Risque et durabilité : la viabilité est-elle si loin de l'optimalité ?

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Résumé

L'analyse économique aborde les questions de risque et de long terme dans le cadre de l'utilité espérée actualisée, dans une perspective d'optimalité. La théorie de la viabilité est basée sur des contraintes de soutenabilité, dans une perspective de faisabilité. Nous proposons ici un pont entre ces deux approches en montrant que la viabilité est équivalente à un ensemble de problèmes d'optimisation intertemporelle dégénérés. Ceci rend l'approche plus facilement interprétable en termes économiques, et en particulier du point de vue efficacité. Le cas déterministe est examiné tout d'abord. Nous soulignons les connections entre le noyau de viabilité et la fonction de temps de crise minimal. Nous présentons ensuite la viabilité stochastique, avec les notions de scénario viable et de probabilité maximale de viabilité. Nous montrons que la probabilité maximale de viabilité partage des propriétés de programmation dynamique avec l'optimum de l'utilité espérée actualisée. Les deux approches sont donc cohérentes dynamiquement, ce qui pourrait servir de base pour une axiomatisation de critères pour la prise de décision de long terme en présence de risque.

Mots-clés : Durabilité, incertitude, multicritère, viabilité.

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Risk and Sustainability:
Is Viability that far from Optimality?

Abstract

Economic analysis addresses risk and long-term issues with discounted expected utility, focusing on optimality. Viability theory is based on sustainability constraints to be satisfied over time, focusing on feasibility. We make a bridge between these two approaches by showing that viability is equivalent to an array of degenerate intertemporal optimization problems. This makes the approach more interpretable in economic terms, and especially regarding efficiency. First, the deterministic case is examined. A particular emphasis is put on the connections between the viability kernel and the minimal time of crisis function. Then, we present stochastic viability with the notions of viable scenario and maximal viability probability. We show that the maximal viability probability shares dynamic programming properties with optimal discounted expected utility. Thus, both exhibit time-consistency, which may be a basis for an axiomatization of criteria under risk and long run for public decision-making.

Keywords: Sustainability, uncertainty, multicriteria, viability.

1 Introduction

Dealing with environmental issues such as climate change, biodiversity preservation, or managing natural resources, requires to account for conflicting objectives, dynamics, uncertainty and long-term.

The issue of decision under risk has been widely addressed in the economic literature, back to the fundament of expected utility theory axiomatized by von Neuman and Morgenstern [1947]. However, the expected utility framework is known to exhibit serious limitations in certain situations (e.g., Allais [1953] and Ellsberg [1961] paradoxes), and these limitations are often highly relevant in environmental and resource economics [Shaw and Woodward, 2008].

Dynamic and long-term issues also received a particular focus in the economic literature, especially in growth theory [Koopmans, 1965]. Regarding environmental issues, the sustainability debate has shed a new light on the discounted utility approach. In the literature addressing optimal growth theory with environment, it has been shown that the usual neo-classical criterion may lead to unsustainable economic trajectories [Chevé and Schubert, 2002], mainly because of discounting. Fleurbaey and Michel [1999] emphasize that, in a finite horizon framework, a criterion without discounting can easily be defended, but that, in infinite horizon, discounted approach is used more for practical than for theoretical reasons. The discounted utility criterion has mainly been criticized because it neglects long-run utility, being qualified as a "dictatorship of the present" by Chichilnisky [1996]. The sustainability debate has thus been marked by the introduction of other criteria [Rotillon, 2005].

Given these important contributions on risk on the one hand, and dynamic/long-run issues on the other hand, one could be surprised to note that joint issues of risk and sustainability have received less attention as to the axiomatic fundations. An exception is the book *Sustainability: Dynamics and Uncertainty* by Chichilnisky, Heal, and Beltratti [1998] in which risk and dynamics are addressed mainly in the discounted expected utility framework. In his textbook, *The Economics of Risk and Time* [Gollier, 2001], Gollier provides two arguments to justify the use of discounted expected utility with exponential discounting. First, it is *time-consistent*, which may be considered as a fine property for public decision making [Cohen and Michel, 1988]. Second, it accounts for *pure preference for the present* (impatience) via discounting, which may have a sense in individual decision making but is criticized in the sustainability debate on long-run issues. The discounted expected utility has not been designed to address sustainability issues but, as far as we know, usual sustainability criteria have not yet been applied in a stochastic framework, providing no alternative. However, as stressed in the Stern Review for climate change [Stern, 2006], environmental issues are characterized by both risk and dynamic/long-run, and these issues should be addressed in a single framework. Howarth [1995] emphasizes that, under uncertainty, a deontological approach should be used to address the sustainability issue, and sustainability conditions should be imposed as prior constraints on the maximization of a social welfare function.

According to Gerlagh and Keyser [2003], conservationist policies can be Pareto efficient, and strict resource conservation is equivalent to non-dictatorship of the present. This echoes the following quotation of Marcel Boiteux:

> (...) pour les modèles à long terme, l'approche par les prix n'est pas la meilleure (mieux vaut travailler sur les quantités et trouver les prix par dualité pour orienter ensuite les choix décentralisés des acteurs) [Boiteux, 1976]

Using quantities to deal with the sustainability issue, due to its long-term perspective, is thus an alternative approach to expected utility [Mäler, 2002].

When sustainability objectives are defined using indicators (quantitative measurement of economic or physical meaning) and corresponding thresholds, the problem of the regulator in coping with all the objectives simultaneously is to avoid crisis situations, and sustainability appears closer to a 'satisficing' problem (bounded rationality [Simon, 1957]) than to an optimizing problem [Krawczyk and Kim, 2009]. Actually such a 'satisficing' approach is rather the norm in practice: natural resource management issues are often addressed using quantitative indicators and associated thresholds. For example, the global change issue is addressed with a GHG concentration upper limit, and emission reductions are also defined in quantity terms. The concept of "stewardship" mentioned in the *Stern Review* [Stern, 2006] also stresses similar issues:

> The notion of "stewardship" can be seen as a special form of sustainability. It points to particular aspects of the world, which should themselves be passed on in a state at least as good as that inherited from the previous generation.

Let us also mention the *generalized capacity* of Solow:

> If sustainability means anything more than a vague emotional commitment, it must require that something be conserved for the very long run. It is very important to understand what that thing is: I think it has to be a generalized capacity to produce economic well-being. [Solow, 1993]

If, in an intergenerational equity perspective, the constraints defined by these indicators have to be satisfied throughout time, such sustainability problems can be studied in the viability framework [Martinet and Doyen, 2007, Baumgärtner and Quaas, 2009]. A major mathematical instrument of the viability analysis is the so-called *viability kernel* [Aubin, 1991]. It is composed of all initial states from which economic development paths respecting the constraints can start, under appropriate sequences of decisions. Focusing on such a geometrical tool is not traditional in economic analysis.

Replaced in our discussion on risk, viability can be interpreted in a stochastic framework as the high probability to be above thresholds, following the lead of De Lara and Doyen [2008]. This is related to the psychological process of aspiration assessment: Lopes claims that "sensible people often base their choices on the probability of coming out ahead" [Lopes, 1996]. When stakes are high, as life and death issues illustrated in Dubbins and Savage [1965], this is a fairly "reasonable" approach.

In this paper, we examine how the stochastic viability approach addresses conflicting objectives, uncertainty, dynamic processes and long-term issues. The contribution of this paper is twofold.

First, from a theoretical point of view, we provide a criterion-like description of the viability approach, which allows us to stress its links with the usual economic approaches. In fact, the equivalence is not with one but with several degenerate optimization problems: we provide a description of time additive and time multiplicative criteria, and interpret each kind of formulation. This allows us to draw parallels between the discounted utility criterion and the viability approach. The viability problem being expressed as a dynamic optimization problem appears closer to usual economic representations.

Second, we argue that stochastic viability is a pertinent complementary approach to deal with dynamic problems under uncertainty, and thus to define sustainable management of natural resources or to cope with long-run environmental issues such as climate change. In particular, we show that, if viability in a deterministic framework does not rank all trajectories, this is no longer the case in a stochastic framework: the probability to achieve the sustainability objectives, represented by viability constraints, is a natural currency to rank decision rules.

The paper is organized as follows. In Sect. 2, we describe both the usual discounted utility and the viability approach. We emphasize their conceptual differences, and the implications of these differences for natural resource management. In Sect. 3, we describe how both approaches read in a stochastic framework. We show that, when time and risk interact, the previously exhibited differences reduce and are sources of complementarity, at least to address natural resources management or environmental issues. We conclude on research perspectives on sustainability criteria mixing risk and time in Sect. 4.

2 Deterministic intertemporal optimal choice problem

As stated in the introduction, natural resources management and environmental issues are dynamic in nature. To address these issues, we use a dynamic modeling framework from control theory. We shall consider management of dynamic systems in discrete time. We indeed want to avoid technical difficulties related to continuous time, and concentrate on conceptual issues. In all the sequel, we shall deal with *stationary* problems without explicit dependence upon time, for the sake of simplicity of notations.

2.1 Dynamic economic model

The economy is represented by the following discrete-time control dynamical system

$$x(t+1) = G\big(x(t), c(t)\big) , \quad t = t_0, \dots, T-1 , \quad x(t_0) = x_0 , \qquad (1)$$

where the time index t is discrete ($t \in \mathbb{N}$ is an integer), with t_0 the *initial time* and T the *horizon*, which may be finite ($T < +\infty$) or infinite ($T = +\infty$). The *state* $x(t)$ is a vector belonging to $\mathbb{X} := \mathbb{R}^n$; each capital stock may represent man-made reproducible capital, natural resources (renewable or not) or pollution stocks while $x_0 \in \mathbb{X}$ is the *initial state* for the *initial time* t_0. The *control* $c(t) \in \mathbb{C} := \mathbb{R}^p$ may represent investment, consumption, catches or harvesting effort, or emissions. The mapping $G : \mathbb{X} \times \mathbb{C} \to \mathbb{X}$ stands for the *dynamics* representing the evolution of the various stocks through time.[1] It may include economic models with capital and labor, population dynamic models of natural resources, or pollution accumulation-absorption models.

2.2 Outputs/indicators

Main outputs of this system are given by so-called *indicators* $\mathcal{I}_k\big(x(t), c(t)\big)$, $k = 1, \ldots, K$. An indicator $\mathcal{I}_k : \mathbb{X} \times \mathbb{C} \to \mathbb{R}$ is a state and control function having economic or environmental meaning. From a sustainable development point of view, the indicators are instantaneous measurement of quantities that characterize some aspect of sustainability. Note that an indicator can be reduced to the simplest form, being only one of the stocks or one of the decisions (consumption for instance).

The parallel can be drawn with Lancaster's consumer theory [Lancaster, 1966] as decisions and states are not necessarily a matter of interest in and of themselves, but impact utility by their effects on various *characteristics*, which can be arguments of a multi-attribute utility function. In our case, one could define a function $U(\mathcal{I}_1, ..., \mathcal{I}_K)$ of the indicators.

2.3 Maximal intertemporal discounted utility

In an intertemporal framework, the purpose of economic analysis is to define optimal intertemporal decisions to be applied from a given initial state.

The usual criterion in economics is the discounted return, which is the discounted sum of present and future benefits or values:

$$\max_{c(\cdot)} \sum_{t=t_0}^{T-1} \frac{1}{(1+\delta)^t} U\Big(\mathcal{I}_1\big(x(t), c(t)\big), \ldots, \mathcal{I}_K\big(x(t), c(t)\big)\Big) \ . \tag{2}$$

where $c(\cdot)$ denotes a control path $c(\cdot) := \big(c(t_0), \ldots, c(T-1)\big)$. Here, the utility function U depends on the outputs $\mathcal{I}_1\big(x(t), c(t)\big), \ldots, \mathcal{I}_K\big(x(t), c(t)\big)$

[1] The dynamics G might explicitly depend upon time t. However, we shall treat only the stationary case for the sake of notational simplicity.

of the system, hence is implicitly relying on state and control.[2] Such a program defines an optimal growth trajectory in the terminology of neo-classical economics [Koopmans, 1965]. In the long run, it can lead to unsustainable situations, with utility decreasing toward zero, in particular in models with exhaustible natural resources [Dasgupta and Heal, 1974] or pollution [Chevé and Schubert, 2002].

2.4 The viability approach

Following stewardship or satisficing concerns related to sustainability, suppose now that the decision maker's goal is not to maximize the discounted utility but to maintain some stocks, some aggregate capital or, more generally, some indicators above viability thresholds:[3]

$$\mathcal{I}_k\big(x(t), c(t)\big) \geq \tau_k \ , \quad \forall k = 1, \ldots, K \ . \tag{3}$$

Recall that $\mathcal{I}_k : \mathbb{X} \times \mathbb{C} \to \mathbb{R}$ is an indicator,[4] namely a state and control function having economic or biological meaning; the real τ_k is a threshold. Using these indicators and associated thresholds acting as constraints, a sustainable development path is an economic trajectory that meets all the constraints (eq. 3) at all times t.

Sustainability being defined this way, viability theory determines the conditions for economic trajectories to be sustainable [Martinet and Doyen, 2007]. In particular, the viability analysis describes the conditions on states (economic endowments) and controls (economic decisions) for the resulting trajectory to be viable, that is, to respect all the constraints at all times, given the dynamics of the system. The main mathematical instrument of the viability analysis is the so-called *viability kernel* [Aubin, 1991]. It is composed of all initial states from which viable trajectories can start, i.e., all states from which there are intertemporal decisions resulting in trajectories which satisfy the constraints. From the mathematical point of view, the

[2]The utility function U might explicitly depend upon time t. However, we shall treat only the stationary case for the sake of notational simplicity.

[3]Without loss of generality: a "bad" indicator, such as pollution, can be represented by its negative value.

[4]The indicator $\mathcal{I}_k$ might explicitly depend upon time t. This can include absence of constraints (take $\mathcal{I}_k$ having constant value greater than τ_k, or final target constraint (take $\mathcal{I}_k(t, x, c) \geq \tau_k$ for all $t = t_0, \ldots, T-1$ but not for $\mathcal{I}_k(T, x, c)$).

viability kernel at initial time t_0 reads

$$\mathbb{V}\mathrm{iab}(t_0) = \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \text{there exist controls } \big(c(t_0), \ldots, c(T-1)\big) \\ \text{ such that} \\ \forall k = 1, \ldots, K \ , \quad \forall t = t_0, \ldots, T-1 \ , \\ \mathcal{I}_k\big(x(t), c(t)\big) \geq \tau_k \\ x(t+1) = G\big(x(t), c(t)\big) \\ x(t_0) = x_0 \end{array} \right\} . \qquad (4)$$

A basic viability problem consists in characterizing this set. Handling such a geometrical tool is not usual in economic analysis, which favors optimization approaches. Nevertheless, we show in next section how a viability problem can expressed as a (degenerate) dynamic optimization problem, closer to usual economic representations and efficiency.

2.5 Viability as a degenerate optimization problem

Numerous formulations of viability problems in terms of optimality have been provided. In the continuous case, Aubin [1991] especially focuses on exit time functions together with support or indicator functions for Hamiltonian characterizations. In [Martinet and Doyen, 2007], links with maximin criterion are pointed out. We shall now present alternate equivalent forms of the viability problem (3)-(4) and give their interpretation. These forms will be formulated as optimization problems. However, these latter are *degenerate* optimization problems with no unique solution in general. Indeed, they involve *characteristic functions* as follows.

Reformulating viability with characteristic functions

The viability kernel (4) is the set of initial states from which start trajectories satisfying conditions (3) at every period. These states are called *viable states*. They can be characterized by solving optimization problems, using characteristic functions.

Denote by $\mathbf{1}_A$ the *characteristic function* of the set A, which is equal to one when its argument belongs to A, and to zero otherwise. Using such a tool, the quantity $\mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t), c(t)\big)\Big)$ represents the effectiveness of the economy to satisfy sustainability objective k at time t, i.e., whether the constraint k is satisfied at the given time or not. Using this *characteristic* *function formulation*, condition (3) can be described in multiplicative[5] form, as $\prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big) = 1$. If any of the indicator is below the associated threshold, the characteristic function is equal to zero for this indicator, and the product is nil. If all of the constraints are respected, it is equal to one.

There are at least two different ways to describe the viability kernel in optimality terms using the characteristic functions. On the one hand, one can use a time-additive form, the criterion being to minimize the following value[6]

$$\sum_{t=t_0}^{T-1} \left(1 - \prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big)\right) , \tag{5}$$

which counts the number of time periods during which (at least) one constraint is not respected along a given trajectory. In our sustainability issue, it can be interpreted as the number of generations that do not achieve the sustainability objectives. It is equal to zero when all the constraints are respected at all times along the trajectory defined by the given controls. This time-additive form has an easy interpretation based on the *time of crisis* criterion developed by Doyen and Saint-Pierre [1997], Béné et al. [2001], Martinet et al. [2007]. The optimal control problem associated with this criterion termed minimal time of crisis corresponds to[7]

$$\mathrm{C}(t_0,x_0) = \min_{c(\cdot)} \sum_{t=t_0}^{T-1} \left(1 - \prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big)\right) .$$

It turns out that the viability kernel of the problem is composed of all initial states where the minimal time of crisis is nil:

Proposition 1 $x_0 \in \mathbb{V}\mathrm{iab}(t_0) \iff \mathrm{C}(t_0,x_0) = 0.$

On the other hand, one can use a time-multiplicative form, the criterion being to maximize the product upon time of the product of characteristic

[5]An equivalent additive form is $\sum_{k=1}^{K}\Big(1 - \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big)\Big) = 0$, and the minimum form is $\min_{k=1,\dots,K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big) = 1$.

[6]As we are summing nonnegative numbers, the given sum is mathematically well-defined whether the time horizon is finite, i.e., $T < +\infty$, or infinite, i.e., $T = +\infty$.

[7]See footnote 6.

functions[8]

$$V(t_0,x_0) = \max_{c(\cdot)} \prod_{t=t_0}^{T-1} \prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t),c(t)\big)\Big) \ . \tag{6}$$

As soon as one of the constraints is not respected at some time period, the criterion is equal to zero. It is equal to one when all the constraints are respected at all times, characterizing a viable state:

Proposition 2 $x_0 \in \mathbb{V}\mathrm{iab}(t_0) \iff V(t_0,x_0) = 1$.

Even if it has little economic meaning in the deterministic case, this form will appear useful in the stochastic case that we shall address in the next section.

Both the criteria (5) and (6) will provide the same information by characterizing the viability kernel of the problem. Nevertheless, the multiplicative form only gives a boolean information, whereas the minimum time of crisis indicates, for states characterized by a strictly positive value function, the minimal number of crisis period the economy is going to face. It thus provides a meaningful information on what happens outside the viability kernel, and how to reach it [Martinet et al., 2007].

Altogether, the viability approach focuses on feasibility, and not on optimality, defining efficient trajectories with respect to the given objectives. Hence, an economic objection to this approach is that it does not rank all the various trajectories, or, more specifically, that all viable trajectories have the same value. Note however that one can apply another (economic) criterion to select an optimal trajectory among those which are efficient in meeting the viability constraints. In the next section, we show that this critic vanishes in a stochastic framework.

3 Stochastic intertemporal optimal choice problem

In this section, we proceed to the same description of both discounted utility and viability in a stochastic framework. We show how a dynamic programming structure occurs in both approaches.

[8]As we are multiplying numbers within $[0,1]$, the given product is mathematically well-defined either the time horizon is finite, i.e., $T < +\infty$, or infinite, i.e., $T = +\infty$.

3.1 Dynamic economic model

Consider the following discrete-time control dynamical system

$$x(t+1) = G\big(x(t), c(t), w(t)\big)\ , \quad t = t_0, \dots, T-1\ , \quad x(t_0) = x_0\ , \qquad (7)$$

where $w(t) \in \mathbb{W} := \mathbb{R}^q$ denotes an *uncertainty* or *disturbance* which affects the dynamics at time t. The initial state x_0 is supposed to be deterministic and known; however, if needed, x_0 could be added to the uncertainties. We define

$$\Omega := \mathbb{W}^{T-t_0} \qquad (8)$$

as the set of *scenarios*, the notation for a scenario being $w(\cdot) := \big(w(t_0), \dots, w(T-1)\big)$. From now on, we shall assume that the set Ω is equipped with a *probability*[9] $\mathbb{P}$ which measures the likelihood of subsets of scenarii. The notation $w(\cdot) = \big(w(t_0), \dots, w(T-1)\big)$ still denotes a generic point in Ω; however, it may also be interpreted as a sequence of random variables when $w(\cdot)$ is identified with the identity mapping from Ω to Ω. The *mathematical expectation* with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

3.2 Decision rules

Let us define a *decision rule* as a mapping $\mathfrak{c} : \mathbb{N} \times \mathbb{X} \to \mathbb{C}$. A decision rule is a *(state) feedback* which assigns a control $c = \mathfrak{c}(t, x) \in \mathbb{C}$ to any state x for any time t. With such a definition, we implicitly assume that the state is (at least partially) measured.[10]

Given a decision rule $\mathfrak{c}$, a scenario $w(\cdot) \in \Omega$, an initial state $x_0 \in \mathbb{X}$ and an initial time t_0, the *solution state* $x(\cdot) = (x(t_0), \dots, x(T))$ is well defined as the solution of the difference equation

$$x(t+1) = G\big(x(t), \mathfrak{c}\big(t, x(t)\big), w(t)\big) \text{ with } x(t_0) = x_0$$

The *solution control* $c(\cdot) = \big(c(t_0), c(t_0+1), \dots, c(T-1)\big)$ is the associated decision path where $c(t) = \mathfrak{c}\big(t, x(t)\big)$.

[9] The probability $\mathbb{P}$ is defined on the Borel product σ-field of $\mathbb{W}^{T-t_0}$. The mappings G, $\mathcal{I}_1$, …, $\mathcal{I}_k$, and all decision rules $\mathfrak{c}$ (see below) are supposed to be measurable.

[10] As a consequence, we shall not consider the case where only a corrupted observation of the state is available to the decision-maker (as it is nevertheless the case in practical situations).

3.3 Discounted expected utility

There are plenty of criteria to make choices under risk (see for instance Savage [1972]), but less attention has been paid to criteria for dynamical problems mixing time and risk. The usual criterion in economics is the discounted *expected* return, which is the expected discounted sum of present and future benefits or values:

$$\max_{\mathfrak{c}} \mathbb{E}\left[\sum_{t=t_0}^{T-1} \frac{1}{(1+\delta)^t} U\big(\mathcal{I}_1\big(x(t),c(t)\big),\ldots,\mathcal{I}_K\big(x(t),c(t)\big)\big)\right]. \tag{9}$$

The criterion (9) is built upon two well axiomatized theories, the discounted intertemporal utility [Koopmans, 1965] and the expected utility [von Neuman and Morgenstern, 1947]. However, we are not aware of an axiomatization of the expected discounted intertemporal utility. This criterion is widely used and offers interesting applicability properties as time consistency and dynamic programming. However, discounting makes this criterion problematic for long-run issues such as resources and environmental management.

3.4 Viable scenarios and viability probability

We now describe how the viability approach described in the previous Sect. 2 can be extended to the stochastic framework of this section.[11]

We impose viability constraints in this uncertain case as follows

$$\mathcal{I}_k\big(x(t),c(t),w(t)\big) \geq \tau_k,\ k=1,\ldots,K.$$

For these constraints to make sense in the stochastic case, we need to introduce the notion of a viable scenario. A scenario $w(\cdot)$ is said to be *viable under a given decision rule* $\mathfrak{c}$ if the state and control trajectory driven by the decision rule $\mathfrak{c}$, introduced in §3.2, satisfies the constraints. It means that, if the given scenario occurs, the economic trajectory defined by this decision rule is viable.

[11] Mathematical materials for stochastic viability can be found in Aubin and Prato [1998], Buckdahn et al. [2004] but they focus on the continuous time case. Contributions for discrete time systems are Doyen et al. [2007], Béné and Doyen [2008], De Lara and Doyen [2008], De Lara and Martinet [2009].

For any decision rule $\mathfrak{c}$, initial state x_0, and initial time t_0, let us define the set of *viable scenarios* by:

$$\Omega_{\mathfrak{c},t_0,x_0} := \left\{ w(\cdot) \in \Omega \;\middle|\; \begin{array}{l} x(t_0) = x_0 \\ x(t+1) = G\big(t, x(t), c(t), w(t)\big) \\ c(t) = \mathfrak{c}\big(t, x(t)\big) \\ \mathcal{I}_k\big(x(t), c(t), w(t)\big) \geq \tau_k \\ k = 1, \ldots, K \\ t = t_0, \ldots, T-1 \end{array} \right\} . \tag{10}$$

This is the set of scenarios for which the given decision rule would result in viable economic trajectories. The larger the set, the larger the number of scenarios in which the given decision rule succeeds in meeting the viability constraints. Once a probability is defined on the set of scenarios, the viability problem becomes one of probability maximization, as follows.

We say that $\mathbb{P}\left(\Omega_{\mathfrak{c},t_0,x_0}\right)$ is the *viability probability* associated to the initial time t_0, the initial state x_0 and the decision rule $\mathfrak{c}$. Given initial time t_0 and initial state x_0, the *maximal viability probability* is defined by

$$\mathrm{PV}(t_0, x_0) = \sup_{\mathfrak{c}} \mathbb{P}\left(\Omega_{\mathfrak{c},t_0,x_0}\right) \ .$$

From that point of view, the stochastic viability approach aims at characterizing the decision rules which maximize the probability to satisfy the constraints over time.

The concept of viability kernel can then be expanded through the maximal viability probability. At a given confidence level $\beta \in [0, 1]$, it reads as follows:

Proposition 3 $x_0 \in \mathbb{V}\mathrm{iab}_\beta(t_0) \iff \mathrm{PV}(t_0, x_0) \geq \beta$.

3.5 Dynamic programming and time-consistency

As the viability probability can be written in the following expected intertemporal form,

$$\mathbb{P}\left[\Omega_{\mathfrak{c},t_0,x_0}\right] = \mathbb{E}\left[\prod_{t=t_0}^{T-1} \prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x(t), c(t), w(t)\big)\Big)\right] \tag{11}$$

a stochastic dynamic programming equation can be derived [De Lara and Doyen, 2008]. Whenever the probability $\mathbb{P}$ is a product of its marginals (independence), the maximal viability probability satisfies the following backward induction

$$\mathrm{PV}\big(t,x\big) = \max_{c} \mathbb{E}_w \left[\prod_{k=1}^{K} \mathbf{1}_{[\tau_k,+\infty[}\Big(\mathcal{I}_k\big(x,c,w\big)\Big)\mathrm{PV}\big(t+1, G(x,c,w)\big)\right] \ . \quad (12)$$

Such dynamic programming equation is connected to time-consistency. Define, loosely, *time consistency* as the following property. Today, at time t_0, I formulate an optimization problem with criterion $\pi\big(t_0, x(\cdot), c(\cdot), w(\cdot)\big)$; this yields a sequence of optimal decision rules $\mathfrak{c}^{t_0}(t_0, x), \mathfrak{c}^{t_0}(t_0+1, x), \ldots$ Tomorrow, at time t_0+1, I will formulate an optimization problem with criterion $\pi\big(t_0+1, x(\cdot), c(\cdot), w(\cdot)\big)$; this will yield a sequence of optimal decision rules $\mathfrak{c}^{t_0+1}(t_0+1, x), \mathfrak{c}^{t_0+1}(t_0+2, x), \ldots$ Time consistency holds true whenever *my "today rule for tomorrow" coincides with my "tomorrow rule for tomorrow"*: $\mathfrak{c}^{t_0+1}\big(t_0+1, x\big) = \mathfrak{c}^{t_0}(t_0+1, x)$. This happens to be the case when optimal decision rules are given by a dynamic programming equation. In the economic literature, time consistency is rather defined with reference to temporal lotteries as in [Hammond, 1989], and consistency may also be related to stationarity of policy design [Cohen and Michel, 1988]. This is not our point of view here, which may explain discrepancies.

4 Conclusion

Managing natural resources or dealing with environmental issues imply to consider both risk and dynamics/long-term. The main economic approach combining risk and time is the discounted expected utility, which is criticized in the sustainability debate as long-run issues are neglected due to exponential discounting.

The present paper points out how viability, and especially stochastic viability approach, can contribute to this debate. Viability aims at defining the conditions for sustainability constraints to be satisfied over time. Using the stochastic viability approach to address sustainability issue is interpreted as maximizing the probability to be above thresholds. In "life and death" or irreversibility situations, this can be a reasonable view as in the famous casino case of Dubbins and Savage.[12]

> Imagine yourself at a casino with \$1,000. For some reason, you desperately need \$10,000 by morning; anything less is worth nothing for your purpose. What ought you do? The only thing possible is to gamble away your last cent, if need be, in an attempt to reach the target sum of \$10,000. [Dubbins and Savage, 1965]

Stochastic viability defines the decision rules that minimize the risk to violate the constraints, and thus aims at avoiding related catastrophic outcomes. For example, in the climate change issue, it would consist in favoring decision rules which result in thin tail temperature distributions, thus reducing the probability of climate disasters. It echoes the conclusion in [Weitzman, 2007] to focus on fat-tailed distributions issues.

We have shown that viability can be formulated as a degenerate optimization problem, either in a time-additive form (which defines the minimum time of crisis of an economic development path) or in a time-multiplicative form (well-adapted to probabilities). These two formulations share common interesting properties with the discounted expected utility (time-consistency, dynamic programming).

In discounted expected utility, the utility function embodies the relative importance of objectives $\mathcal{I}_1, \ldots, \mathcal{I}_k$. In the viability framework, the relative importance of objectives is embodied in the definition of the thresholds $\tau_1, \ldots, \tau_k$. The definition of such sustainability objectives is behind the scope of the present paper. However, we conclude on different ways to deal with this selection issue.

One can define a preference relationship on the thresholds on the one hand, and describe the necessary trade-offs between thresholds on the other hand. Choosing among them requires a deontological approach to sustainability [Martinet, 2009].

Another way, which strengthens the links between viability and optimality emphasized in this paper, consists in replacing the indicator functions by more regular functions by "smoothing" thresholds. This can be done by making $\tau_1, \ldots, \tau_k$ random [Gollier, 2001] or by concavification.[13] In a sense, this amounts to going from quantiles toward more regular risk measures with

[12] Imagine, in the following tale, that a contract has been put on your head and that you desperately need money to pay the killer before!

[13] A characteristic function $\mathbf{1}_{[\tau_k,+\infty[}(\mathcal{I}_k)$ being both non concave and discontinuous, one might think of concavify and smoothing it by replacing $\mathbf{1}_{[\tau_k,+\infty[}(\mathcal{I}_k)$ by $\min(\tau_k, \mathcal{I}_k)$.

better properties, such as the coherent risk measures [Artzner et al., 1999, Föllmer and Schied, 2002].

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