta)\beta^t + \omega_t - A^t \lambda_0 A^{-1} = 0, \quad t = 1.$$

So

$$\omega_0 = \lambda_0 A^{-1} - (1 - \theta)$$

$$\omega_t = A^t \lambda_0 A^{-1} - (1 - \theta)\beta^t \text{ for } t = 1, 2, \dots$$

Assume

$$A < 1$$

ie

$$x_0 < (\alpha)^{\alpha/(1-\alpha)} \equiv x^{GR}$$

For ω_0 to be non-negative, we need to choose λ_0 such that

$$\lambda_0 A^{-1} \geq 1 - \theta \quad (30)$$

Assume inequality (30) holds. Then for ω_t to be non-negative for all t , we need

$$A^t \geq \beta^t$$

ie

$$x_0 \geq (\alpha\beta)^{\alpha/(1-\alpha)} \equiv x^{mgr}$$

Summing over all ω_t , we need to satisfy

$$\theta = \lambda_0 A^{-1} \sum_{t=0}^{\infty} A^t - (1-\theta) \sum_{t=0}^{\infty} \beta^t = \frac{\lambda_0 A^{-1}}{1-A} - \frac{1-\theta}{1-\beta}$$

ie

$$\theta + \frac{1-\theta}{1-\beta} = \frac{\lambda_0 A^{-1}}{1-A}$$

ie

$$\left[\theta + \frac{1-\theta}{1-\beta} \right] (1-A) = \lambda_0 A^{-1}$$

Since we can choose $\lambda_0 A^{-1}$ as long as (30) holds, we require

$$\left[\theta + \frac{1-\theta}{1-\beta} \right] (1-A) \geq (1-\theta)$$

ie

$$1-A \geq \frac{(1-\theta)(1-\beta)}{\theta(1-\beta) + (1-\theta)}$$

ie

$$A \leq \frac{\theta(1-\beta) + (1-\theta) - (1-\theta)(1-\beta)}{\theta(1-\beta) + (1-\theta)} = \frac{\theta(1-\beta) + \beta(1-\theta)}{\theta(1-\beta) + (1-\theta)} \equiv \hat{A} < 1.$$

H Proof of Proposition 6

With one jump, we jump from x_0 to $x_1 < x_0$, and then stay at x_1 for ever. then

$$x_t = (x_1 - c_1)^\alpha = x_1 \text{ for } t = 2, 3, \dots$$

The necessary conditions are

$$\begin{aligned} (1-\theta) - \lambda_0 \alpha (x_0 - c_0)^{\alpha-1} &= 0, \\ (1-\theta)\beta^t + \omega_t - \lambda_t \alpha (x_1 - c_1)^{\alpha-1} &= 0, \quad t = 1, 2, \dots \quad (31) \\ -\lambda_{t-1} + \lambda_t \alpha (x_t - c_t)^{\alpha-1} &= 0, \quad t = 1, 2, \dots, \end{aligned}$$

So

$$(1-\theta) - \lambda_0 \alpha (x_0 - c_0)^{\alpha-1} = 0, \quad (32)$$

$$(1-\theta)\beta^t + \omega_t - \lambda_{t-1} = 0, \quad t = 1, 2, \dots \quad (33)$$

$$\lambda_1 = \lambda_2 \alpha (x_2 - c_2)^{\alpha-1} = \lambda_2 \alpha (x_1)^{(\alpha-1)/\alpha} \equiv \lambda_2 A_1^{-1}$$

$$\lambda_0 = \lambda_1 \alpha (x_1 - c_1)^{\alpha-1} = \lambda_1 \alpha (x_1)^{(\alpha-1)/\alpha} \equiv \lambda_1 A_1^{-1}$$

$$\lambda_1 = \lambda_0 A_1$$

$$\lambda_2 = \lambda_1 A_1 = \lambda_0 A_1^2$$

and

$$\lambda_t = \lambda_0 A_1^t$$

where, from (32)

$$\lambda_0 A_1^{-1} = 1 - \theta$$

Equation (33) gives

$$\omega_t = \lambda_t A_1^{-1} - (1 - \theta)\beta^t \text{ for } t = 1, 2, \dots$$

ie

$$\omega_t = [\lambda_0 A_1^t] A_1^{-1} - (1 - \theta)\beta^t \text{ for } t = 1, 2..$$

$$\omega_t = (1 - \theta) [A_1^t - \beta^t] \text{ for } t = 1, 2..$$

Assume

$$A_1 > \beta$$

Summing over $t = 1, 2, \dots$

$$\theta = (1 - \theta) \left[\frac{A_1}{1 - A_1} - \frac{\beta}{1 - \beta} \right]$$

$$\theta + \frac{\beta(1 - \theta)}{1 - \beta} = (1 - \theta) \frac{A_1}{1 - A_1}$$

$$\frac{(1 - \beta)\theta + \beta(1 - \theta)}{1 - \beta} = (1 - \theta) \frac{A_1}{1 - A_1}$$

$$\frac{[(1 - \beta)\theta + \beta(1 - \theta)]}{(1 - \beta)(1 - \theta)} = \frac{A_1}{1 - A_1}$$

$$[(1 - \beta)\theta + \beta(1 - \theta)](1 - A_1) = (1 - \beta)(1 - \theta)A_1$$

$$[(1 - \beta)\theta + \beta(1 - \theta)] = A_1 [(1 - \beta)(1 - \theta) - (1 - \beta)\theta + \beta(1 - \theta)]$$

$$A_1 = \frac{\theta(1 - \beta) + \beta(1 - \theta)}{\theta(1 - \beta) + (1 - \theta)} = \hat{A}$$

I Proof of Proposition 7

First order conditions give:

$$(1 - \theta)\beta^t + \omega_t - \lambda_t \alpha x_0^{\frac{\alpha-1}{\alpha}} = 0, \quad t = 0, 1, \dots \quad (34)$$

$$-\lambda_{t-1} + \lambda_t \alpha x_0^{\frac{\alpha-1}{\alpha}} = 0, \quad t = 1, 2, \dots,$$

$$\theta = \sum_{t=0}^{\infty} \omega_t.$$

Calling $A = x_0^{\frac{1-\alpha}{\alpha}}/\alpha$, we obtain that: $\lambda_t = A\lambda_{t-1}$ and so,

$$\lambda_t = \lambda_0 A^t.$$

Note that $A < 1$ when $x_0 < x_\infty$. Summing equation (34) in t we obtain

$$\frac{1-\theta}{1-\beta} + \theta - \frac{\lambda_0}{1-A} \alpha x_0^{\frac{\alpha-1}{\alpha}} = 0,$$

then

$$\lambda_0 = \left(\frac{1-\theta}{1-\beta} + \theta \right) \frac{1-A}{\alpha x_0^{\frac{\alpha-1}{\alpha}}}.$$

$$\omega_t = \left(\frac{1-\theta}{1-\beta} + \theta \right) (1-A)A^t - (1-\theta)\beta^t \geq 0 \iff \left(\frac{1-\theta}{1-\beta} + \theta \right) \frac{1-A}{1-\theta} \geq \left(\frac{\beta}{A} \right)^t$$

for all t . This last inequality is not true for all t because $\frac{\beta}{A} > 1$.

J Proof of Proposition 8

Proof. As $c_t > \underline{c}, \forall t > h$, from (21) $\omega_t = 0, \forall t > h$. From (19)

$$\lambda_t \alpha (x_t - c_t)^{\alpha-1} = (1-\theta)\beta^t, \quad \forall t > h.$$

Then

$$\left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{x_{t+1} - c_{t+1}}{x_t - c_t} \right)^{\alpha-1} = \frac{\beta^{t+1}}{\beta^t} = \beta, \quad \forall t > h \quad (35)$$

From (20)

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\alpha (x_{t+1} - c_{t+1})^{\alpha-1}},$$

Replacing in (35)

$$\frac{1}{\alpha (x_t - c_t)^{\alpha-1}} = \beta, \quad \forall t > h$$

Therefore

$$\frac{1}{\alpha (x_{t+1})^{(\alpha-1)/\alpha}} = \beta, \quad \forall t > h$$

ie

$$x_{t+1} = (\alpha\beta)^{\alpha/(1-\alpha)} \equiv x^{mgr}, \quad \forall t > h$$

Then

$$c_{t+1} = x_{t+1} - x_{t+2}^{1/\alpha} = x^{mgr} - (x^{mgr})^{1/\alpha} \equiv c_\infty$$

■

K Proof of Proposition 9

As $c_t > \underline{c}, \forall t \geq T$, from (21) $\omega_t = 0, \forall t \geq T$. From (19) and (20)

$$\lambda_{t-1} = (1 - \theta)\beta^t, \quad \forall t \geq T.$$

Replacing in (20)

$$(\beta\alpha)^{\alpha/(1-\alpha)} = x_t = x^{mgr} \quad \forall t \geq T + 1.$$

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