What Can Rational Investors Do About Excessive Volatility and Sentiment Fluctuations?

Bernard Dumas
INSEAD, Wharton, CEPR, NBER

Alexander Kurshev
London Business School

Raman Uppal
London Business School, CEPR

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Our objective

► Agents in financial markets claimed to exhibit behavior that deviates from rationality – overconfidence leading to “excessively volatility”

► Suppose a Bayesian, intertemporally optimizing investor (“smart money”) operates in this financial market:

► We wish to understand:

1. What investment strategy this investor will undertake?
2. What effect this strategy will have on equilibrium prices?
3. Whether this will ultimately eradicate the source of excess volatility?

► We do this by building an equilibrium model of investor sentiment.
What we do: Contribution

1. **Model**: Equilibrium of financial market with two populations:
   - Bayesian (rational) learners; Imperfect (irrational) Bayesian learners
     (general equilibrium, risk averse agents, shortsales allowed)

2. **Effect on prices, volatility and correlation**
   - A few rational investors are not enough to eliminate the effect of irrational traders

3. **Optimal portfolios**
   - Profit from predictability, but more sophistication is needed

4. **Survival of irrational traders** (Kogan-Ross-Wang-Westerfield; Yan)
   - Their rate of impoverishment is quite slow
Model: Output and information structure

- **Exogenous process for aggregate output**
  - Output uncertainty: first source of risk ($\delta$ shock)
    
    \[
    \frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dZ_\delta^\delta,
    \]

  - Expected value of rate of growth of dividends $f$ is stochastic
    
    \[
    df_t = -\zeta (f_t - \bar{f}) dt + \sigma_f dZ_t^f; \quad \zeta > 0,
    \]

- **Expected growth rate is not observed** by any investor; investors continuously form (filter) estimates of it, based on $\delta$ and a signal $s$:
  
  \[
  ds_t = f_t dt + \sigma_s dZ_t^s,
  \]
Population $A$ is deluded

▶ **Group A: Irrational traders**

- They believe steadfastly that
  - innovations in signal have correlation $\phi \geq 0$ with innovations in $f$, when, in fact, true correlation is zero

\[
d s_t = f_t dt + \sigma_s \phi dZ^f_t + \sigma_s \sqrt{1 - \phi^2} dZ^s_t.
\]

- They overreact to signal and cause excess volatility in stock market
- Otherwise, behave optimally
- Degree of irrationality captured by a single parameter: $\phi$

▶ **Group B: Rational traders** ("smart money").
Result of filtering (in terms of B’s Wiener)

\[
\begin{align*}
\dot{\hat{f}}^A_t &= -\zeta \left( \hat{f}^A - \bar{f} \right) + \left( \frac{\gamma^A}{\sigma^2} \right) \left( \hat{f}^B - \hat{f}^A \right) dt + \gamma^A \frac{\sigma^2}{\sigma^2} \sigma dW_{\delta,t}^B + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma^2} \sigma dW_{s,t}^B \\
\dot{\hat{f}}^B_t &= -\zeta \left( \hat{f}^B - \bar{f} \right) dt + \frac{\gamma^B}{\sigma^2} \sigma dW_{\delta,t}^B + \frac{\gamma^B}{\sigma^2} \sigma dW_{s,t}^B.
\end{align*}
\]

▶ Group A is called "overconfident" because the steady-state variance of \( f \) as estimated by Group A, \( \gamma^A \), decreases as \( \phi \) rises.

▶ Group A has more volatile beliefs than Group B because conditional variance of \( \hat{f}^A \) monotonically increasing in \( \phi \).

▶ Difference of opinion: \( \hat{g} \triangleq \hat{f}^B - \hat{f}^A \)

So, \( \hat{g} > 0 \) implies Group B relatively optimistic compared to Group A.
Sentiment

▶ **Change from** \(B\) **to** \(A\)'s probability measure** given by \(\eta\):

\[
\frac{d\eta_t}{\eta_t} = -\hat{g}\left(\frac{dW^B_{\delta,t}}{\sigma_{\delta}} + \frac{dW^B_{s,t}}{\sigma_{s}}\right).
\]

▶ \(\eta\) is a measure of **sentiment** – shows how Group \(A\) over- or under-estimates the probability of a state relative to Group \(B\).

▶ Girsanov’s theorem tells how current disagreement gets encoded into \(\eta\):
  
  - For instance, if \(A\) is currently comparatively optimistic \((\hat{f}^A > \hat{f}^B)\), Group \(A\) views positive innovations in \(\delta\) as more probable than \(B\).
  - This is coded by Girsanov as positive innovations in \(\eta\) for those states of nature where \(\delta\) has positive innovations.
Diffusion matrix of state variables

- Four state variables \( \{ \delta, \eta, \hat{f}^B, \hat{g} \} \).
  Driven by only two Brownians, \( W^B_\delta \) and \( W^B_s \) because \( f \) is unobserved.

\[
\begin{pmatrix}
\delta \\
\eta \\
\hat{f}^B \\
\hat{g}
\end{pmatrix}
\begin{bmatrix}
\delta \sigma_\delta > 0 & 0 \\
-\eta \frac{\hat{g}}{\sigma_\delta} & -\eta \frac{\hat{g}}{\sigma_s} \\
\gamma^B \frac{\sigma}{\sigma_\delta} > 0 & \gamma^B \frac{\sigma}{\sigma_s} > 0 \\
\gamma^B - \gamma^A \frac{\sigma}{\sigma_\delta} \geq 0 & \gamma^B - \left( \phi \sigma_s \sigma f + \gamma^A \right) \frac{\sigma}{\sigma_s} \leq 0
\end{bmatrix}
\]

- Two distinct effects of imperfect learning:
  1. Instantaneous: \( \hat{g} \) has nonzero diffusion, so disagreement is stochastic.
  2. Cumulative: \( \hat{g} \) affects diffusion of \( \eta \), so disagreement drives sentiment.
Objective functions

Market is assumed complete; use static formulation of dynamic problem

Problem of Group $B$:

$$\sup_c \mathbb{E}^B \int_0^\infty e^{-\rho t} \frac{1}{\alpha} (c_t^B)^\alpha \, dt,$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^B \, dt = \bar{\theta}^B \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t \, dt,$$

Group $A$’s problem under $B$’s measure

$$\sup_c \mathbb{E}^B \int_0^\infty \eta_t \times e^{-\rho t} \frac{1}{\alpha} (c_t^A)^\alpha \, dt,$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^A \, dt = \bar{\theta}^A \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t \, dt.$$
Complete-market equilibrium

Definition: An equilibrium is a price system and a pair of consumption-portfolio processes such that

1. investors choose their optimal consumption-portfolio strategies, given their perceived price processes;
2. the perceived security price processes are consistent across investors;
3. commodity and securities markets clear.

The aggregate resource constraint is:

\[
\delta_t = c_t^A + c_t^B
\]

\[
\delta_t = \left( \frac{\lambda^A \xi^B e^{\rho t}}{\eta_t} \right)^{\frac{1}{\alpha-1}} + \left( \lambda^B \xi^B e^{\rho t} \right)^{\frac{1}{\alpha-1}}.
\]
Pricing measure and consumption-sharing rule

\[ \xi_t^B = e^{-\rho t} \delta_t^{\alpha-1} \left[ \left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \]

\[ c_t^A = \delta_t \times \omega(\eta_t) \quad c_t^B = \delta_t \times (1 - \omega(\eta_t)) \]

\[ \omega(\eta_t) \triangleq \frac{\left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}}} \]

- **Linear** consumption-sharing rule because same degree of risk aversion.
- **Stochastic slope** because of the improper use of signal by Group A.
Solving for equilibrium

► Can solve for pricing measure and consumption as a function of $\delta_t$, and current value of change of measure, $\eta_t$.

\[ \xi_t^i = \delta_0^{\alpha-1} \exp \left( - \int_0^t r \, dt - \frac{1}{2} \int_0^t \|\kappa^i\|^2 \, dt - \int_0^t (\kappa^i)^\top dW^i \right). \]

► Given the constant multipliers $\lambda^A$ and $\lambda^B$, and given exogenous process for $\delta$ and $\eta$, we have now characterized the complete-market equilibrium.

► To relate the Lagrange multipliers $\lambda^A$ and $\lambda^B$ to initial endowments, requires the calculation of the wealth of each group.
Securities markets implementation of complete-market equilibrium

► Financial securities available:

1. Equity, which is a claim on total output
2. Consol bond
3. Instantaneously riskless bank deposit

► The equilibrium price of a security, with \( \text{payoff} \in \{1, \delta_u, c_u^B\} \):

\[
\text{Price} \left( \delta, \eta, \hat{f}^B, \hat{g}, t \right) \triangleq \mathbb{E}^B_{\delta, \eta, \hat{f}^B, \hat{g}} \int_t^{\infty} \frac{\xi_u^B}{\xi_t} \times \text{payoff} \, du.
\]
Computing expected values to obtain prices and wealth

▶ To compute equity and bond prices and wealth, need the joint conditional distribution of $\eta_u$ and $\delta_u$, given $\delta_t, \eta_t, \hat{f}_t^A, \hat{g}_t$ at $t$.

▶ Not easy to obtain joint distribution but its characteristic function $\mathbb{E}^{B}_{\hat{f}^B, \hat{g}} \left[ \left( \begin{array}{c} \delta_u \\ \delta \end{array} \right)^\varepsilon \left( \begin{array}{c} \eta_u \\ \eta \end{array} \right)^\chi \right]; \varepsilon, \chi \in \mathbb{C}$ can be obtained in closed form.

▶ Three effects:

1. Effect of growth and variance of $\delta$
2. Effect of variance of $\eta$ ($\varepsilon = 0$)
3. Effect of correlation between $\delta$ and $\eta$
Results

The interest rate

▶ Average belief

\[ \hat{f}^M \triangleq \hat{f}^A \times \omega(\eta) + \hat{f}^B \times (1 - \omega(\eta)). \]

▶ Holding \( \hat{f}^M \) fixed, \( \hat{g} \) represents the effect of pure dispersion of beliefs

▶ The rate of interest can then be written as:

\[
r(\eta, \hat{f}^M, \hat{g}) = \rho + (1 - \alpha) \hat{f}^M - \frac{1}{2} (1 - \alpha) (2 - \alpha) \sigma_\delta^2 \times \hat{g}^2 \times \omega(\eta) \times [1 - \omega(\eta)].
\]

▶ The interest rate is increasing in \( \hat{f}^M \) (for all \( \alpha \)) and \( \hat{g} \) (for \( \alpha < 0 \)).
Market Prices of Risk

The market prices of risk in the eyes of Population $B$ and $A$ are:

$$\kappa^B (\eta, \hat{g}) = \begin{bmatrix} (1 - \alpha) \sigma_\delta \\ 0 \end{bmatrix} + \hat{g} \times \omega(\eta) \times \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix},$$

$$\kappa^A (\eta, \hat{g}) = \begin{bmatrix} (1 - \alpha) \sigma_\delta \\ 0 \end{bmatrix} - \hat{g} \times [1 - \omega(\eta)] \times \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}.$$

Under agreement ($\hat{g} = 0$), the prices of risk include a reward for output risk $W_\delta$, but zero reward for signal risk $W_s$.

With disagreement, investors realize that probability measure of other population will fluctuate randomly. Hence, require a risk premium for vagaries of others.
Benchmark Parameter Values

The parameter values that we specify are based on estimation of models similar to ours in Brennan-Xia (2001).

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters for aggregate endowment and the signal</td>
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<td></td>
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<tr>
<td>Long-term average growth rate of aggregate endowment</td>
<td>$\bar{f}$</td>
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<td>Volatility of expected growth rate of endowment</td>
<td>$\sigma_f$</td>
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<tr>
<td>Volatility of aggregate endowment</td>
<td>$\sigma_\delta$</td>
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<tr>
<td>Mean reversion parameter</td>
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<tr>
<td>Volatility of the signal</td>
<td>$\sigma_s$</td>
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<tr>
<td>Parameters for the agents</td>
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<td></td>
</tr>
<tr>
<td>Agent A’s correlation between signal and mean growth rate</td>
<td>$\phi$</td>
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</tr>
<tr>
<td>Agent B’s correlation between signal and mean growth rate</td>
<td>—</td>
<td>0</td>
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<tr>
<td>Agent A’s initial share of aggregate endowment</td>
<td>$\lambda^B/\lambda^A$</td>
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<tr>
<td>Time-preference parameter for both agents</td>
<td>$\rho$</td>
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</tr>
<tr>
<td>Relative risk aversion for both agents</td>
<td>$1 - \alpha$</td>
<td>3</td>
</tr>
</tbody>
</table>
Plots

▶ All plots have on the $x$-axis
  • Either $\hat{g}$ measuring \textit{disagreement}.
  • Or, $\omega$ measuring \textit{relative size} of irrational group.

▶ All plots have two curves for rationality and \textit{irrationality}:
  • A \textit{red-dotted} curve representing the case of $\phi = 0.00$
  • A \textit{blue-dashed} curve representing the case of $\phi = 0.95$
Irrationality leads to a drop in prices of equity and bonds.

Prices decrease with disagreement.
Even modest population of irrational traders makes sizable difference.

Heterogeneity increases further the drop in prices.
Volatilities: Effect of irrationality and disagreement

Dispersion of beliefs and presence of irrational traders increase volatility (same is true for correlation)
Volatilities: Effect of heterogeneity

- Presence of a few rational investors not sufficient to drive down volatility.
If rationality ($\phi = 0$) and agreement ($\hat{g} = 0$): 100% in equity, 0% in bonds because both investors identical

If rationality but $\hat{g} \neq 0$, $B$ still 100% in equity and speculates on future growth with only bond

Under irrationality, $B$ holds less equity than he/she would in a rational market, (unless wildly optimistic). Scared of noise.
Portfolio of Group $B$: Static and Intertemporal Hedging

- Intertemporal hedge driven mostly by desire to hedge $\hat{g}$ fluctuations
Survival of Population $A$—Irrational agents

This figure shows expected value of Population $A$’s consumption share as a function of time measured in years.

This is survival of traders who are fickle: sometimes overoptimistic, sometimes overpessimistic.
Conclusions

► We have modeled excessive volatility arising from
  • excessive fluctuations of anticipations of irrational investors
► Even a modest-sized irrational population makes quite a difference
► What rational investor can do:
  • Take positions on current differences in beliefs
  • Hedge against future revisions in:
    ★ Market’s beliefs
    ★ Their own beliefs
  • Bonds are useful instruments in doing so
► Irrational traders survive a long time
  • Excessive volatility is not easy to “arbitrage”
  • Excessive volatility, if it is there, is likely to remain