Putting neurons in culture: The cerebral foundations of reading and mathematics

II. Space, time and number: cerebral foundations of mathematical intuitions

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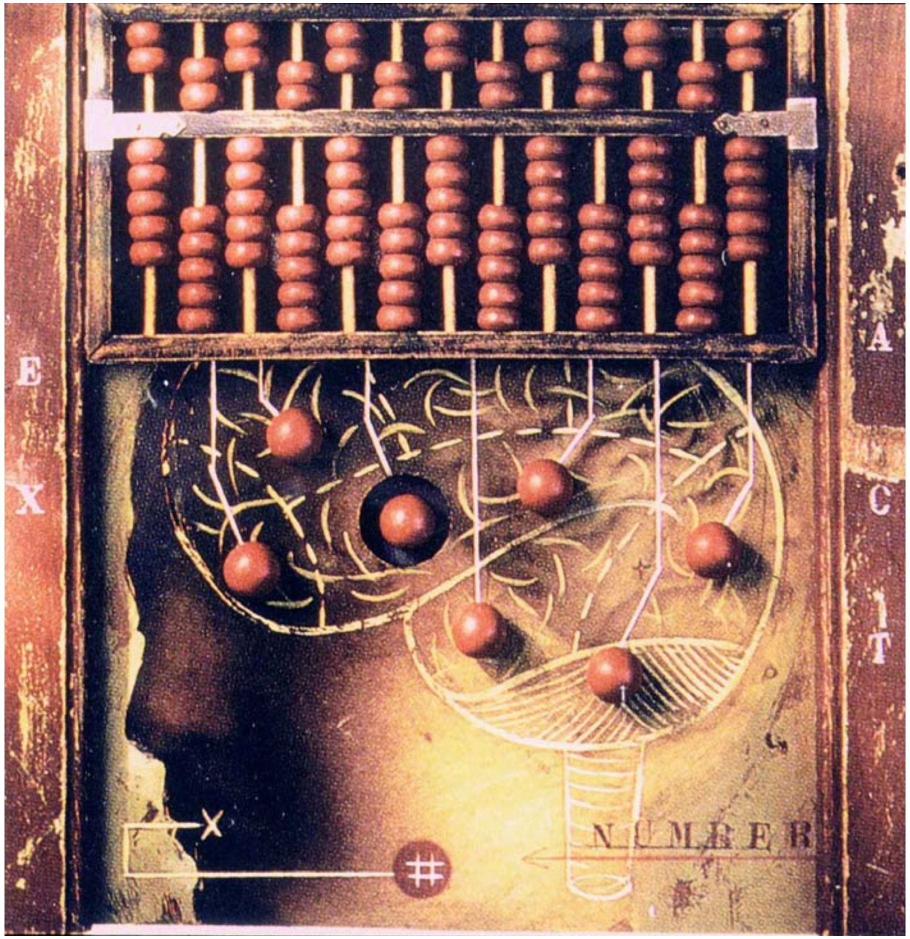
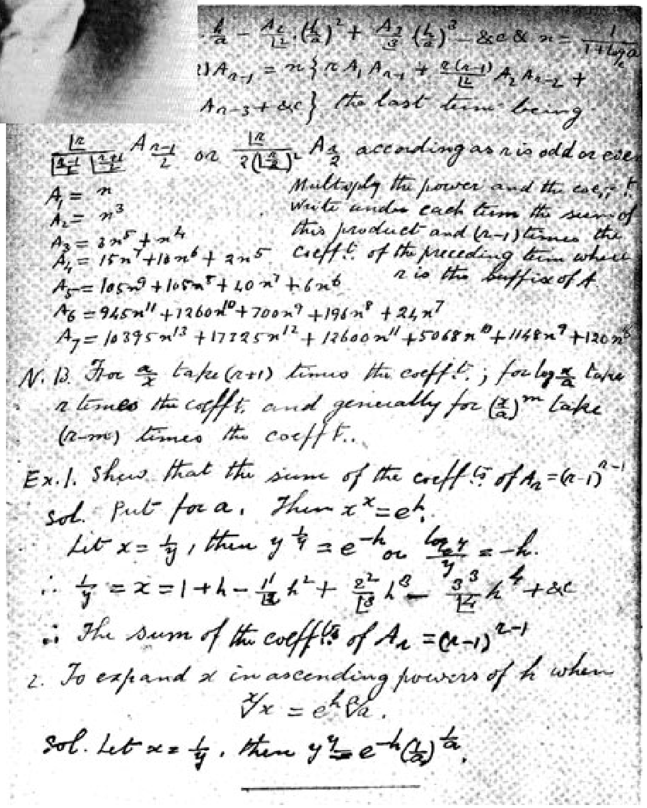


Two mathematicians

Srinivasa Ramanujan
(1887-1920)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$



$\frac{h}{a} - \frac{A_2}{\lfloor 2}(\frac{h}{a})^2 + \frac{A_3}{\lfloor 3}(\frac{h}{a})^3 - \&c \;\&\; n = \frac{1}{1+\log a}$

$A_{n-1} = n\{ n A_1 A_{n-1} + \frac{n(n-1)}{\lfloor 2} A_2 A_{n-2} +$
$A_{n-3} + \&c\}$ the last term being

$\frac{\lfloor n}{\lfloor \frac{n-1}{2} \lfloor \frac{n+1}{2}} A_{\frac{n-1}{2}}$ or $\frac{\lfloor n}{2(\lfloor \frac{n}{2})^2} A_{\frac{n}{2}}$ according as n is odd or even

$A_1 = n$
$A_2 = n^3$
$A_3 = 3n^5 + n^4$
$A_4 = 15n^7 + 10n^6 + n^5$
$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$
$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$
$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

Multiply the power and the coeff.t Write under each term the sum of this product and $(n-1)$ times the coeff.t of the preceding term where n is the suffix of A.

N.B. For $\frac{a}{x}$ take $(n+1)$ times the coeff.t; for $\log\frac{x}{a}$ take n times the coeff.t and generally for $(\frac{x}{a})^m$ take $(n-m)$ times the coeff.t.

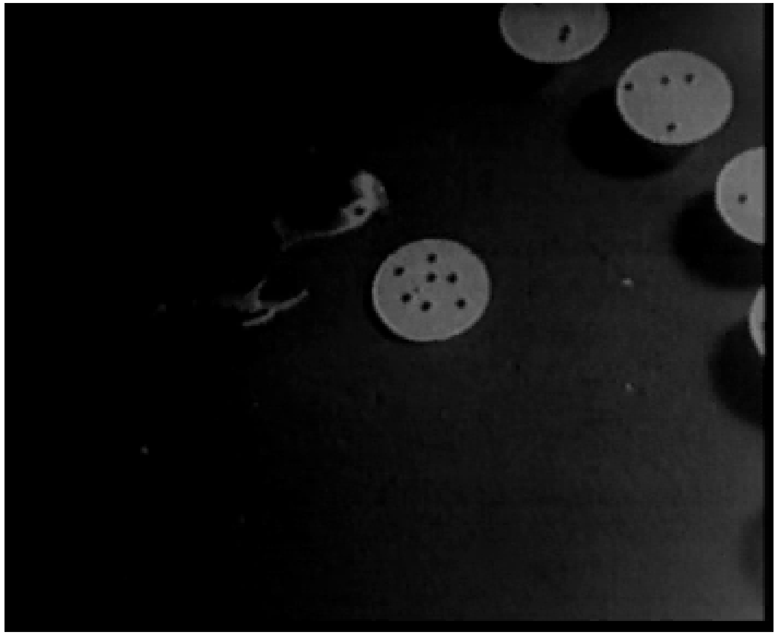
Ex. 1. Show that the sum of the coeff.ts of $A_n = (n-1)^{n-1}$

Sol. Put for a. Then $x^x = e^h$.
Let $x = \frac{1}{y}$, then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{\log y}{y} = -h$.
$\therefore \frac{1}{y} = x = 1 + h - \frac{1^1}{\lfloor 2}h^2 + \frac{2^2}{\lfloor 3}h^3 - \frac{3^3}{\lfloor 4}h^4 + \&c$
$\therefore$ The sum of the coeff.ts of $A_n = (n-1)^{n-1}$

2. To expand x in ascending powers of h when $\sqrt[x]{x} = e^h \sqrt[a]{a}$.

Sol. Let $x = \frac{1}{y}$, then $y^y = e^{-h}(\frac{1}{a})^{\frac{1}{a}}$.

Otto Köhler's parrott (ca. 1955)



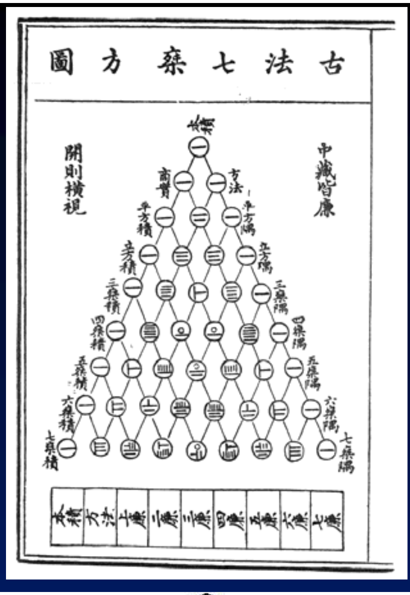
Disagreement over the nature of mathematics

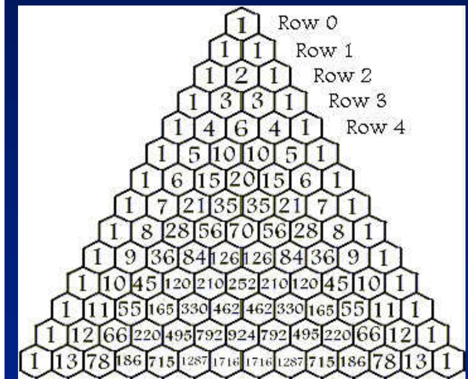
- A corpus of absolute truths, independent of the human mind (Platonism):

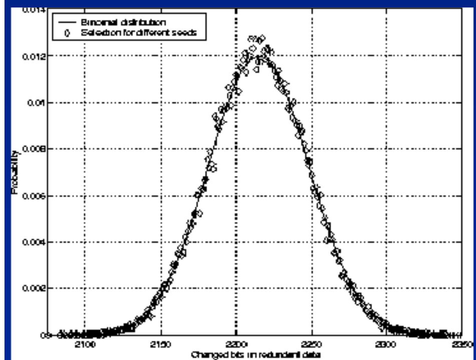
  « I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our “creations,” are simply our notes of our observations » (Hardy)

- A creation of the human brain

  « “Mathematical objects” correspond to physical states of our brain » (Jean-Pierre Changeux)







Two problems in the philosophy of mathematics

- The problem of its "absolute truth"

"Mathematics takes us into the region of absolute necessity, to which not only the actual word, but every possible word, must conform." (Bertrand Russell)

How does a finite and fallible human brain come to know some absolute mathematical truths, agreed upon by all, and seemingly waiting for their discovery since all eternity?

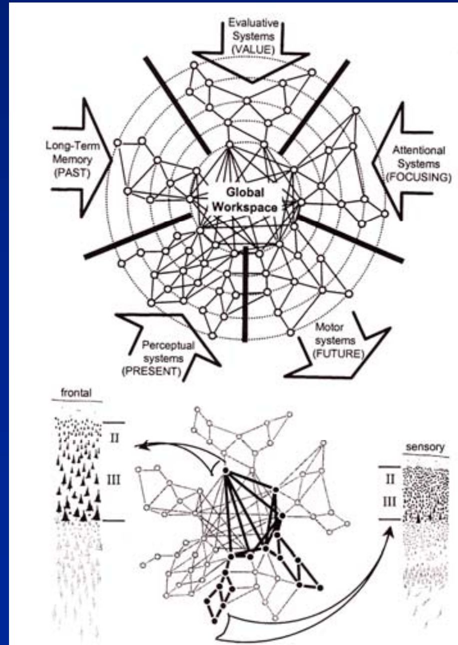
- The problem of its « unreasonable effectiveness in the natural sciences » (Wigner)

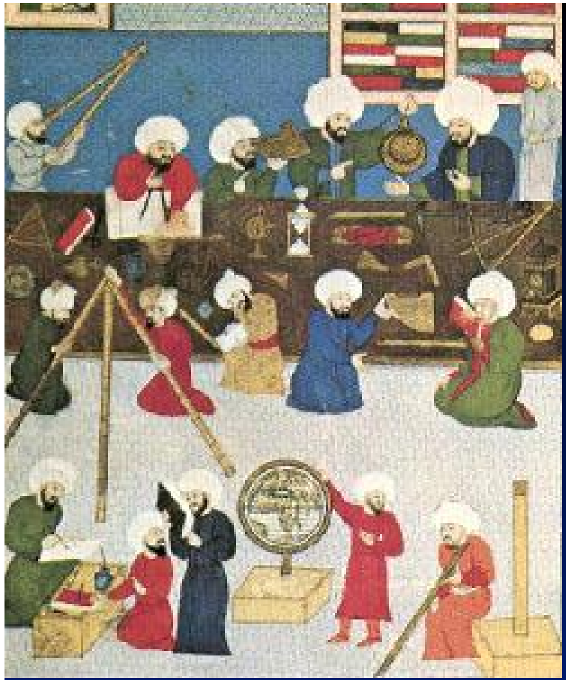
"How is it possible that mathematics, a product of human thought that is independent of experience, fits so excellently the objects of physical reality?" (Einstein).

The origins of mathematics in a cognitive neuroscience perspective

- During its evolution, our primate brain has been endowed with elementary representations that are adequate to certain aspects of the external world.
- These internalized representations of space, time and number, shared with many animal species, provide the **foundations of mathematics**.
- Why are we the only species capable of mathematics?
- Humans may possess a unique ability to mobilize in a top-down manner and reconnect in novel ways their evolutionary ancient brain processors. As a result:

- arbitrary symbols can be attached to quantities and other non-verbal concepts
- disparate concepts can be integrated into an over-arching framework (e.g. the number-space metaphor)





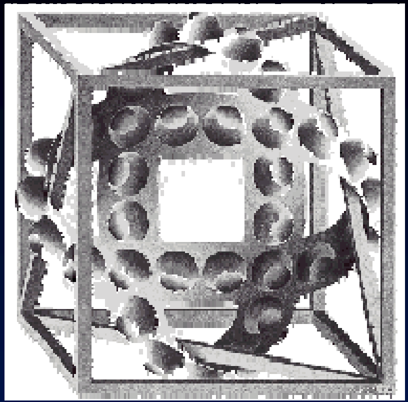
Mathematical reality from a cognitive neuroscience perspective: « absolute truth »

- The ultimate products of mathematics are so tightly constrained by the pre-existing structure of our mental representations that they appear to us as a rigid body of absolute truths.
- However, their cultural construction is fuzzy and chaotic

"Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere." (W.S. Anglin)

- Intuition plays an essential role in the invention of mathematics:

"Even though pure mathematics could do without it, it is always necessary to come back to intuition to bridge the abyss [that] separates symbol from reality." (H. Poincaré, *La logique et l'intuition*, 1889; cited by P. Gallison, 2003)



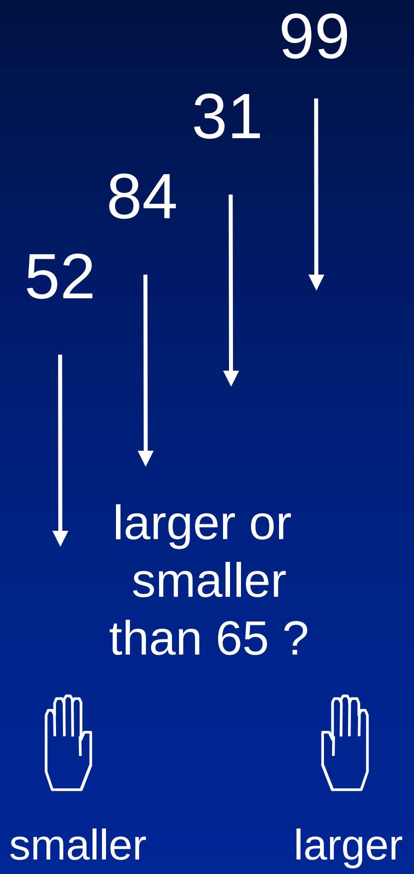
Mathematical reality from a cognitive neuroscience perspective: « unreasonable effectiveness »

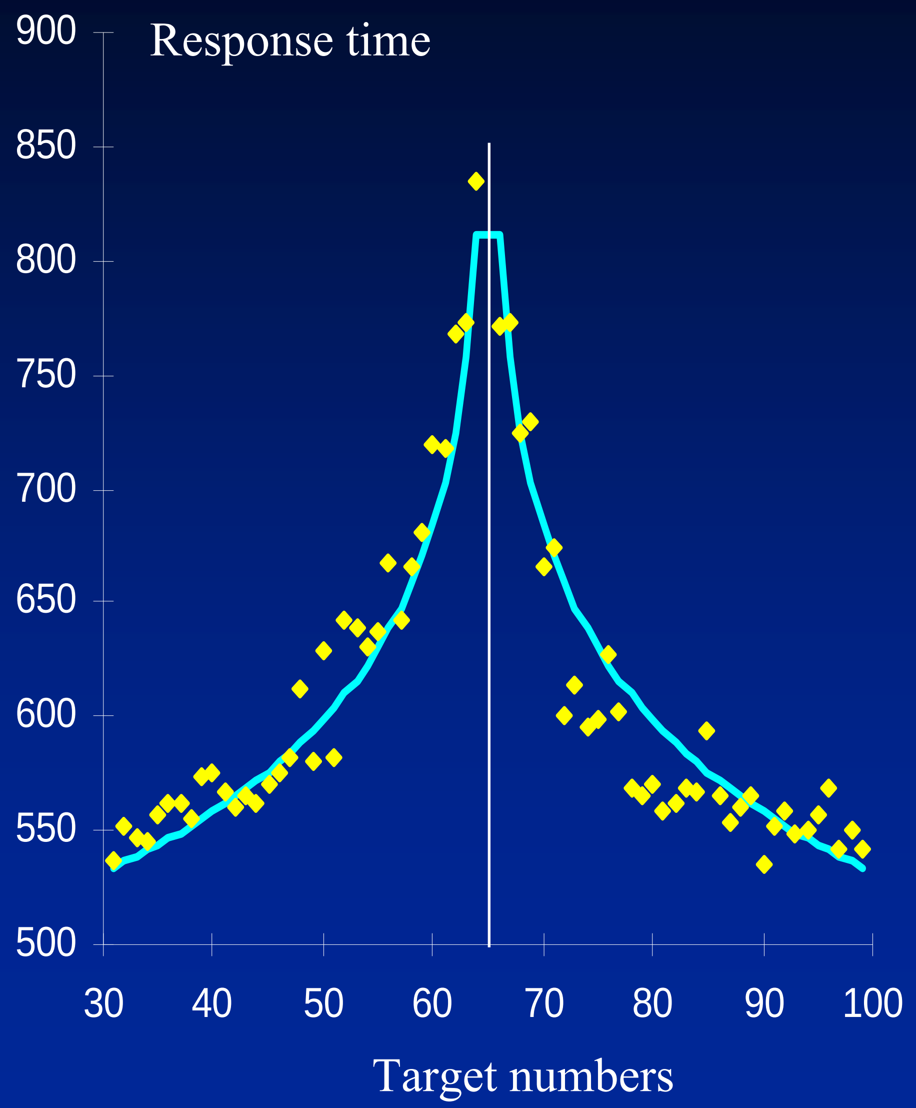
- Mathematicians constantly create new mathematical “objects”, many of which are not adapted to the external physical world
- Some are adapted, however, because
  - They are founded on basic representations which have proven useful during evolution (e.g. sense of number, space, time)
  - Mathematicians and physicists keep selecting them for their explanatory adequacy

"There is nothing mysterious, as some have tried to maintain, about the applicability of mathematics. What we get by abstraction from something can be returned." (R.L. Wilder)

The Distance Effect in number comparison

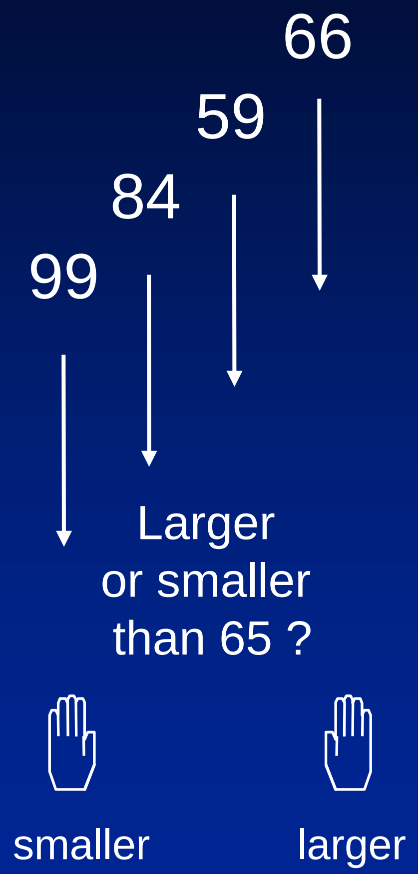
(first discovered by Moyer and Landauer, 1967)

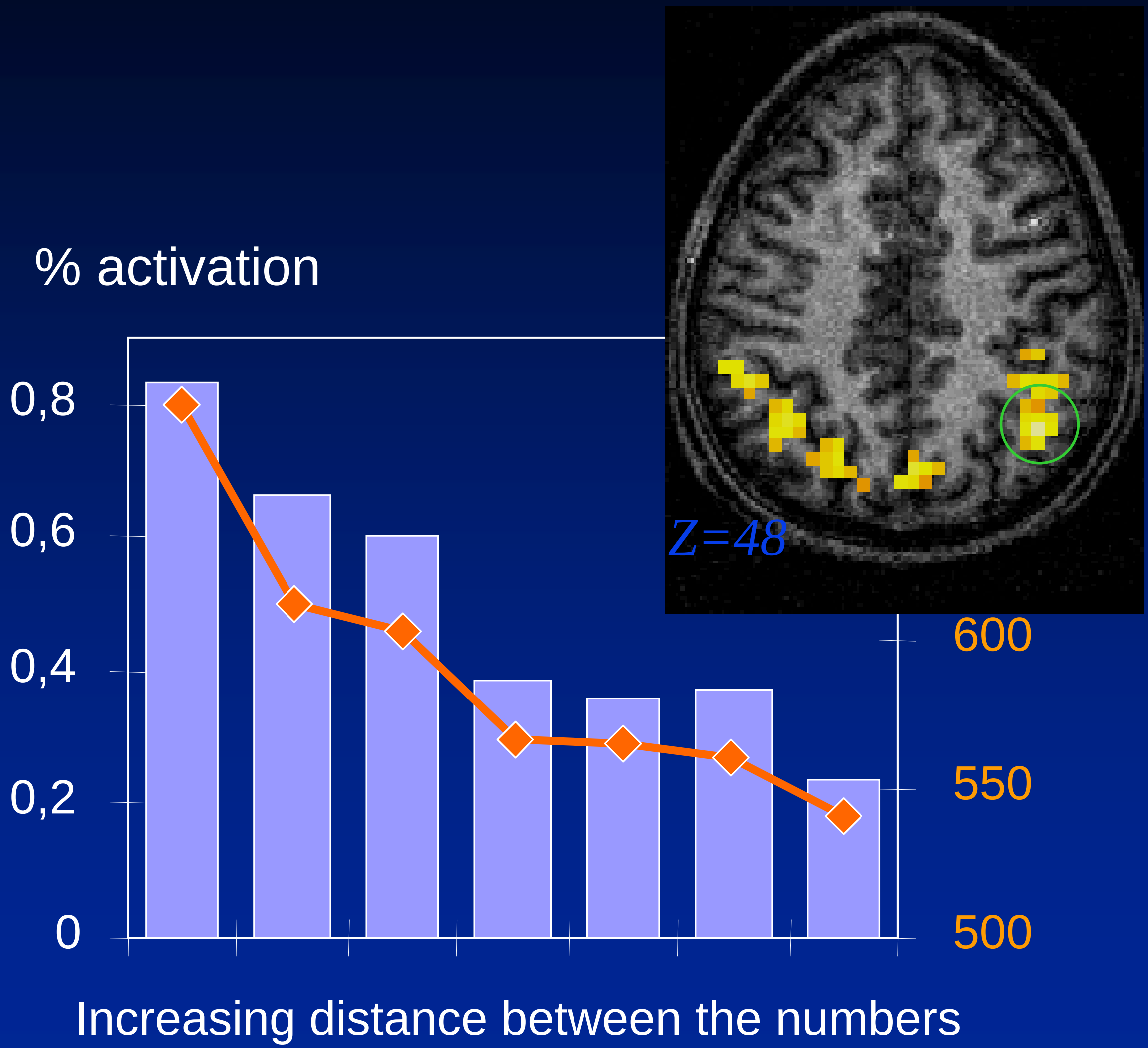




Dehaene, S., Dupoux, E., & Mehler, J. (1990). *Journal of Experimental Psychology: Human Perception and Performance, 16*, 626-641.

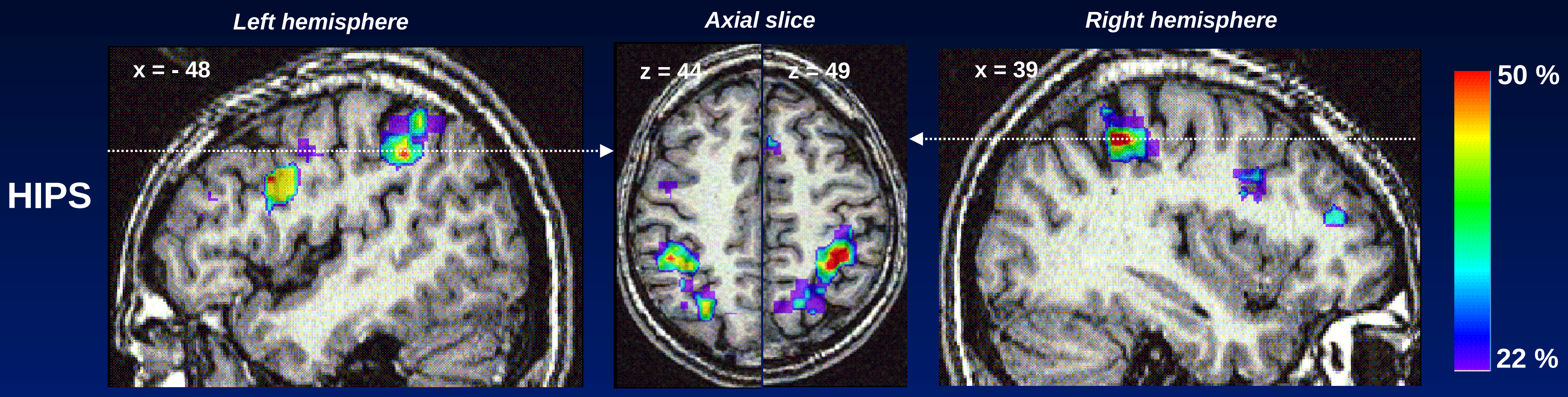
Neural bases of the distance effect





Pinel, P., Dehaene, S., Riviere, D., & LeBihan, D. (2001). *Neuroimage, 14*(5), 1013-1026.

Previous studies of number sense and the horizontal segment of the intraparietal sulcus (HIPS)



- All numerical tasks activate this region
  (e.g. addition, subtraction, comparison, approximation, digit detection…)

- This region fulfils two criteria for a semantic-level representation:

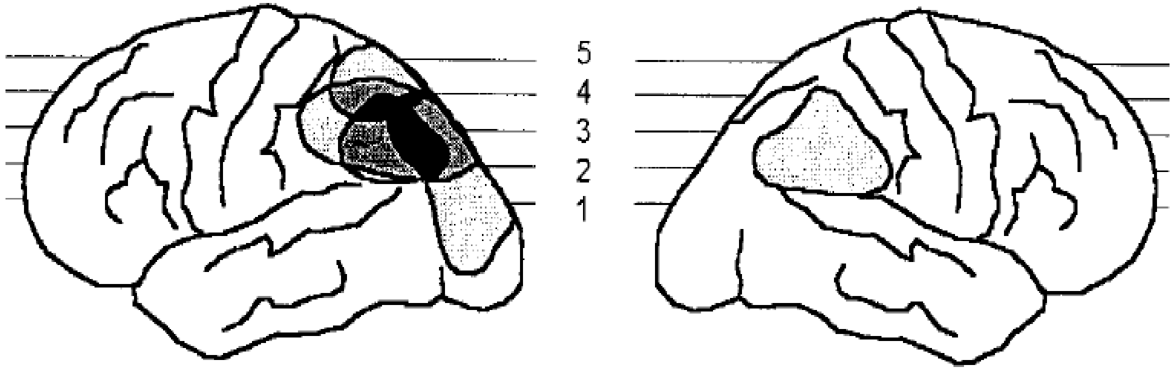
-It responds to number in various formats (Arabic digits, written or spoken words), more than to other categories of objects (e.g. letters, colors, animals…)

-Its activation varies according to a semantic metric (numerical distance, number size)

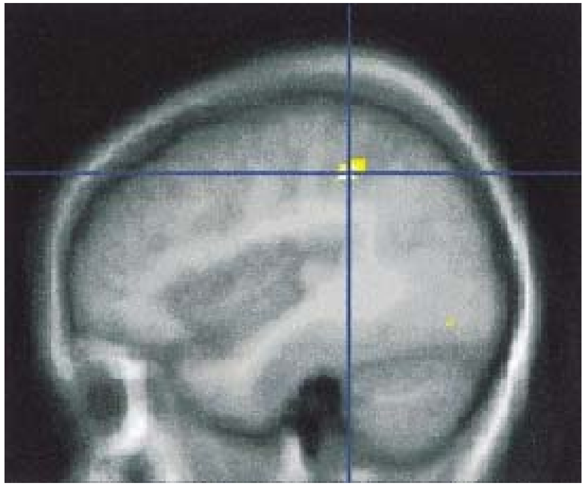
Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). *Cognitive Neuropsychology*

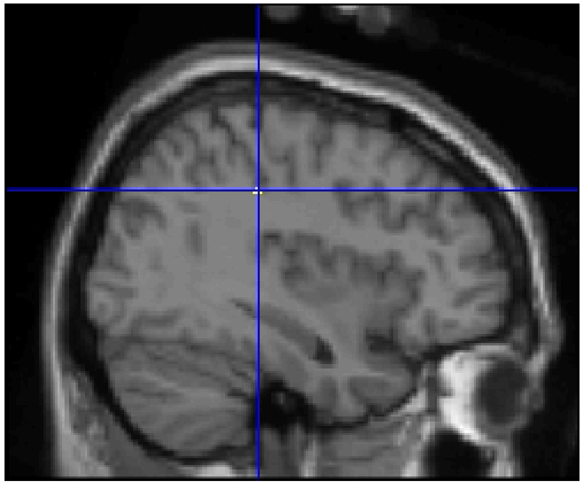
Parietal dysfunction causes impairments in number sense

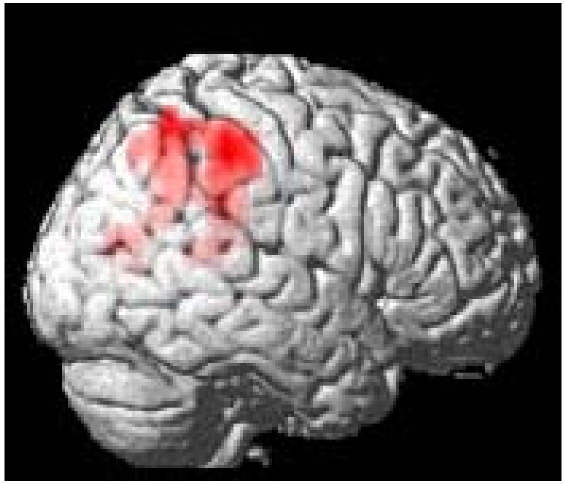
Lesions causing acalculia in adults



Anomalies correlated with developmental dyscalculia in children







Dyscalculic adults born pre-term show missing gray matter in the intraparietal sulcus, compared to non-dyscalculic pre-term controls. (Isaacs et al., 2001)

Turner's syndrome (monosomy 45-X) is frequently associated with dyscalculia. We found that a group of Turners girls showed both structural and functional alterations in the intraparietal sulcus (Molko et al., 2003)

Numerical and visuo-spatial maps in the parietal lobe

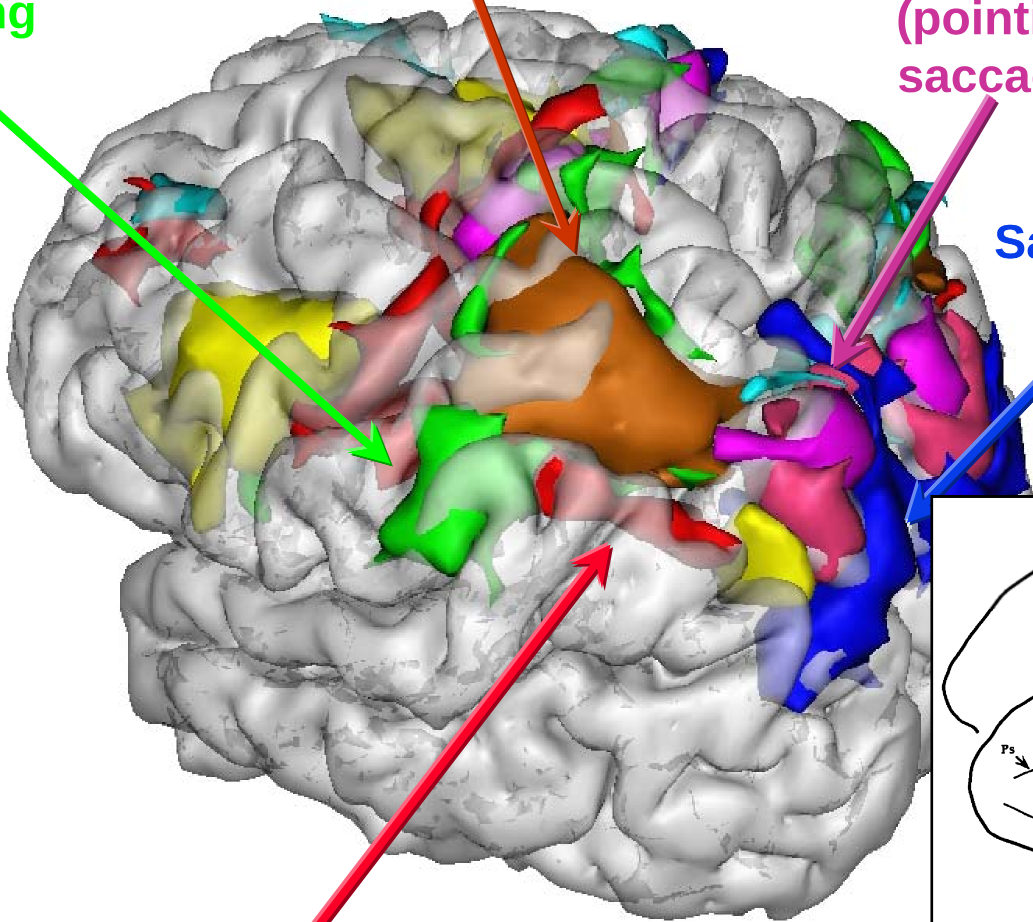
(Simon et al., *Neuron* 2002, *Neuroimage* 2004)

Manual tasks (grasping, pointing)

Grasping

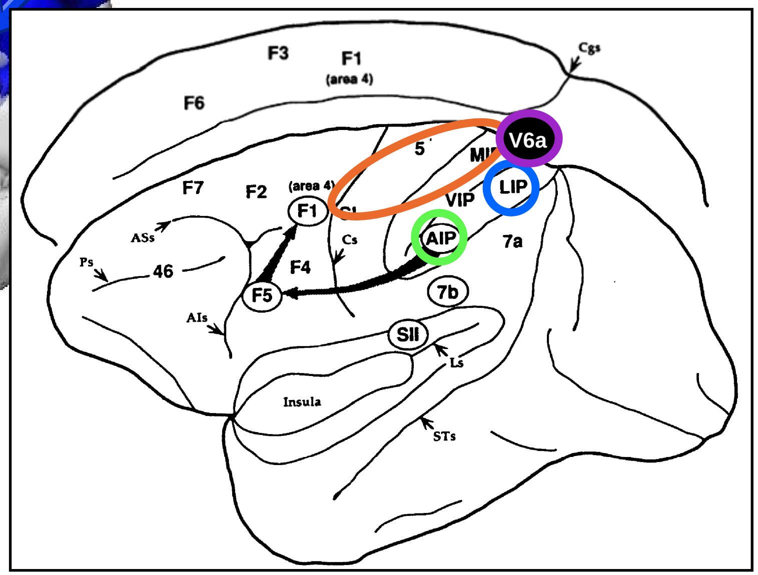
Visuo-spatial tasks (pointing, grasping, saccades, attention

Saccades



Calculation only

Monkey brain

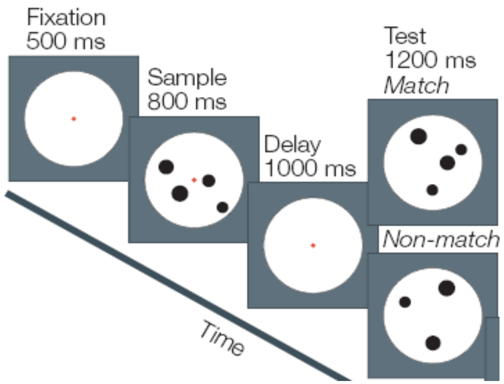


Number neurons in the monkey

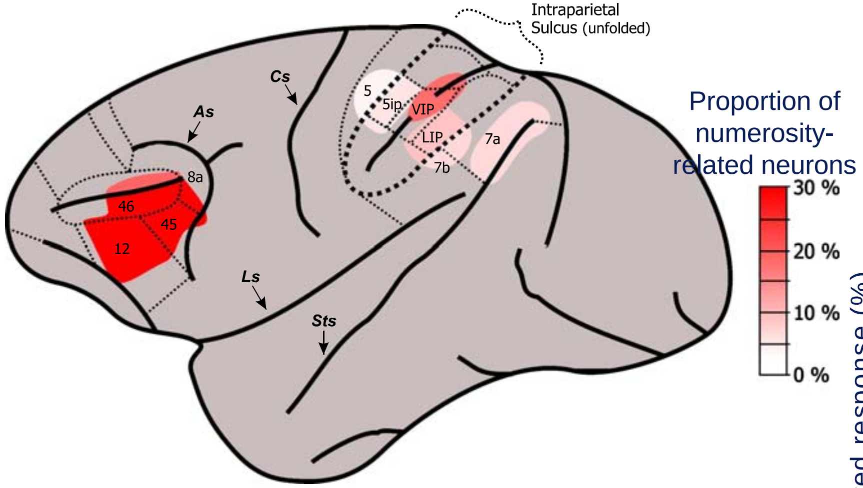
(Nieder, Freedman & Miller, 2002; Nieder & Miller, 2003, 2004, 2005)

Task:

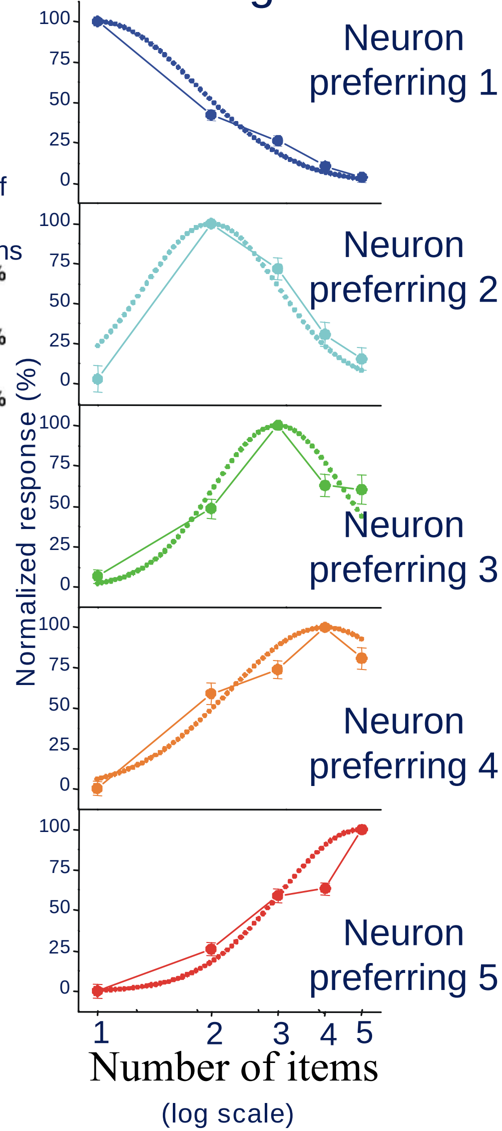
same-different judgement with small numerosities



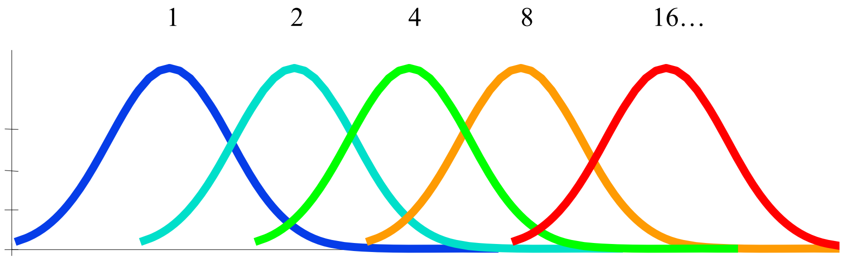
Anatomy



Neuronal firing rates



The Dehaene-Changeux (1993) model:
Coding by Log-Gaussian numerosity detectors



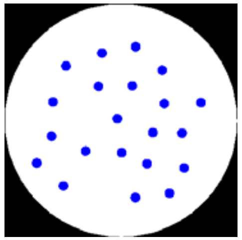
Internal logarithmic scale : log(*n*)

Nieder, A., Freedman, D. J., & Miller, E. K. (2002). Representation of the quantity of visual items in the primate prefrontal cortex. *Science, 297*(5587), 1708-1711.
Nieder, A., & Miller, E. K. (2003). Coding of cognitive magnitude. Compressed scaling of numerical information in the primate prefrontal cortex. *Neuron, 37*(1), 149-157.

From numerosity detectors to numerical decisions: Elements of a mathematical theory

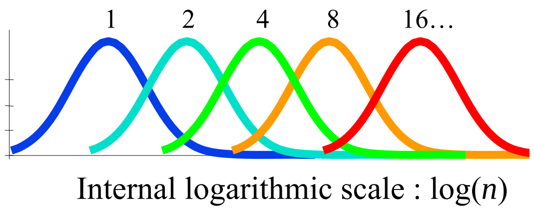
(S. Dehaene, *Attention & Performance*, 2006, in press)

Stimulus of numerosity *n*

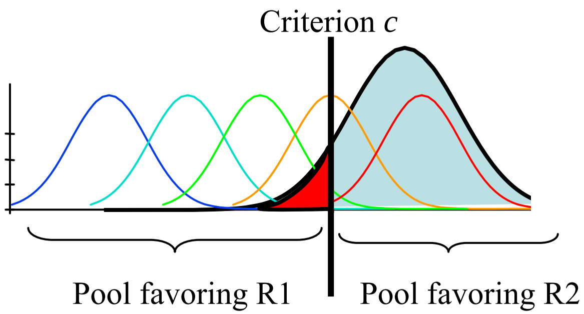


Response in simple arithmetic tasks:
-Larger or smaller than x?
-Equal to x?

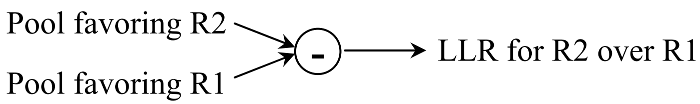
1. Coding by Log-Gaussian numerosity detectors



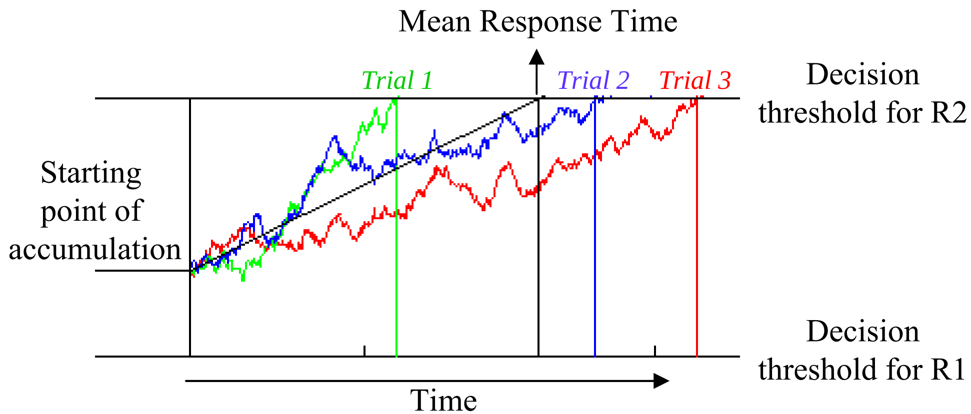
2. Application of a criterion and formation of two pools of units



3. Computation of log-likelihood ratio by differencing

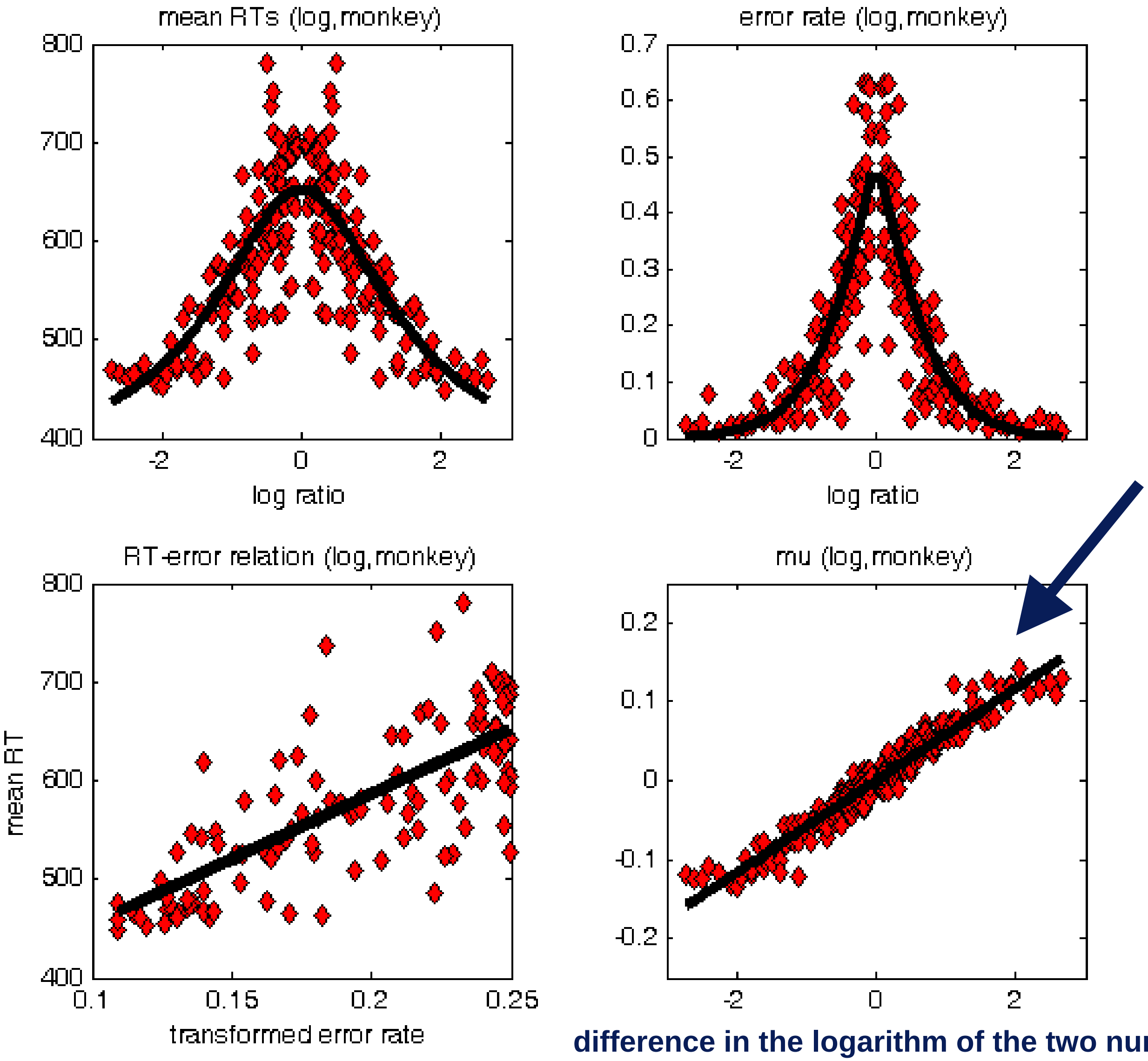


4. Accumulation of LLR, forming a random-walk process



Example: Which of two numerosities is the larger?

Data from Cantlon & Brannon (2006)



difference in the logarithm of the two numbers

Subjects = monkeys
Stimuli = sets of dots

Crucial hidden variable:

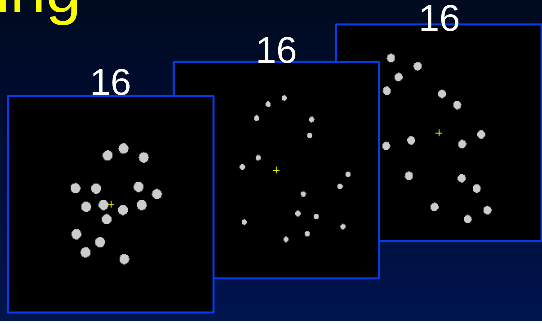
Amount of information accumulated per unit of time

Varies linearly with the difference in the logarithm of the two numbers

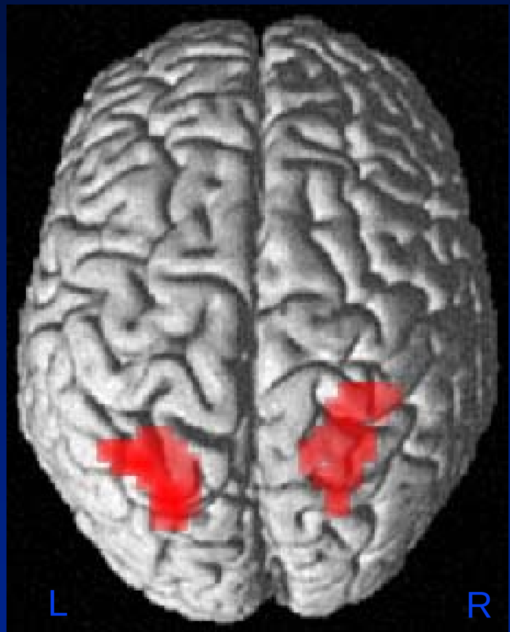
Does human IPS contain number neurons? fMRI adaptation reveals Log-Gaussian turning in the human intraparietal sulcus

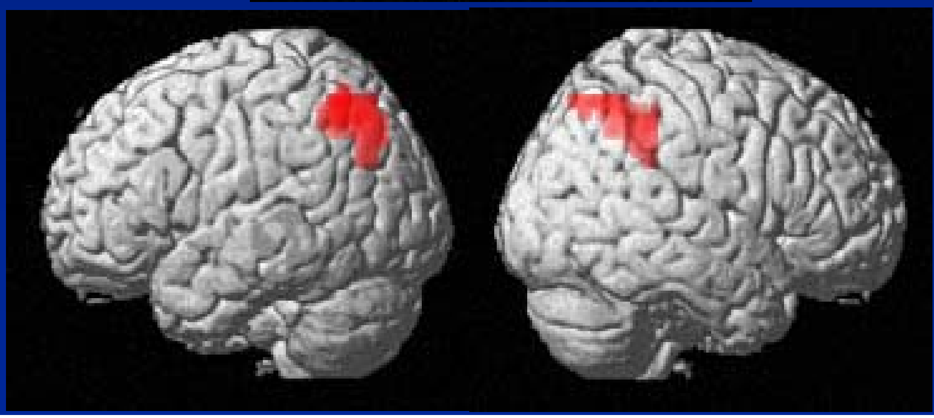
Piazza, Izard, Pinel, Le Bihan & Dehaene, Neuron 2004

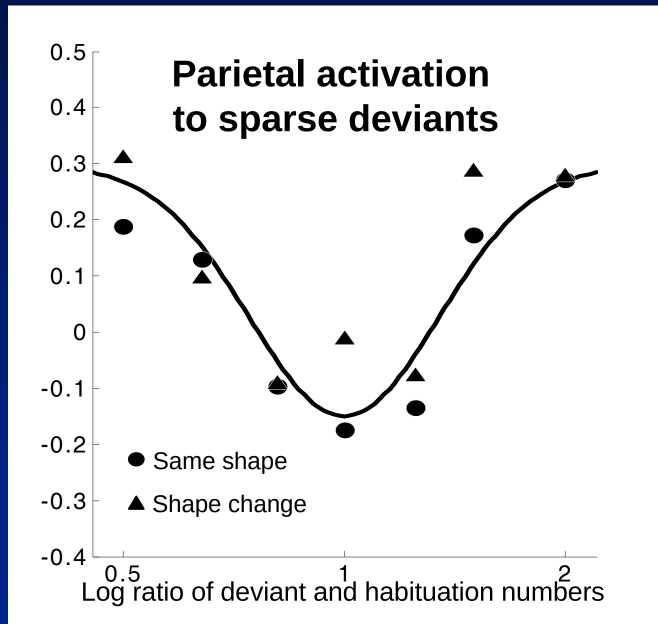
Adaptation to a fixed number

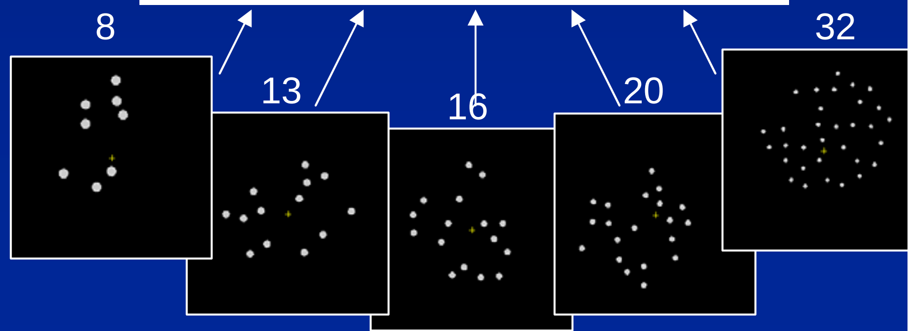


Regions responding to a change in number





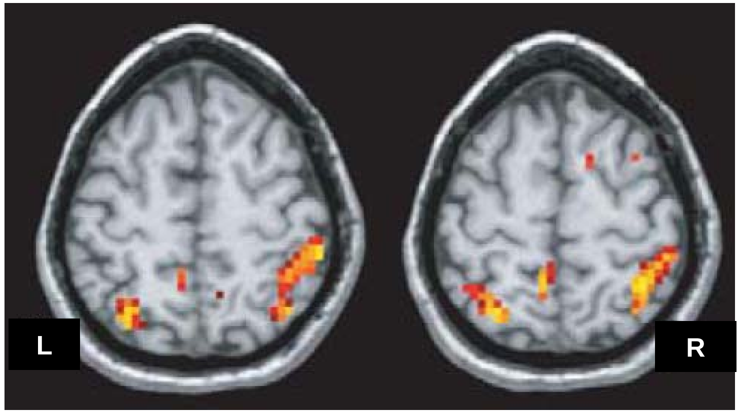




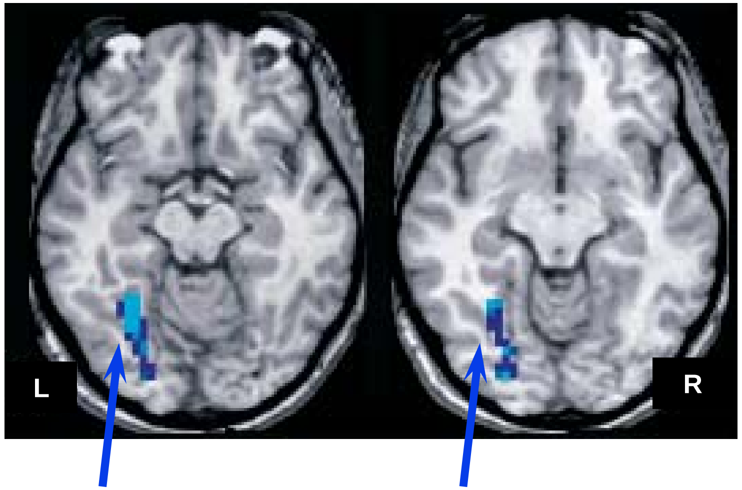
A basic dorsal-ventral organization for shape vs number

Improved FMRI adaptation design by Cantlon, Brannon et al. (PLOS, 2006)

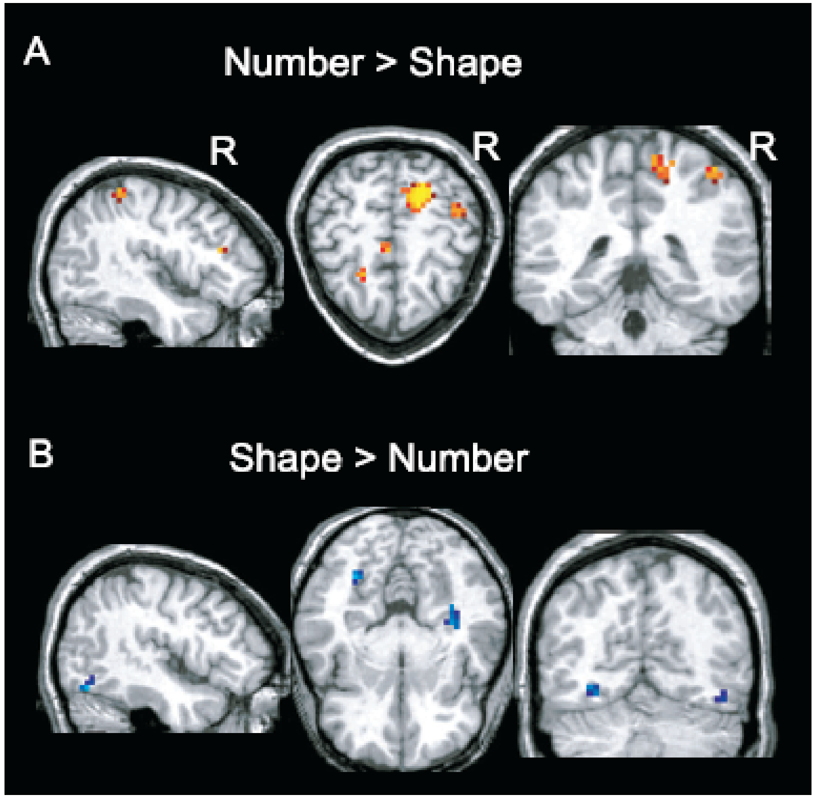
Number change > Shape change



Shape change > Number change

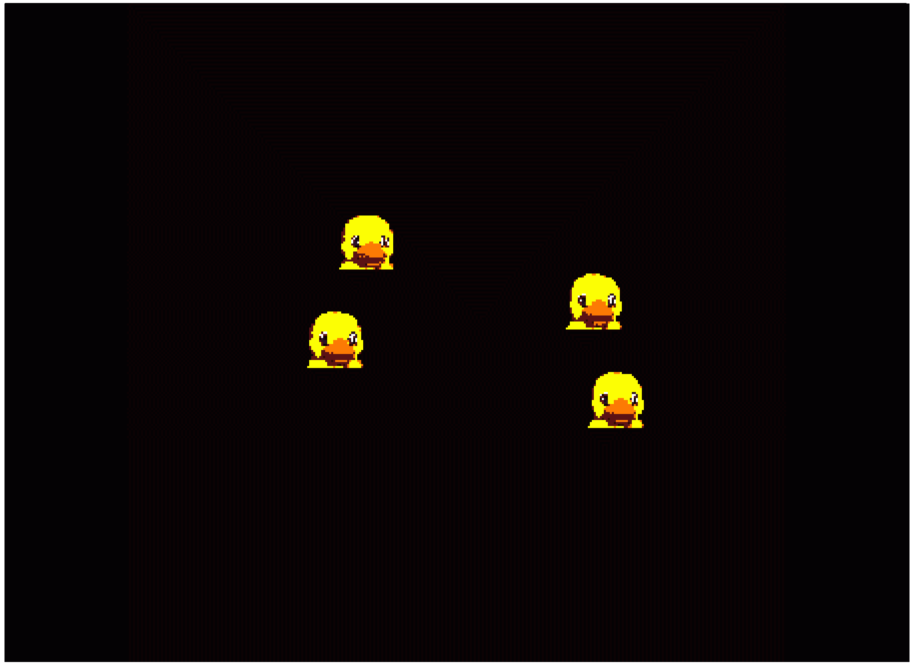


This organization is already present in four-year-olds



Do infants show numerosity adaptation and recovery?

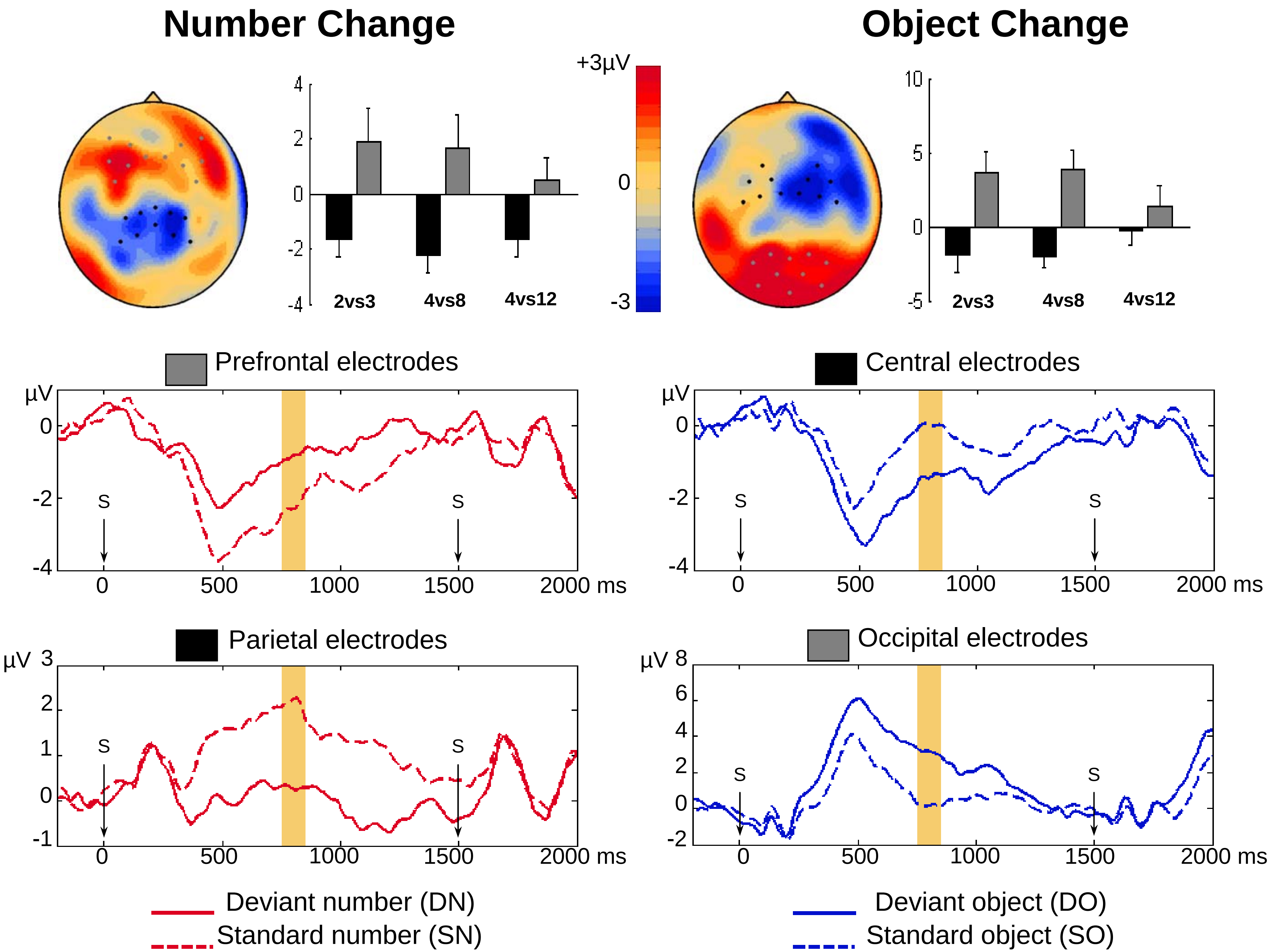
(Izard, Dehaene-Lambertz & Dehaene, submitted)



2 x 2 design : numerosity and/or object change

3 pairs of numerosities:
4 vs 8 ; 4 vs 12 ; 2 vs 3

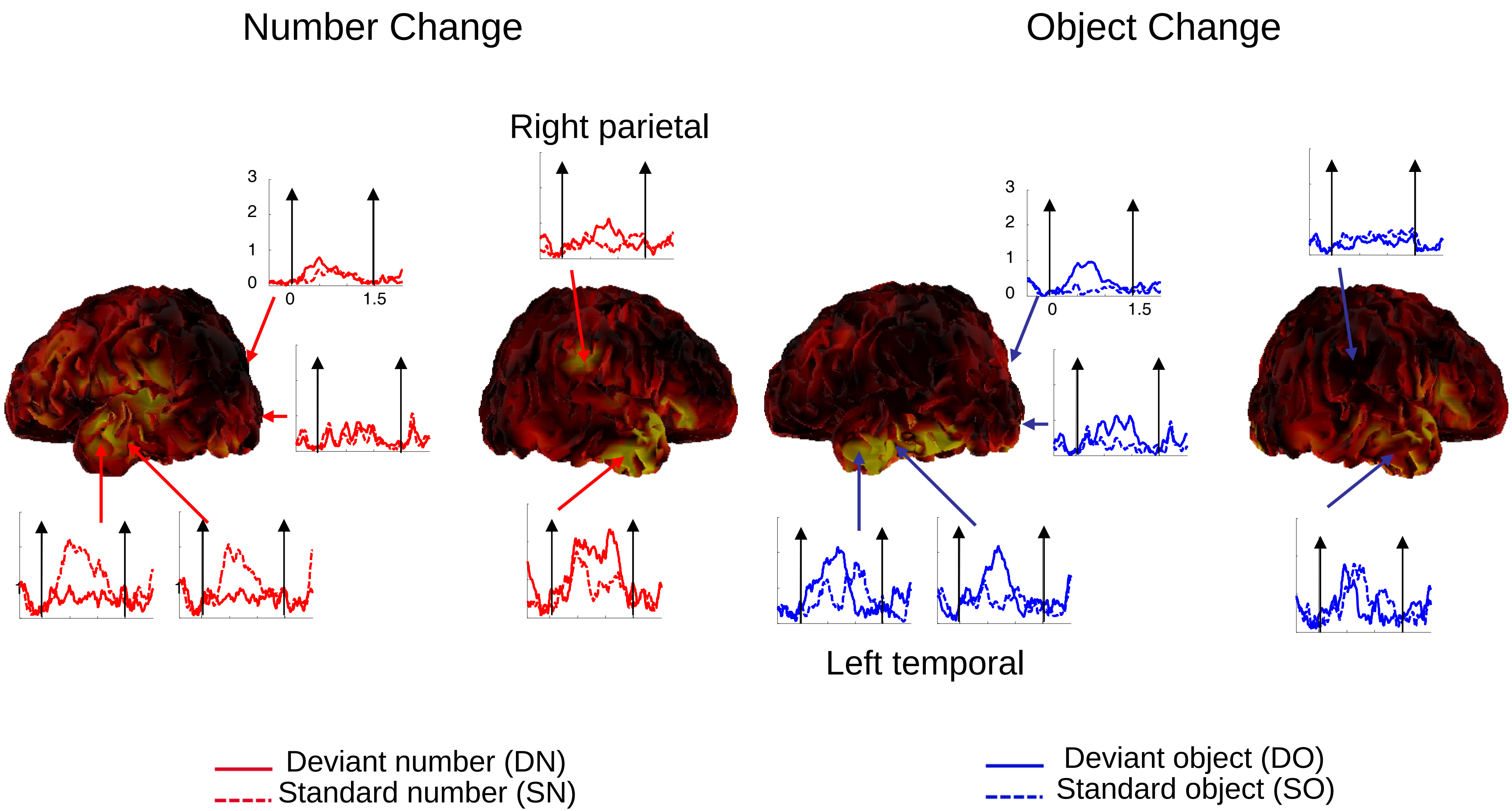
Twelve 3-4 month-old infants in each group



Number Change
Object Change
+3µV
0
-3
2vs3
4vs8
4vs12
Prefrontal electrodes
Central electrodes
Parietal electrodes
Occipital electrodes
µV
S
0
500
1000
1500
2000 ms
Deviant number (DN)
Standard number (SN)
Deviant object (DO)
Standard object (SO)

A basic dorsal / ventral organization in 3-4 month old infants:

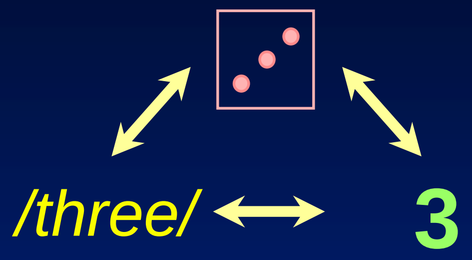
Right parietal response to number, left temporal response to objects



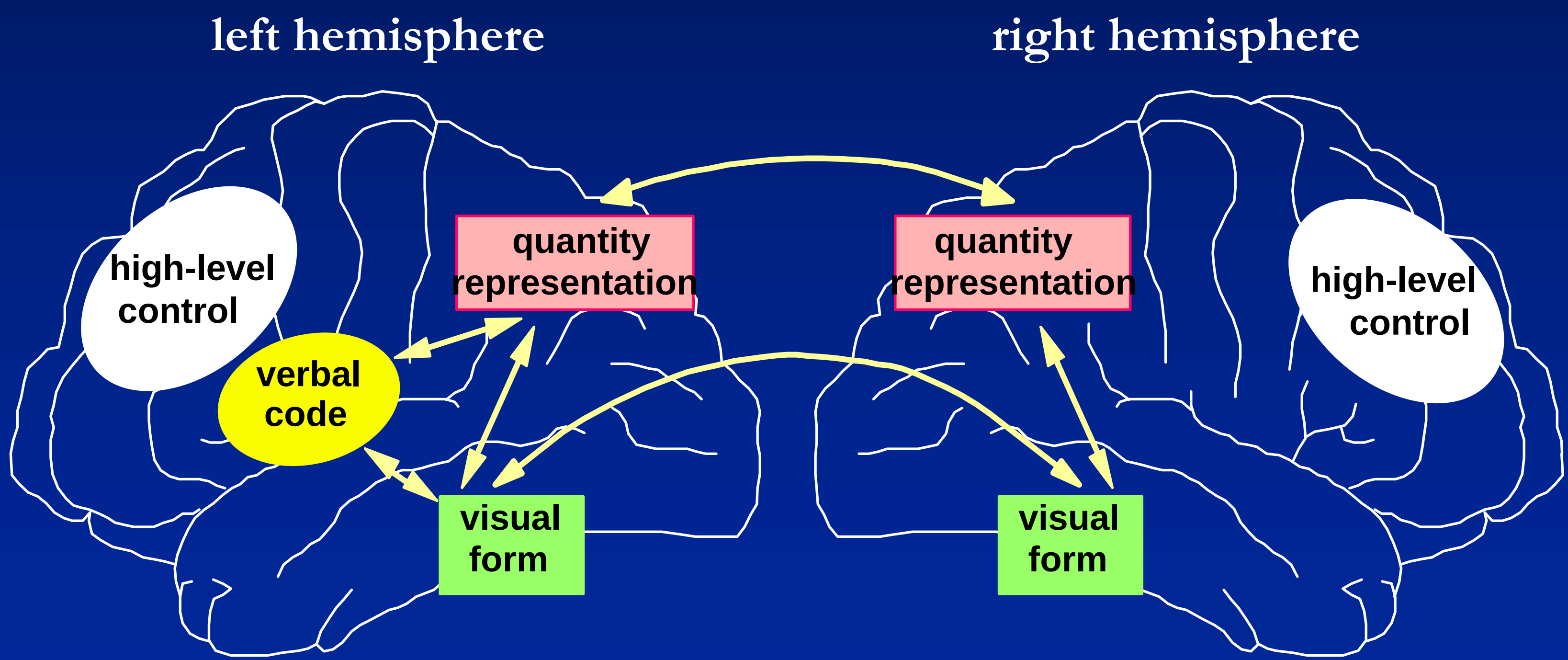
Attaching symbols to quantities:
The triple-code model of number processing

Quantity code
Number comparison; Proximity judgment; Approximation
Quantity manipulations (e.g. subtraction)

Verbal code
Spoken comprehension and production
rote tables (e.g. multiplication)



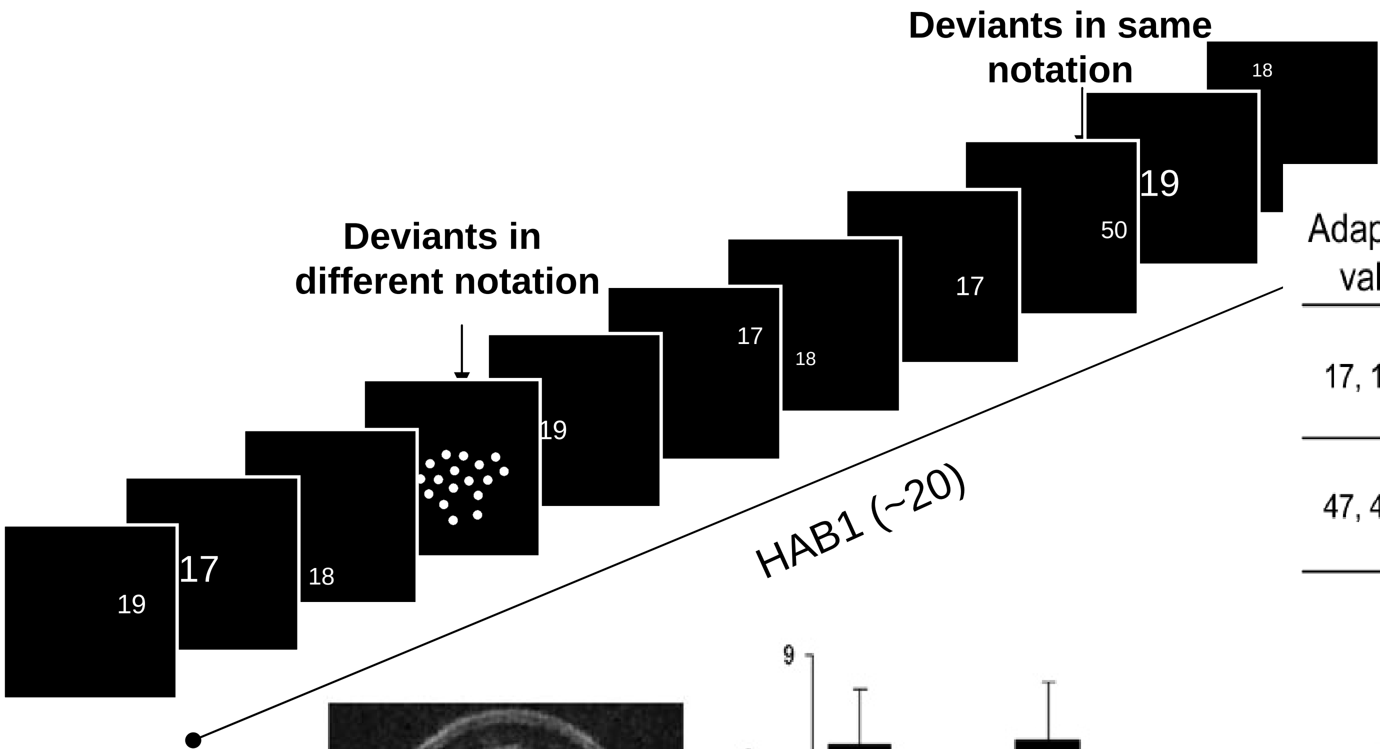
Arabic code
Arabic reading and writing
multidigit calculations



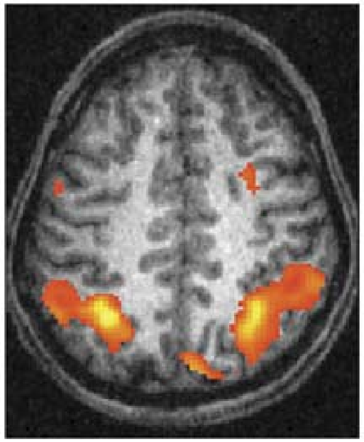
An fMRI study of cross-notation adaptation

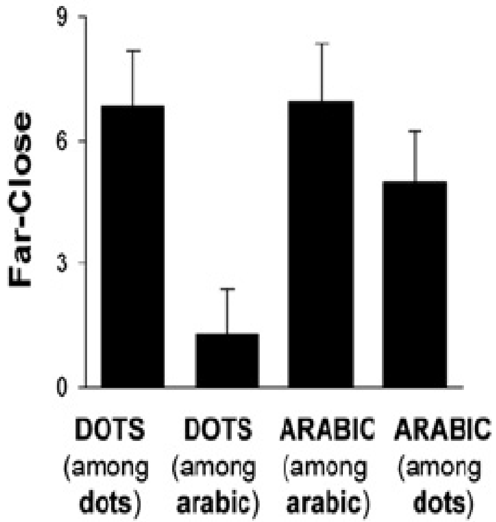
Piazza, Pinel and Dehaene, Neuron 2007

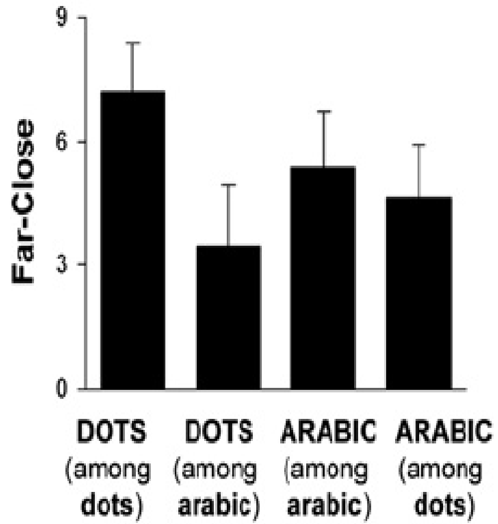
• Do the same neurons code for the symbol 20 and for twenty dots?



Adaptation values	Deviant values 20	Deviant values 50
17, 18, or 19	close	far
47, 48, or 49	far	close

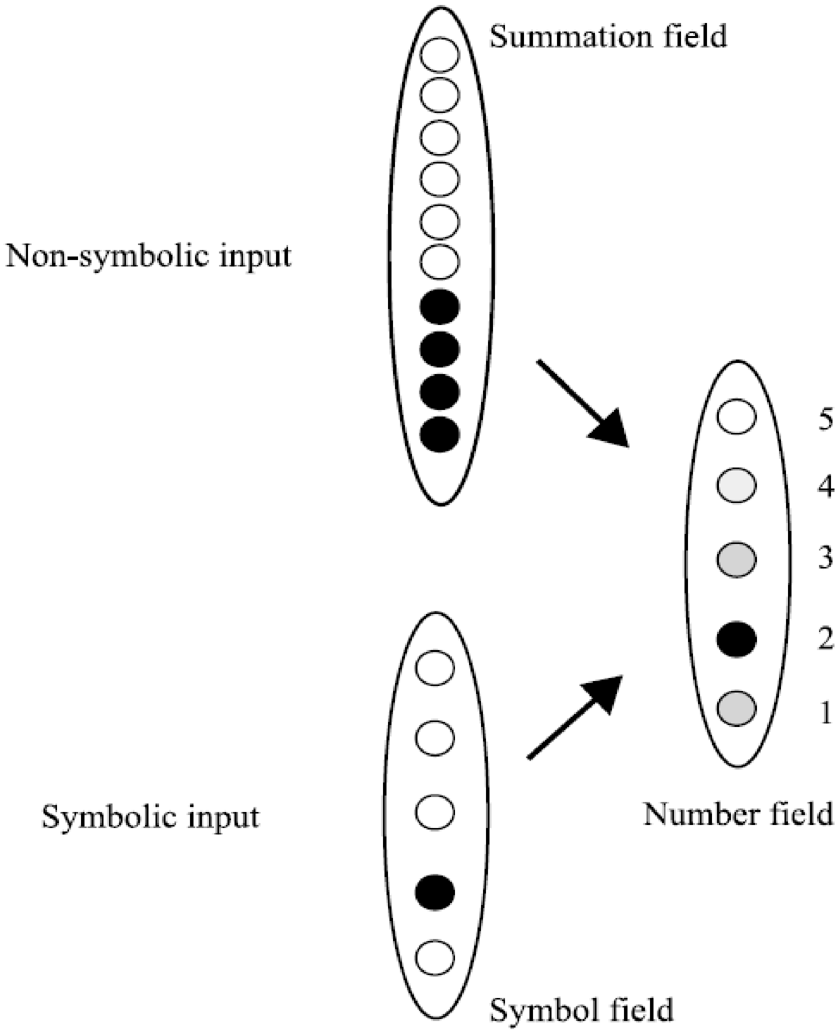






The numerosity representation may be changed by learning symbols

Verguts, T., & Fias, W. (2004). Representation of number in animals and humans: a neural model.
J Cogn Neurosci, 16(9), 1493-1504.

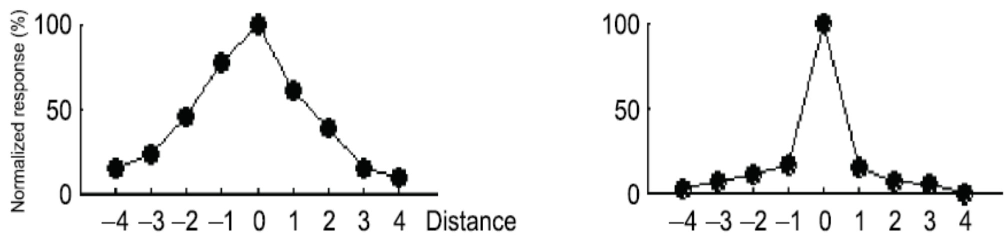


After learning, in the ***same*** neurons…

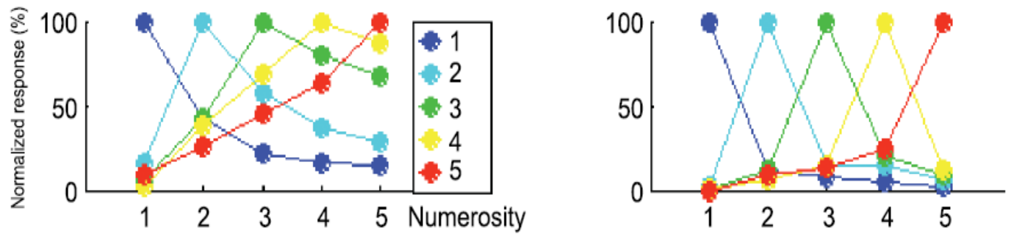
The numerosity tuning curves become narrower

Non-symbolic input

Symbolic input

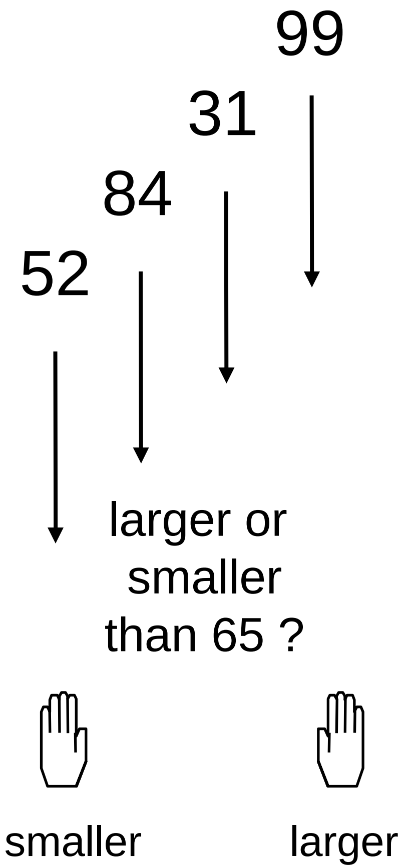


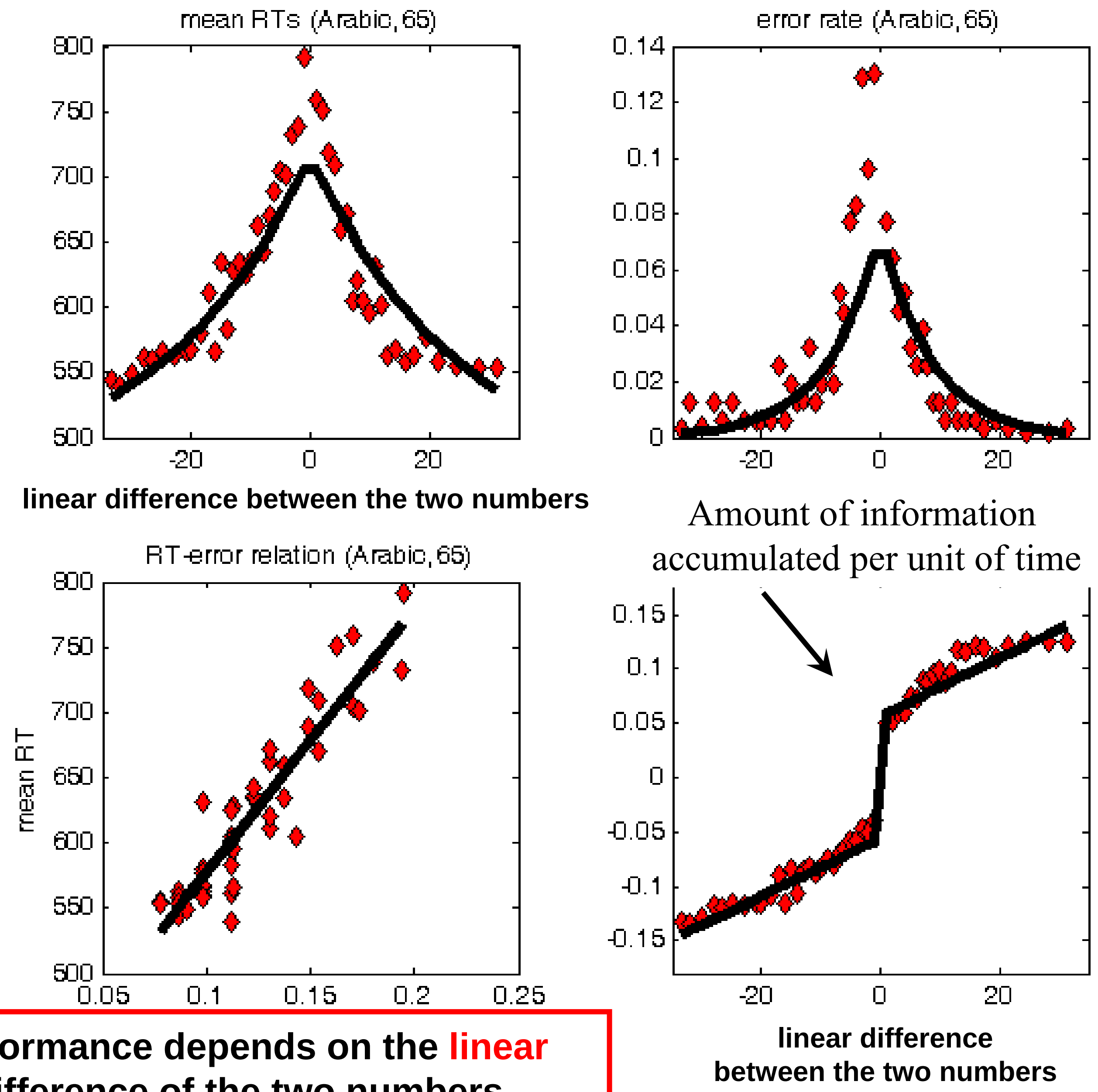
They cease to increase in width with number



Which of two *Arabic numerals* is the larger?

Subjects = humans
Stimuli = Arabic numerals



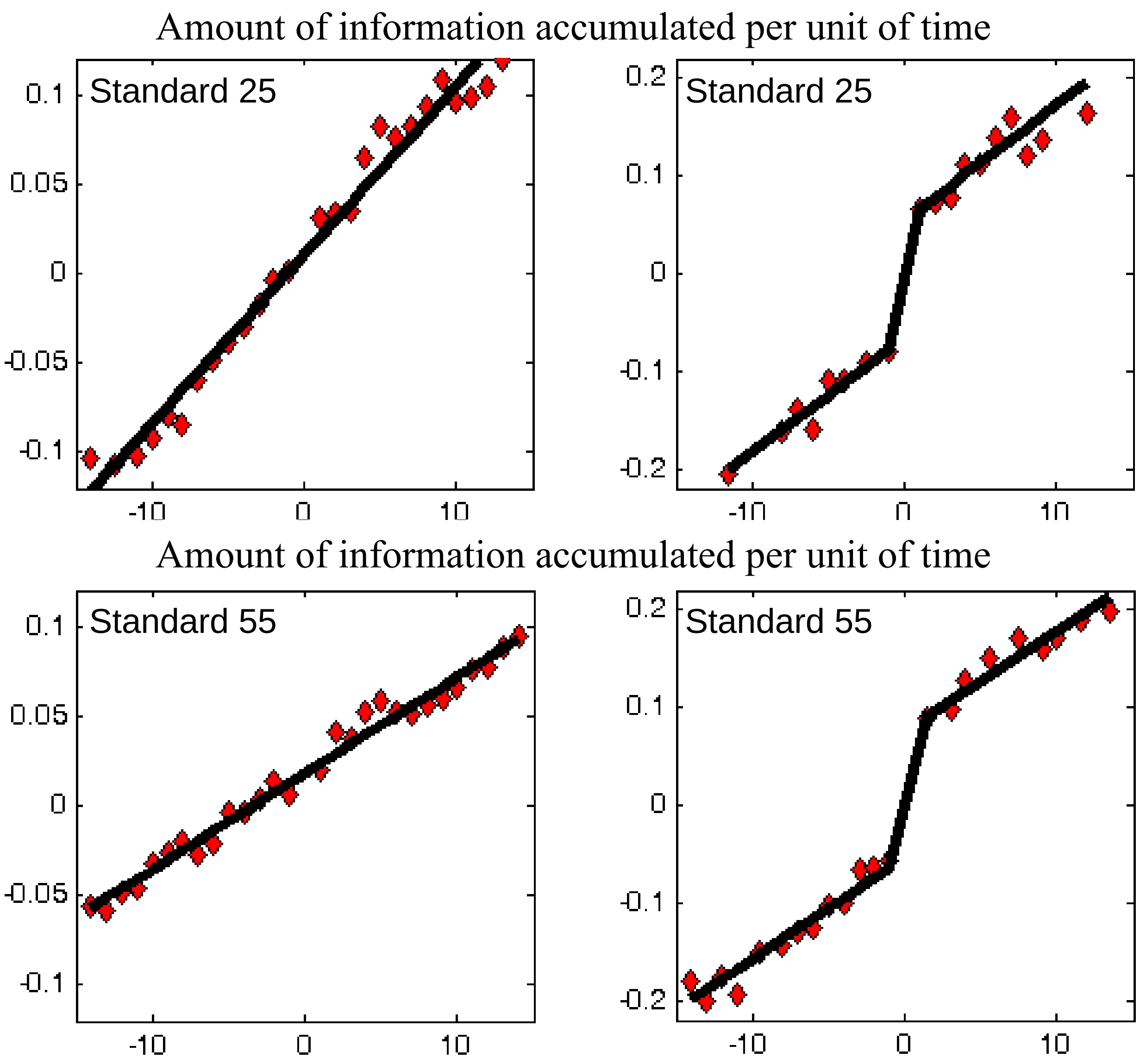


Performance depends on the linear difference of the two numbers

Non-symbolic and symbolic comparison within the same subjects

10 human adults compared sets of dots or Arabic numerals to a fixed reference, either 25 or 55

Non-symbolic comparison | **Symbolic comparison**



Conclusion

The number representation is profoundly different for symbolic and non-symbolic numbers:

-Exact, not approximate representation

-Linear, not logarithmic representation

Development of the linear understanding of number

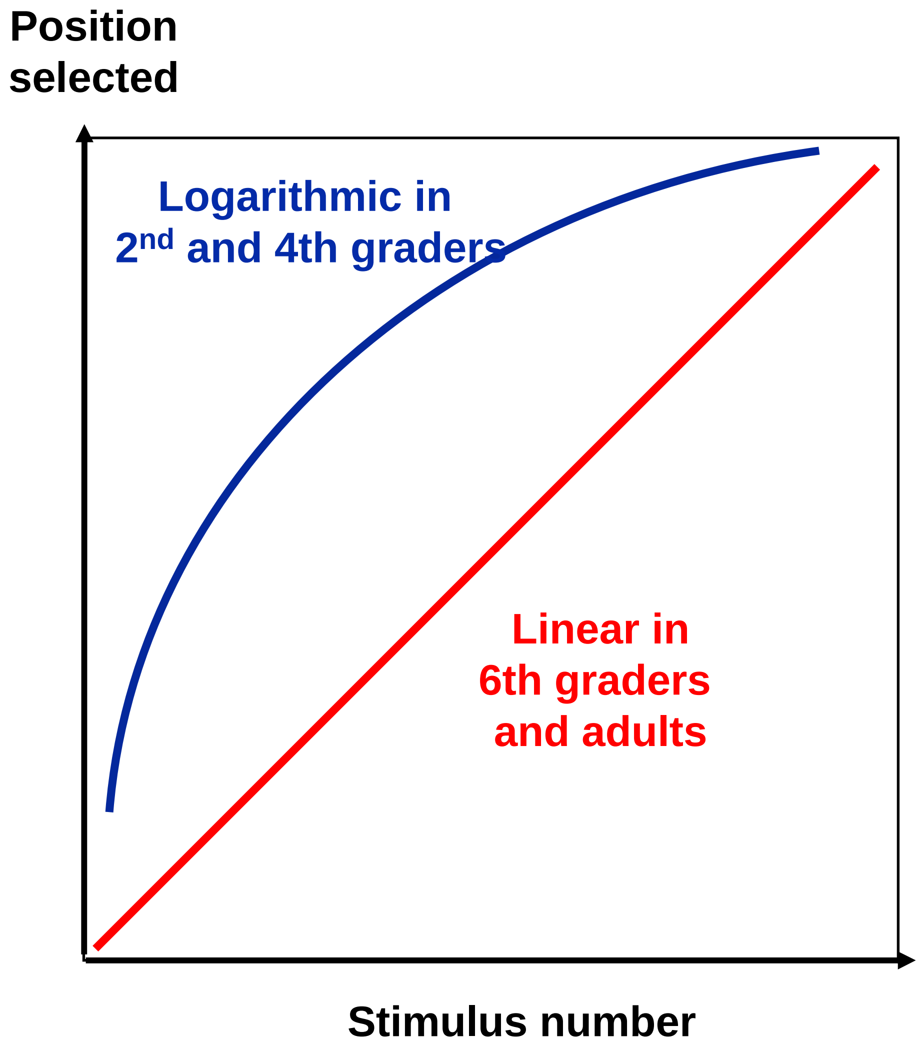
(Siegler & Opfer, 2003)

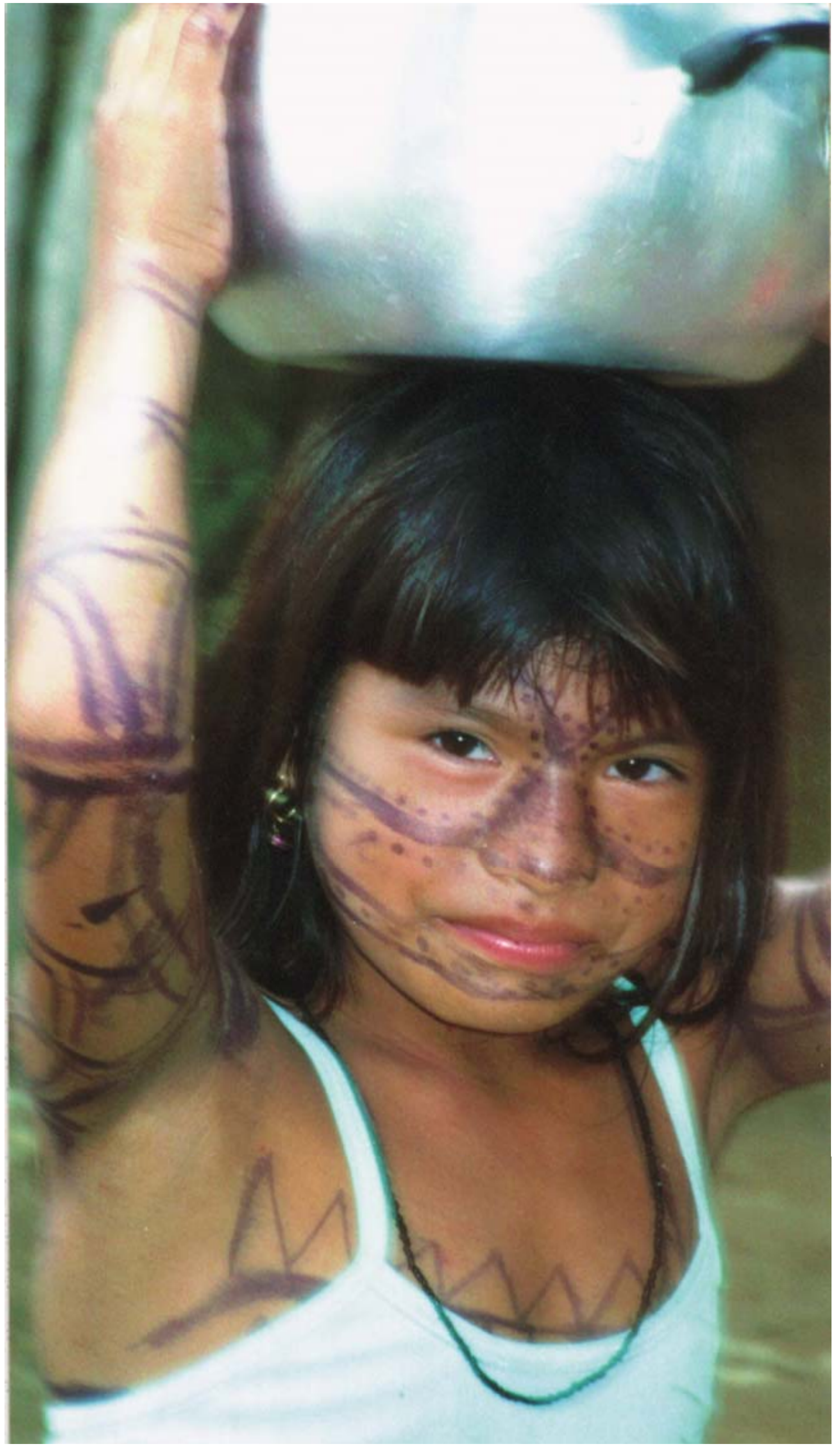
Do children know how numbers map onto space?

« Thermometer » tasks:

Where should number *n* go?

A major change occurs during mathematical education : switch from a logarithmic to a linear understanding of number

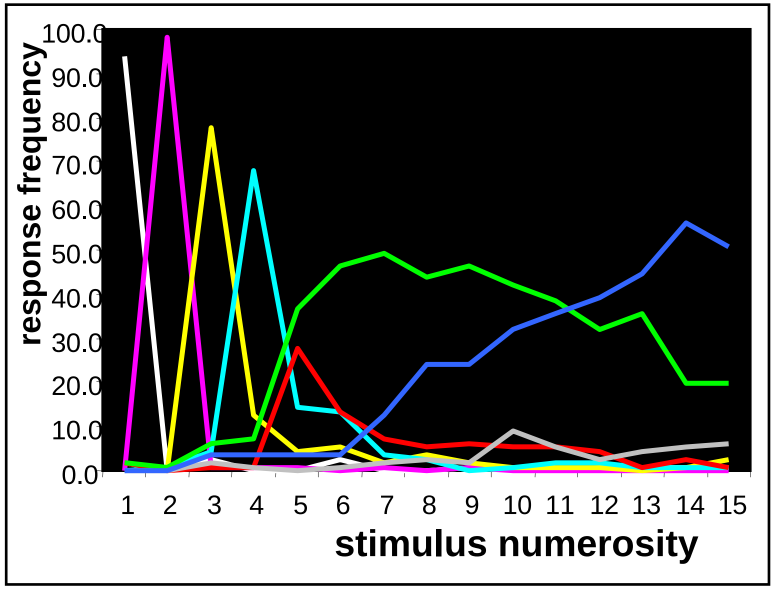




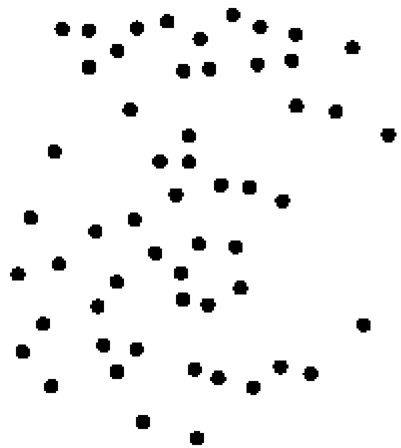
Numerical cognition without words in the Munduruku

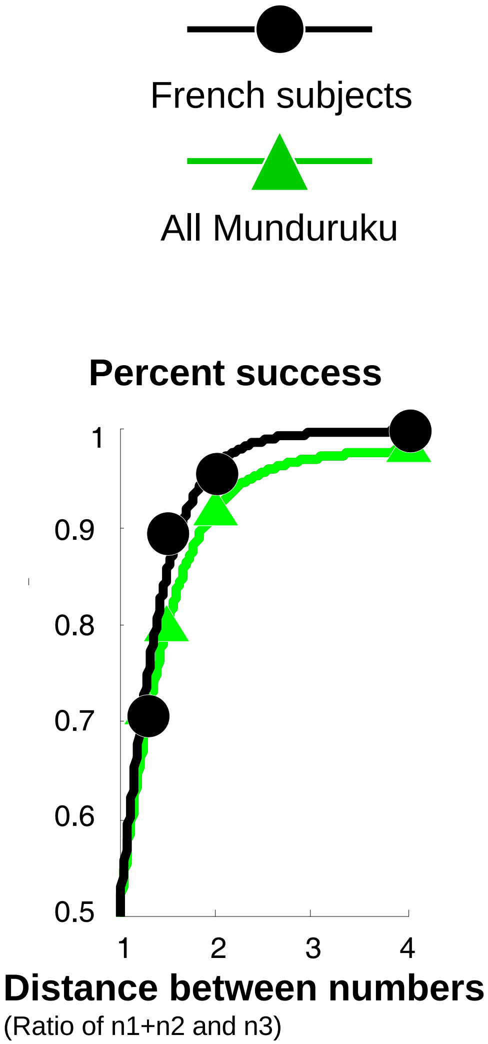
Pica, Lemer, Izard, & Dehaene, Science, 2004

- pug ma = one
- xep xep = two
- ebapug = three
- ebadipdip = four
- pug põgbi = one hand
- xep xep põgbi = two hands
- adesu/ade gu = some, not many
- ade/ade ma = many, really many



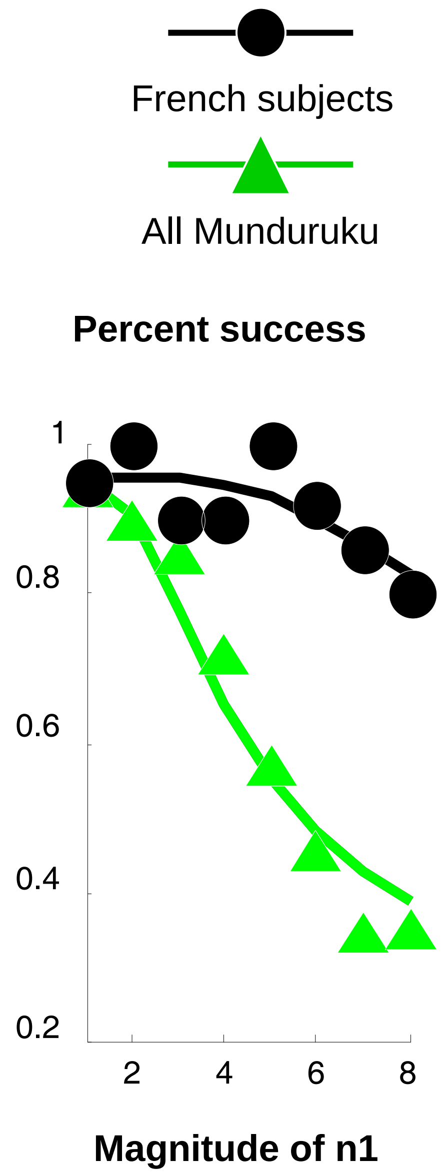
Success in approximate addition and comparison





Pica, Lemer, Izard, & Dehaene, Science, 2004

Failure in exact subtraction of small quantities



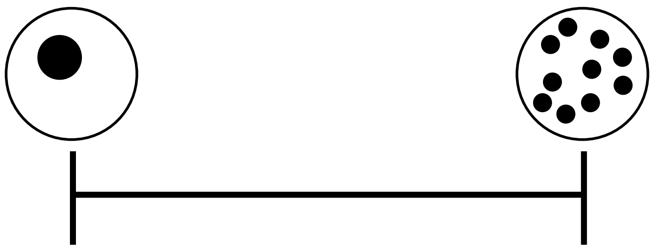
Pica, Lemer, Izard, & Dehaene, Science, 2004

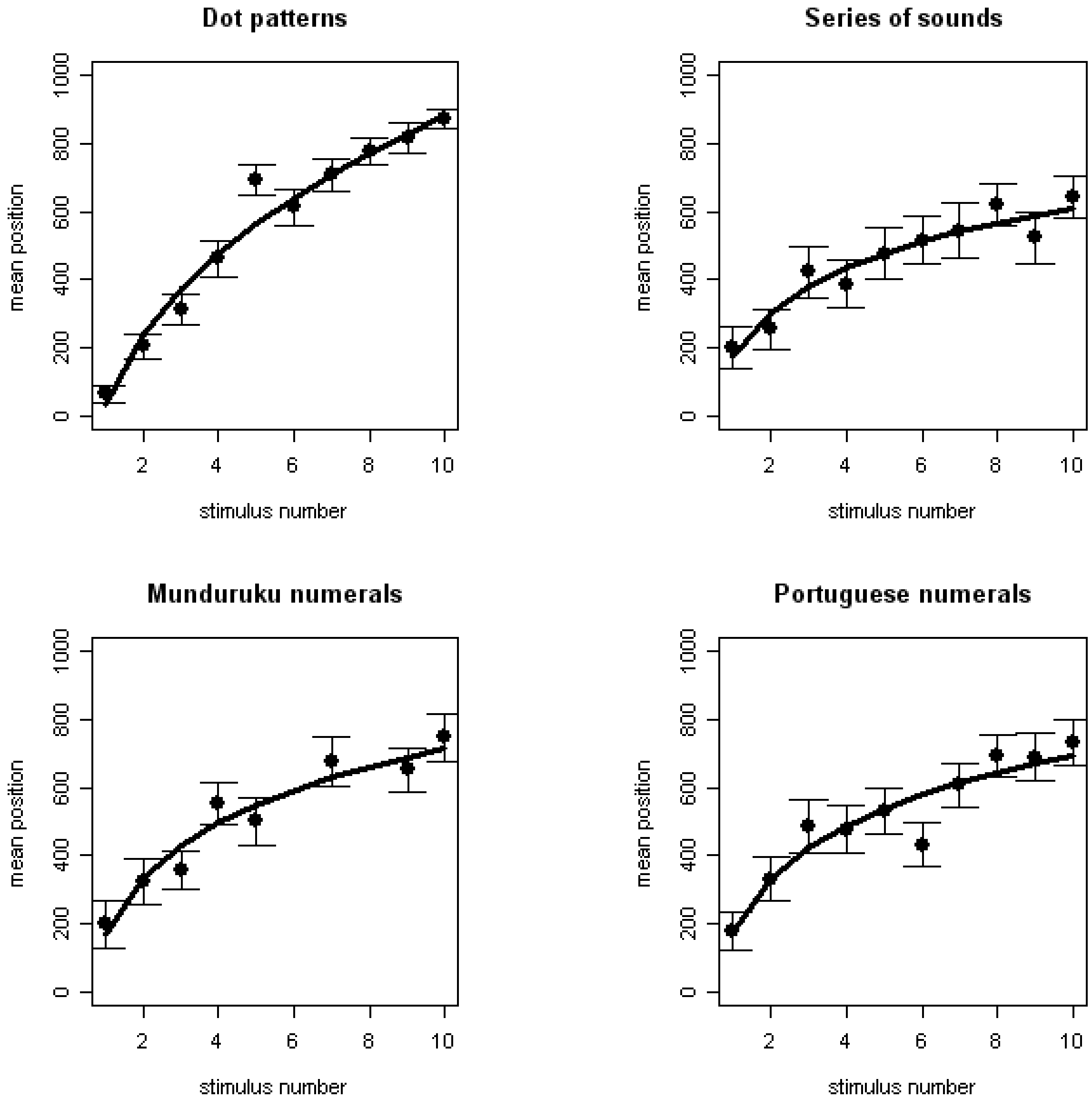
Logarithmic Number-Space mapping in the Munduruku

The Munduruku do not have any measuring system.

-**Can they do the thermometer task?**

- **Would they show a logarithmic bias even in the range 1 through 10?**

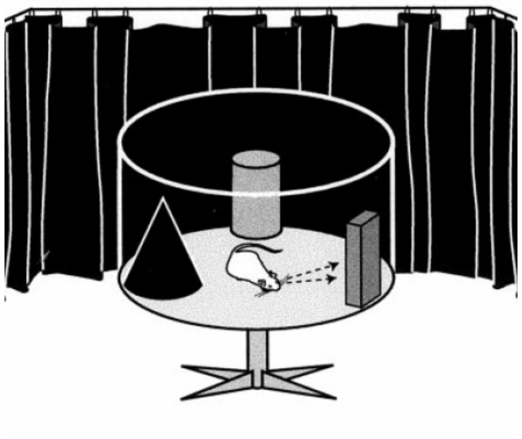




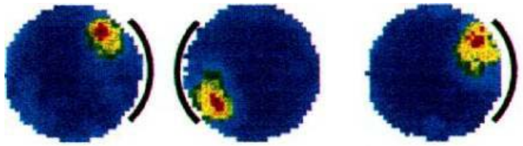
Core knowledge of geometry

Is **geometry**
also part of our evolutionary heritage,
much like number sense is?

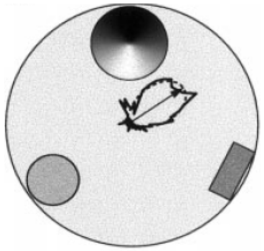
Animal navigation abilities



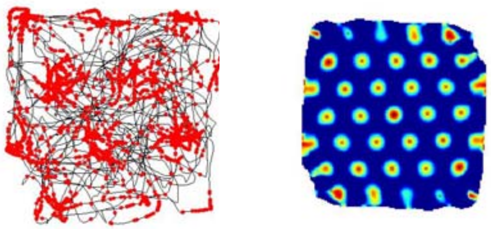
Place cells



Head direction cells



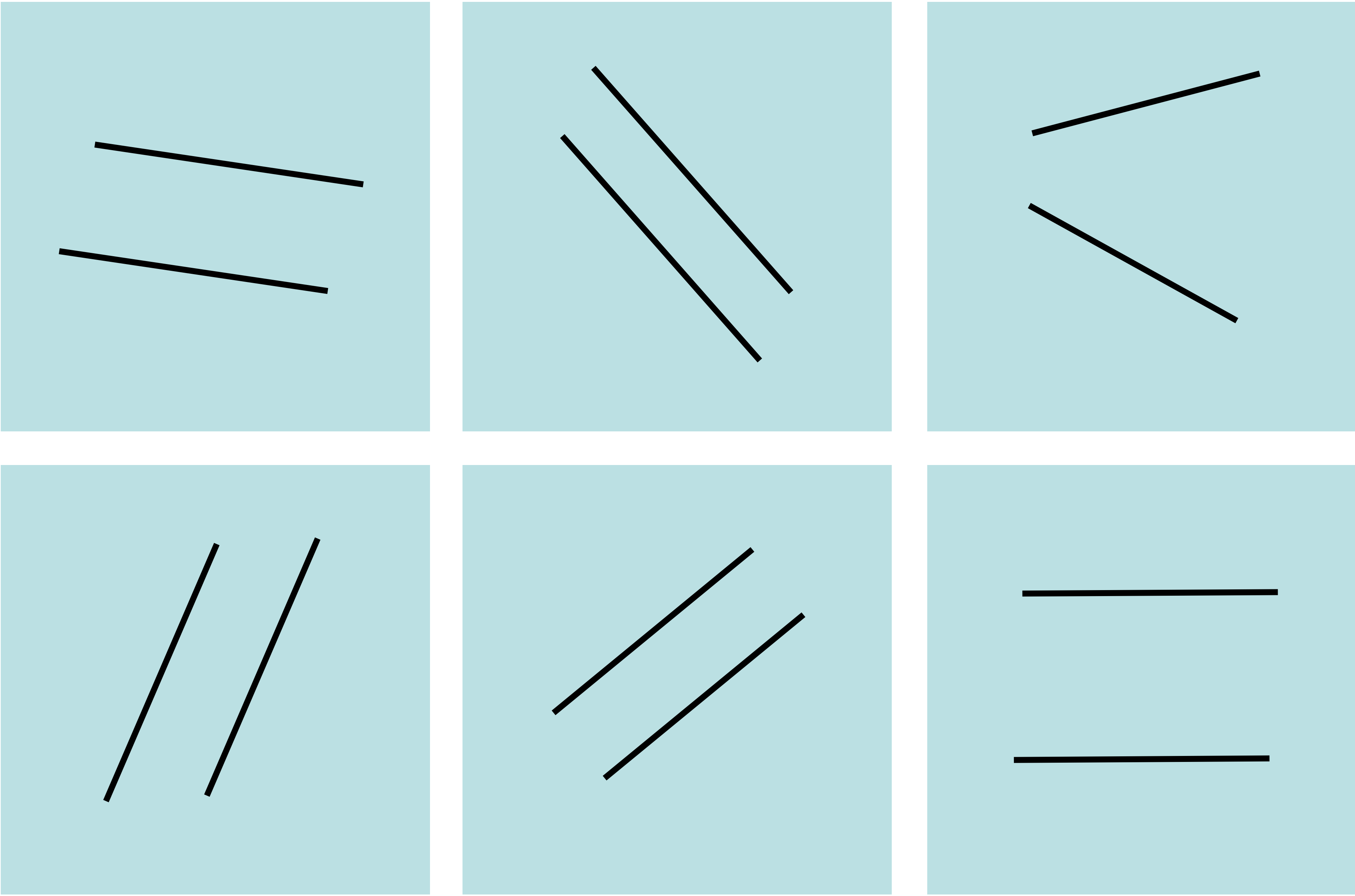
Grid cells

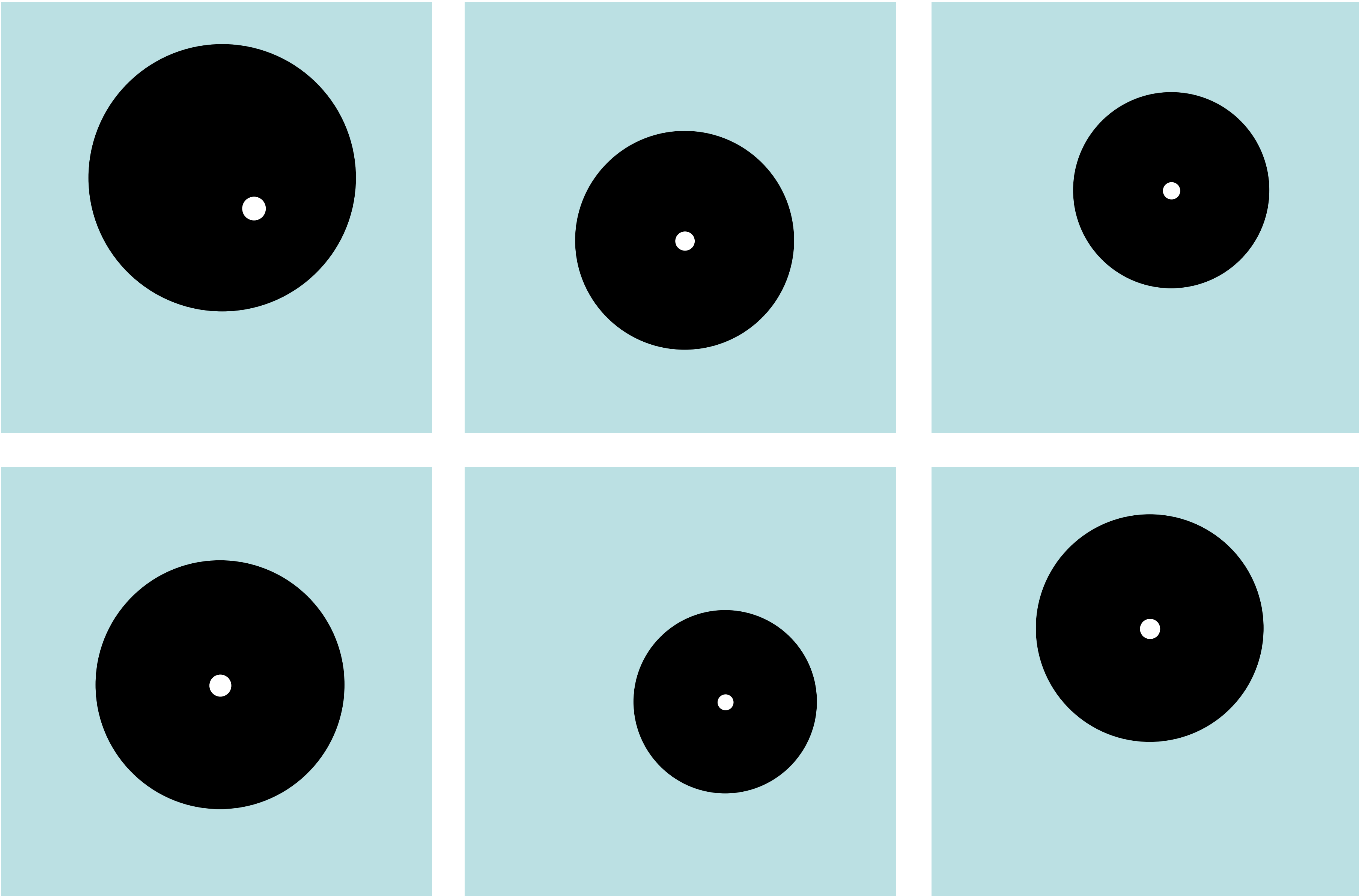


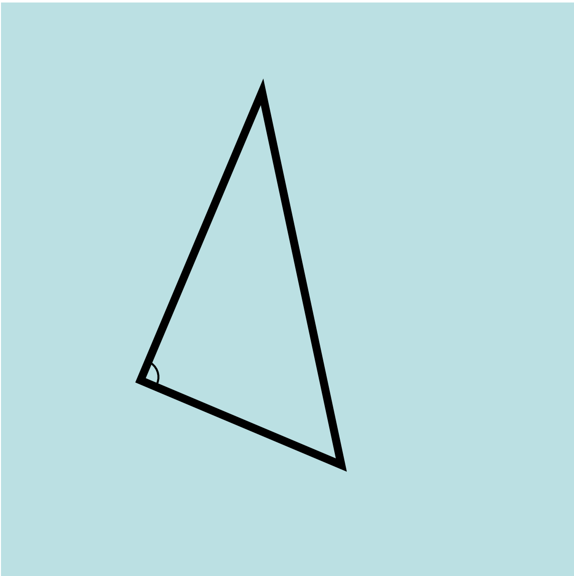
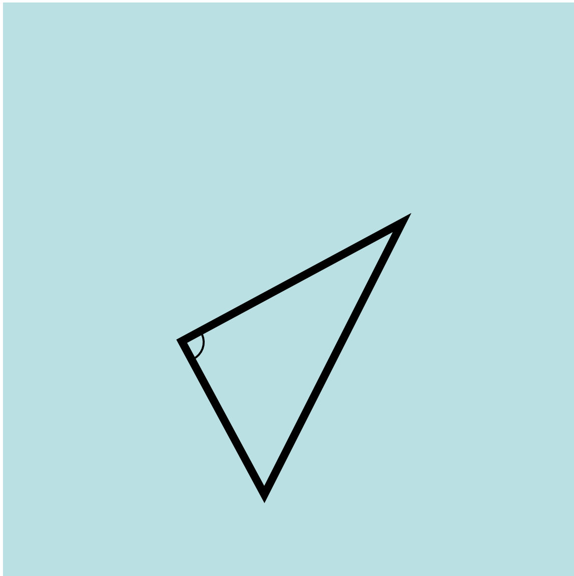
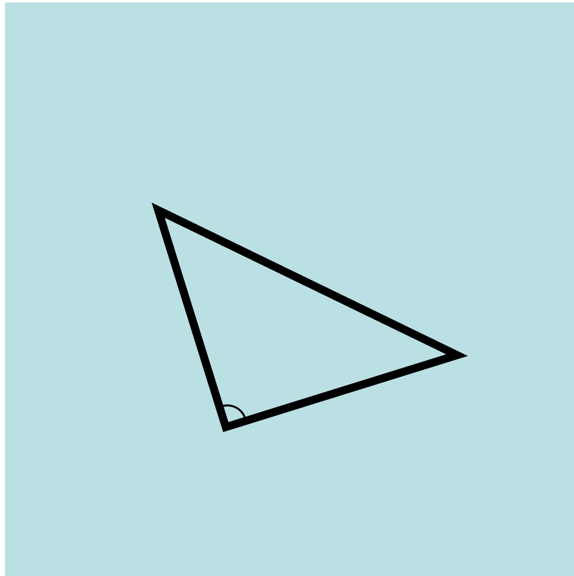
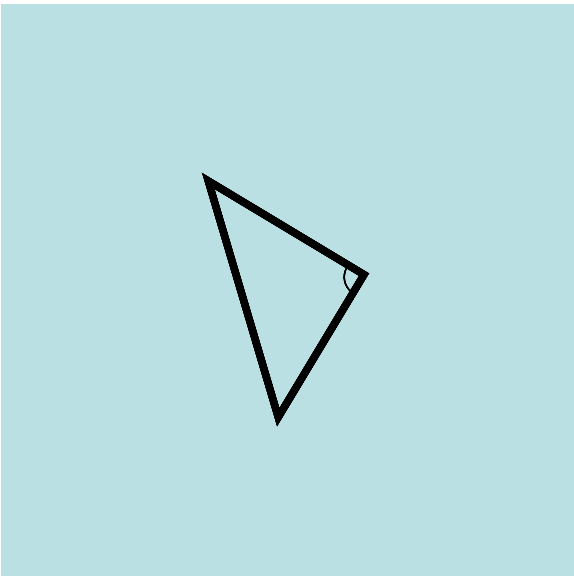
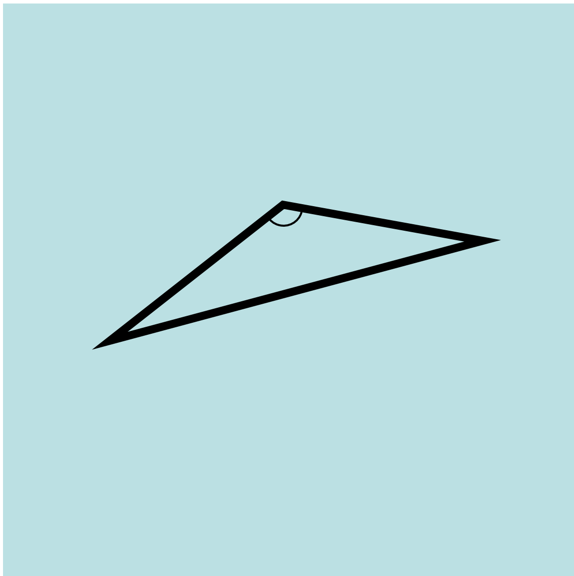
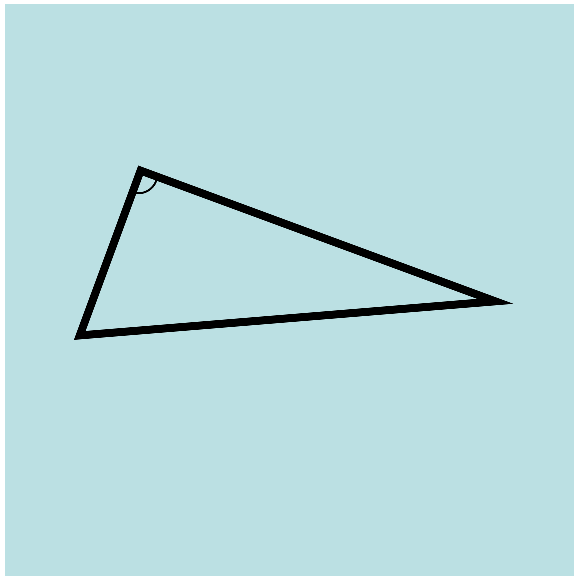
Stanislas Dehaene, Véronique Izard,
Pierre Pica, Elizabeth Spelke

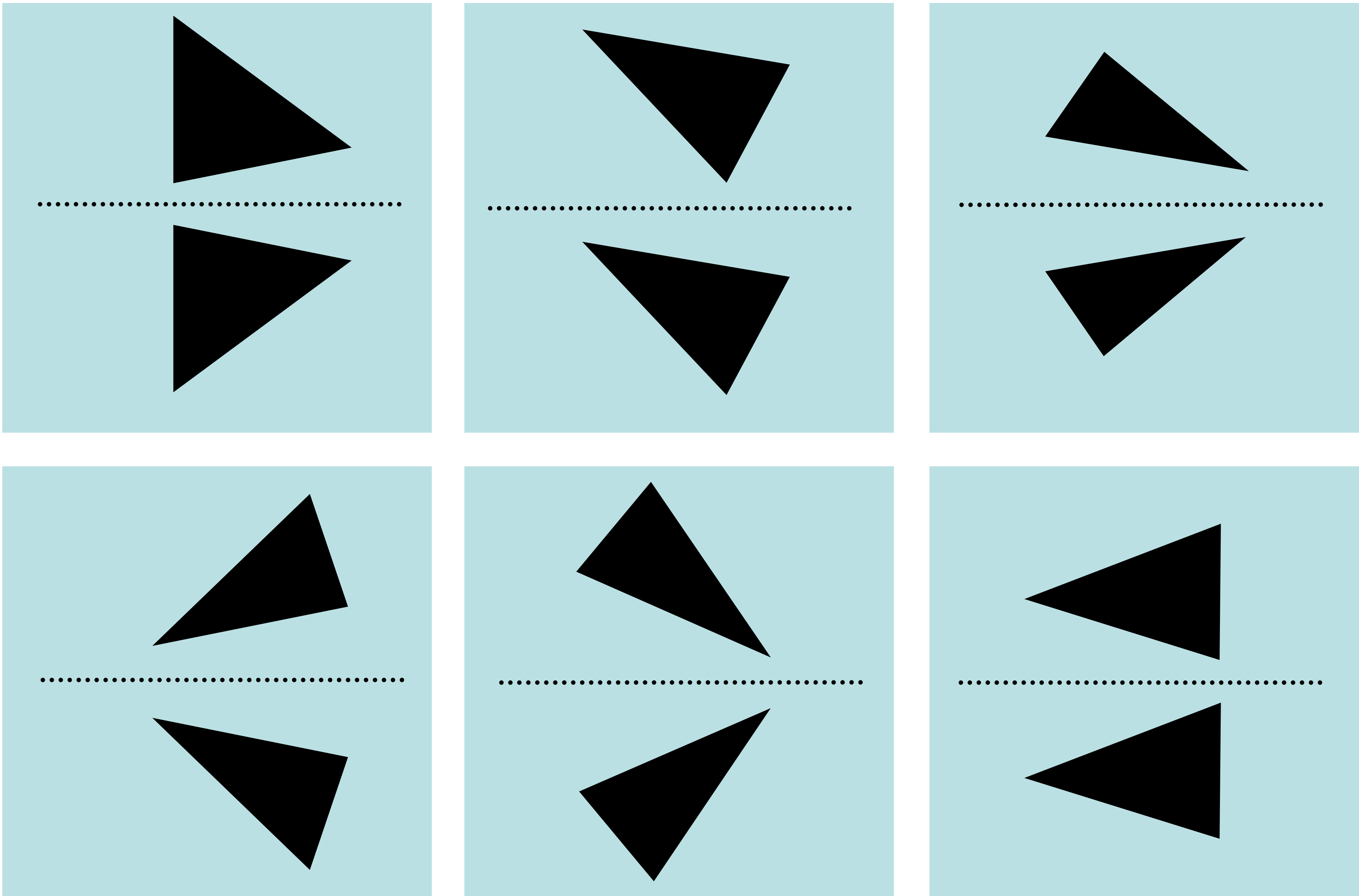
Core knowledge of geometry in an Amazonian indigene group

Science, January 2006

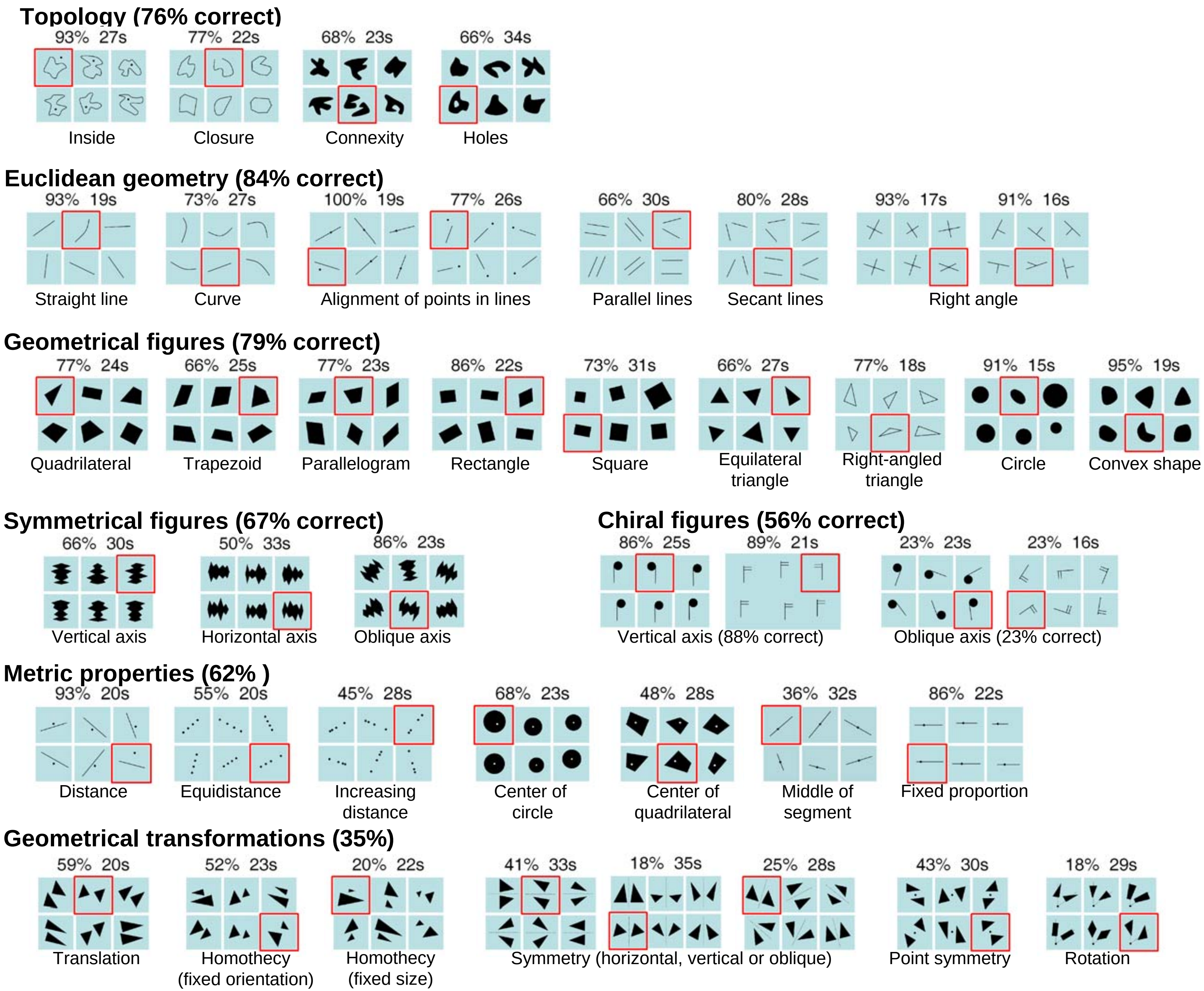




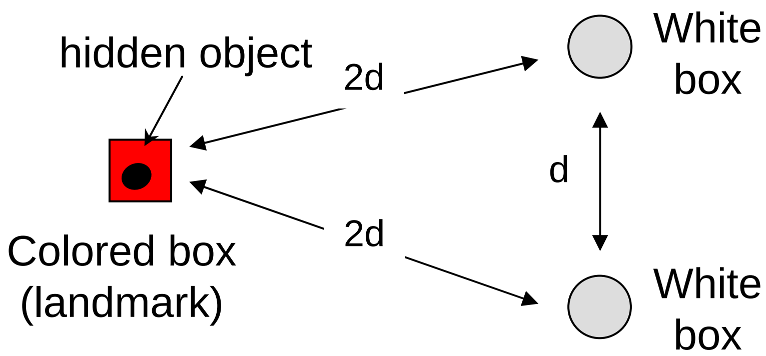


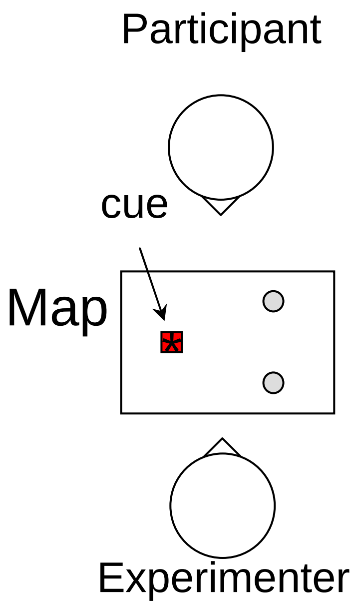
Core concepts of geometry are available to uneducated, monolingual Munduruku indians

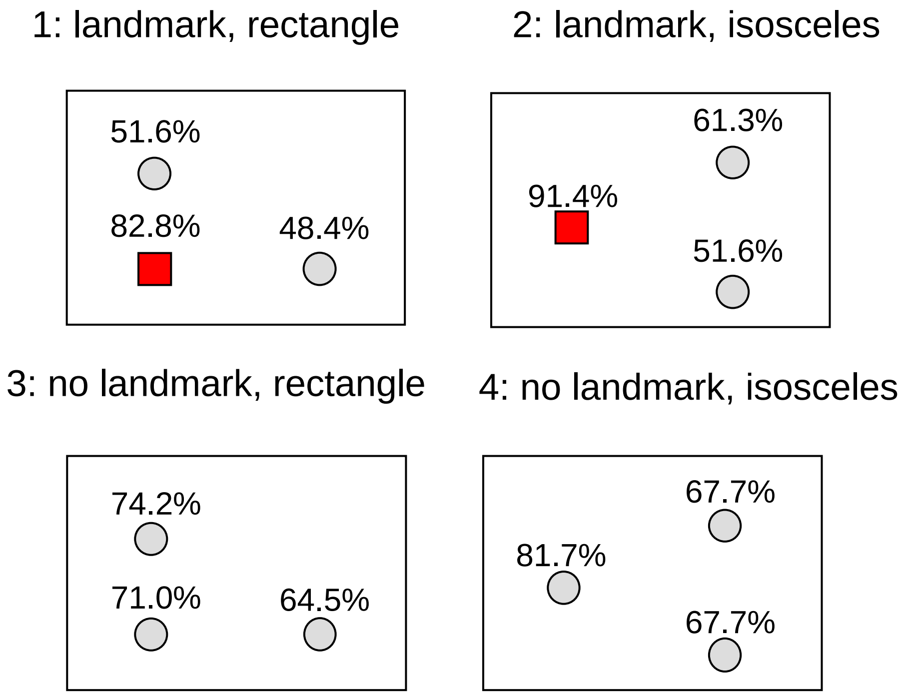


The Munduruku can use geometrical relations in a « map »

Environment

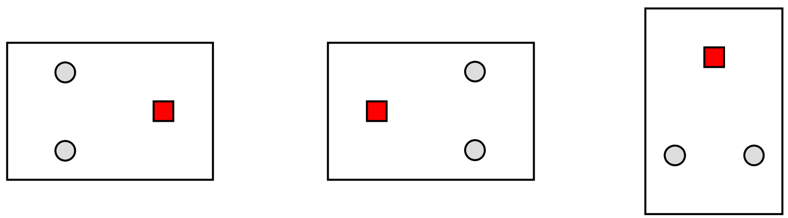




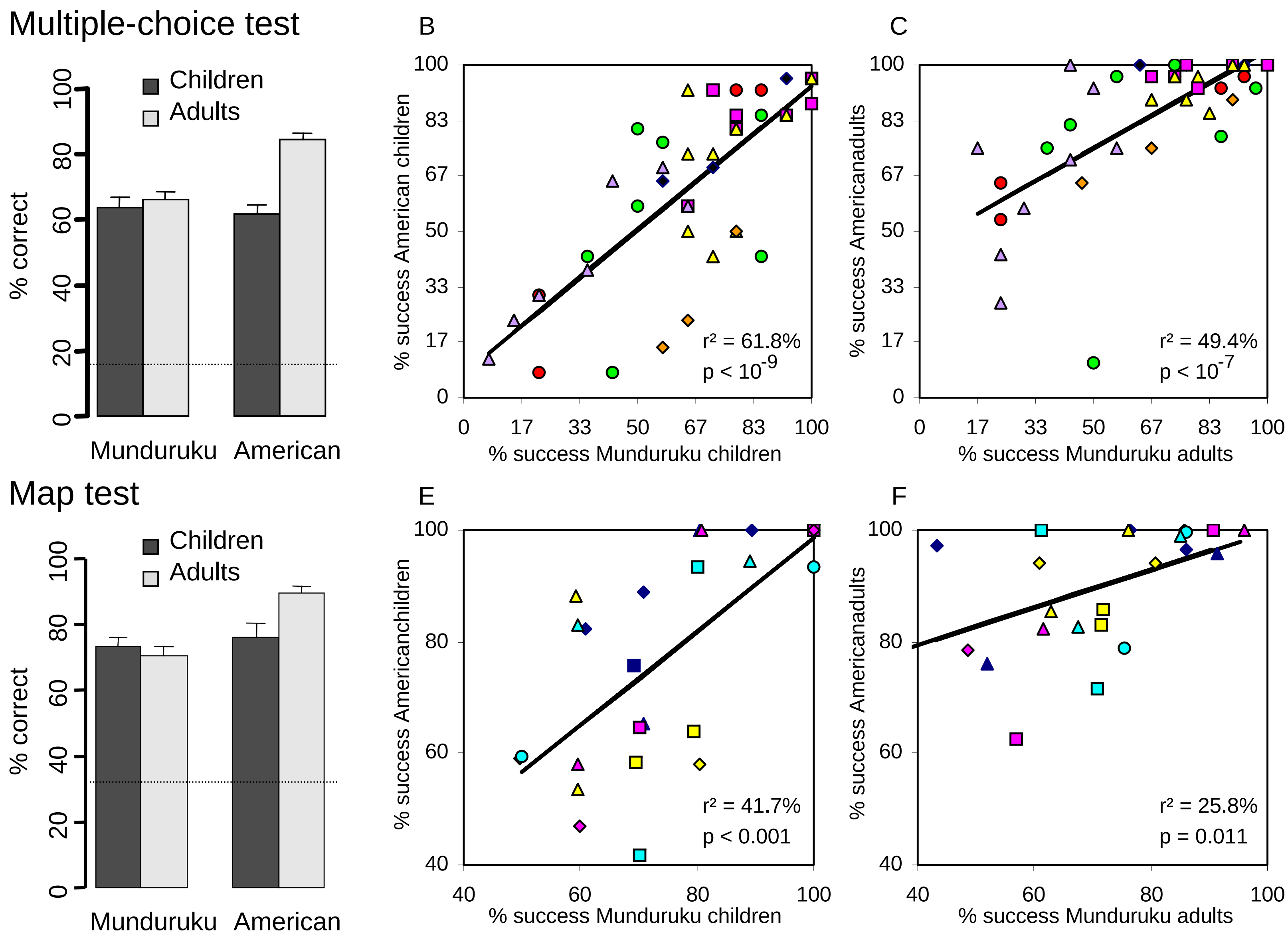


Success regardless of map orientation

Egocentric	Allocentric	Rotated
71.0%	70.6%	72.6%



The geometrical intuitions of Munduruku indians correlate with those of American children and adults



Are the Munduruku's geometrical intuitions Euclidean?

• Euclid included in his geometry an 'ugly' fifth postulate, which boils down to stating that in any triangle, the sum of the angles is always π or 180°.

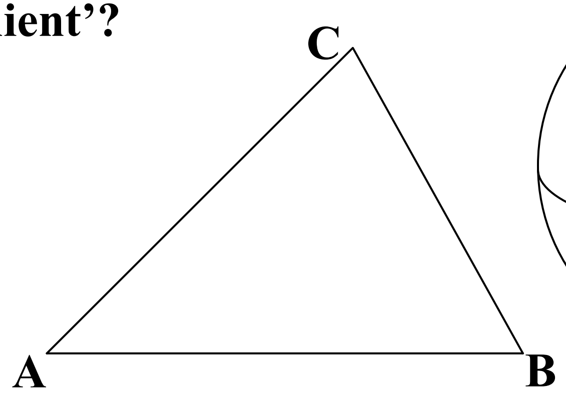
• Saccheri (1733), Lobatchevsky (1829), Bolyai (1832), and Gauss explored the 'imaginary geometry' obtained by contradicting Euclid's fifth postulate

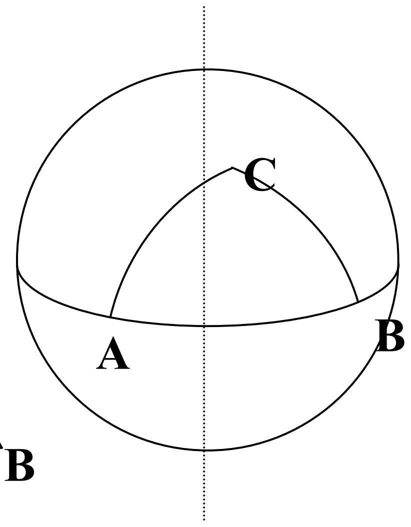
• Riemann, Beltrami, Poincaré finally proved that this 'non-Euclidean' geometry is consistent by providing simple models of hyperbolic and elliptic geometry.

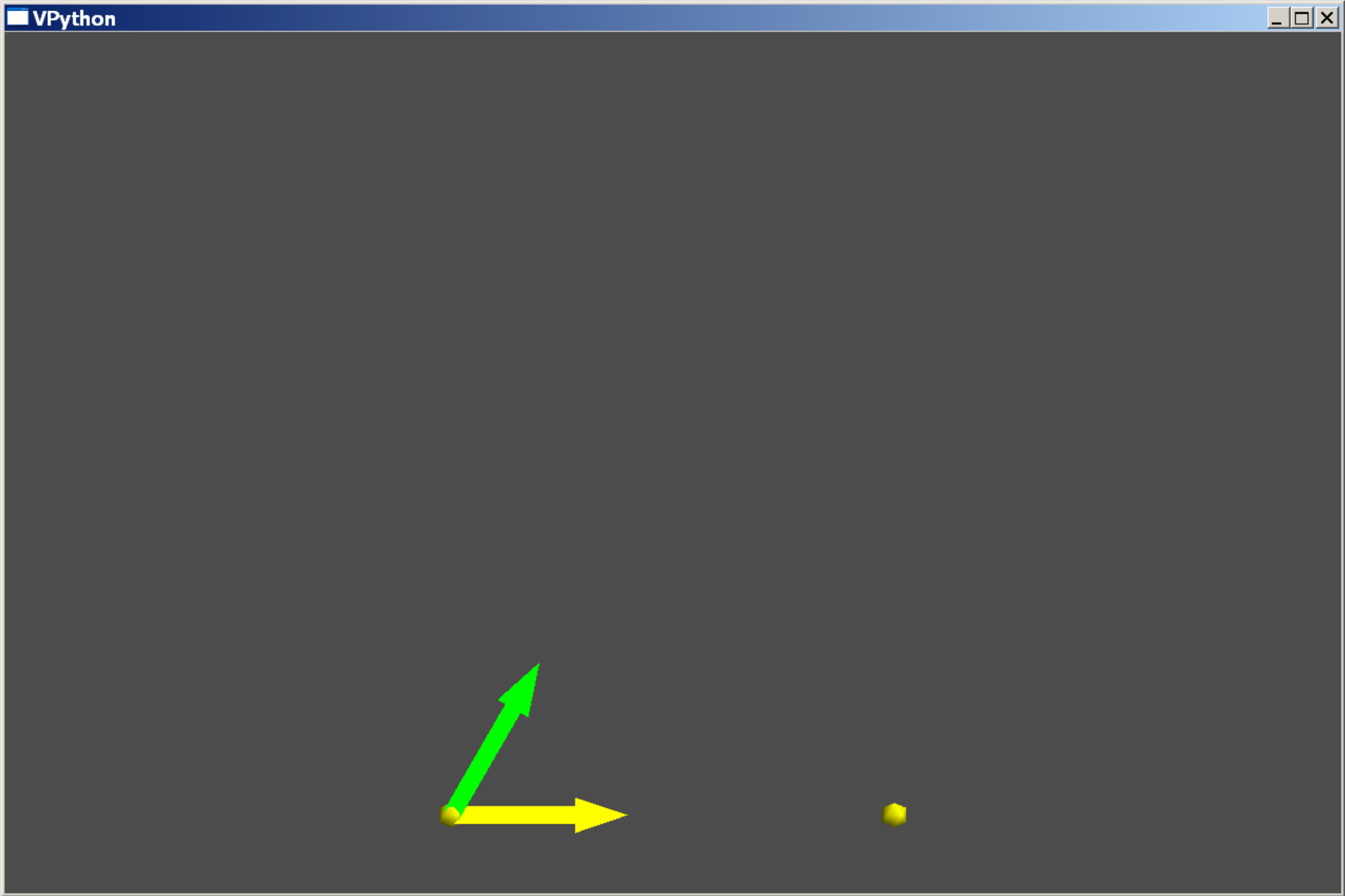
Is our core knowledge of geometry inherently Euclidean?

Or is Euclidean geometry just more 'convenient'?

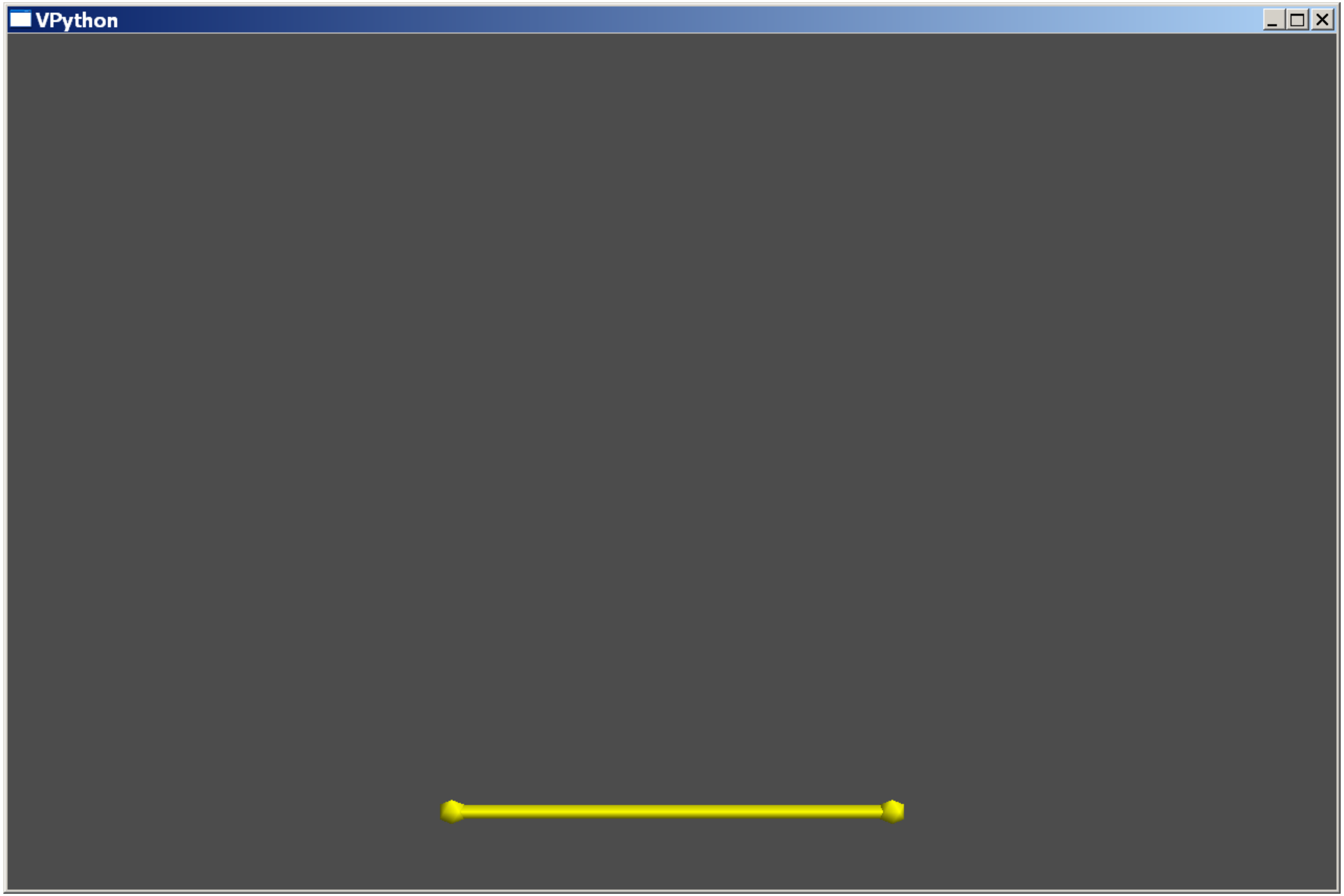
"Through natural selection, our mind has adapted to the conditions of the external world, […] it has adopted the geometry most advantageous to our species; or, in other words, the most convenient."
Henri Poincaré, *La science et l'hypothèse*



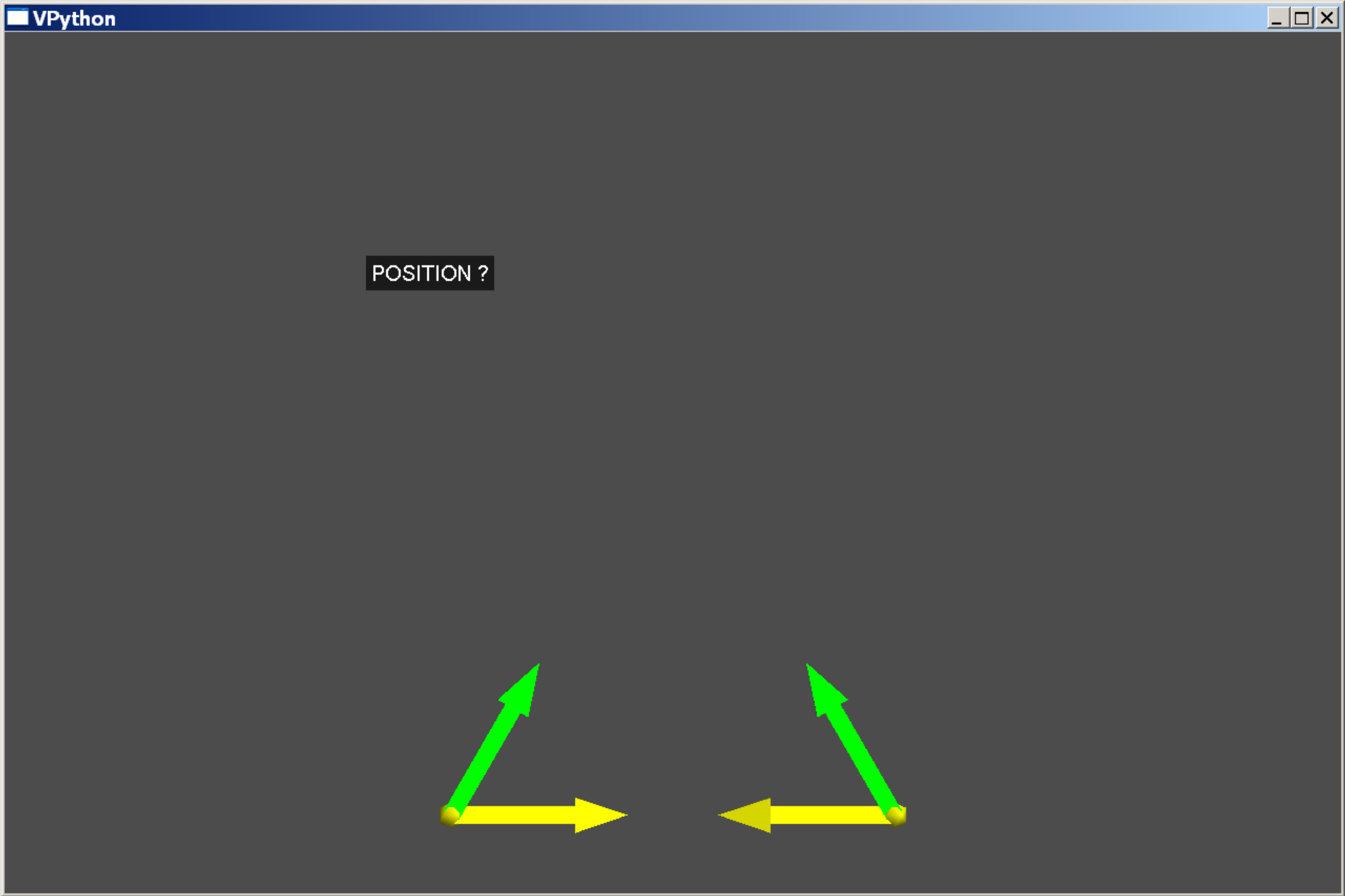




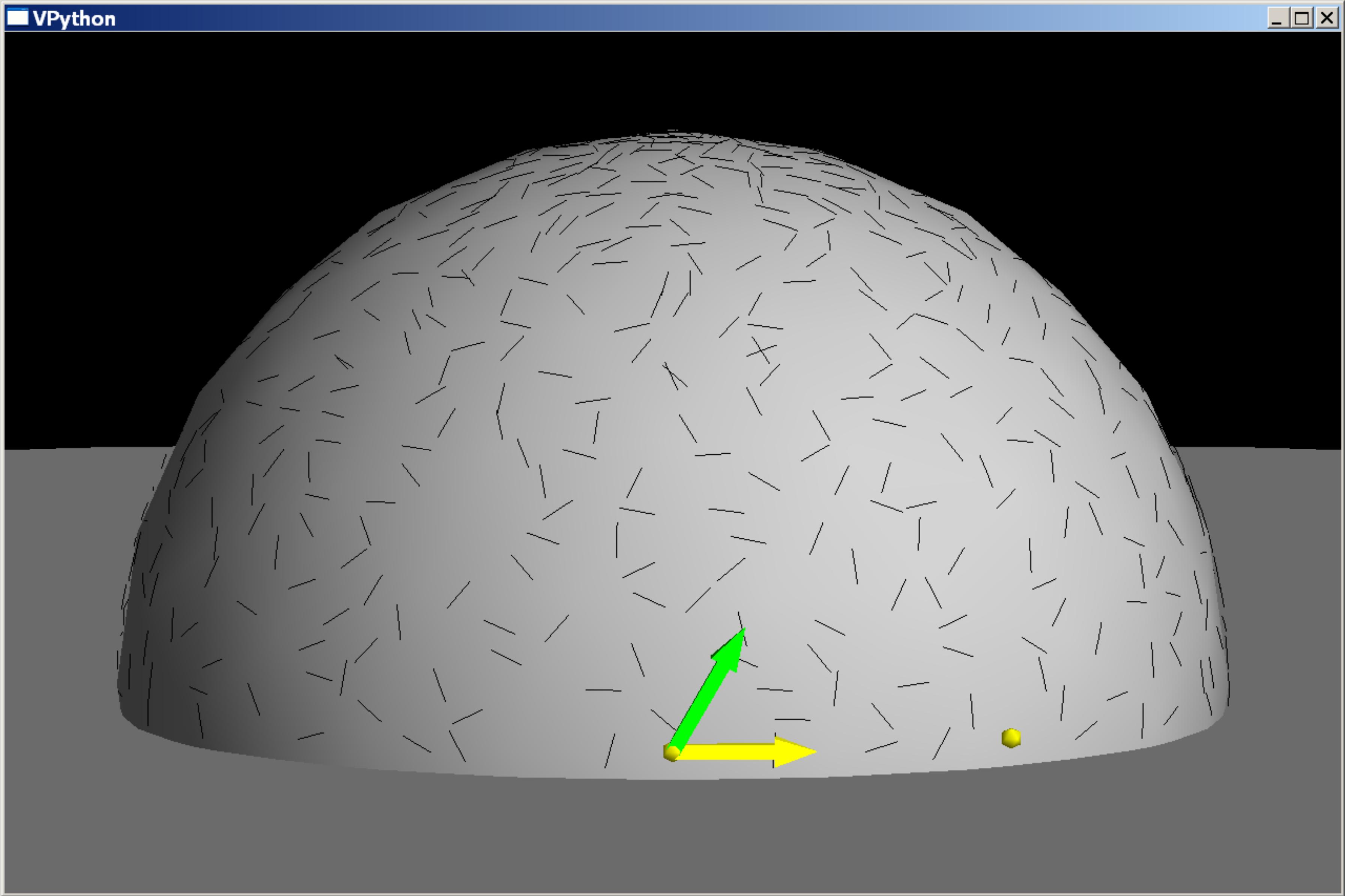
This is a place where the land is very flat.
You can see two villages. From this village here, you can see two paths.



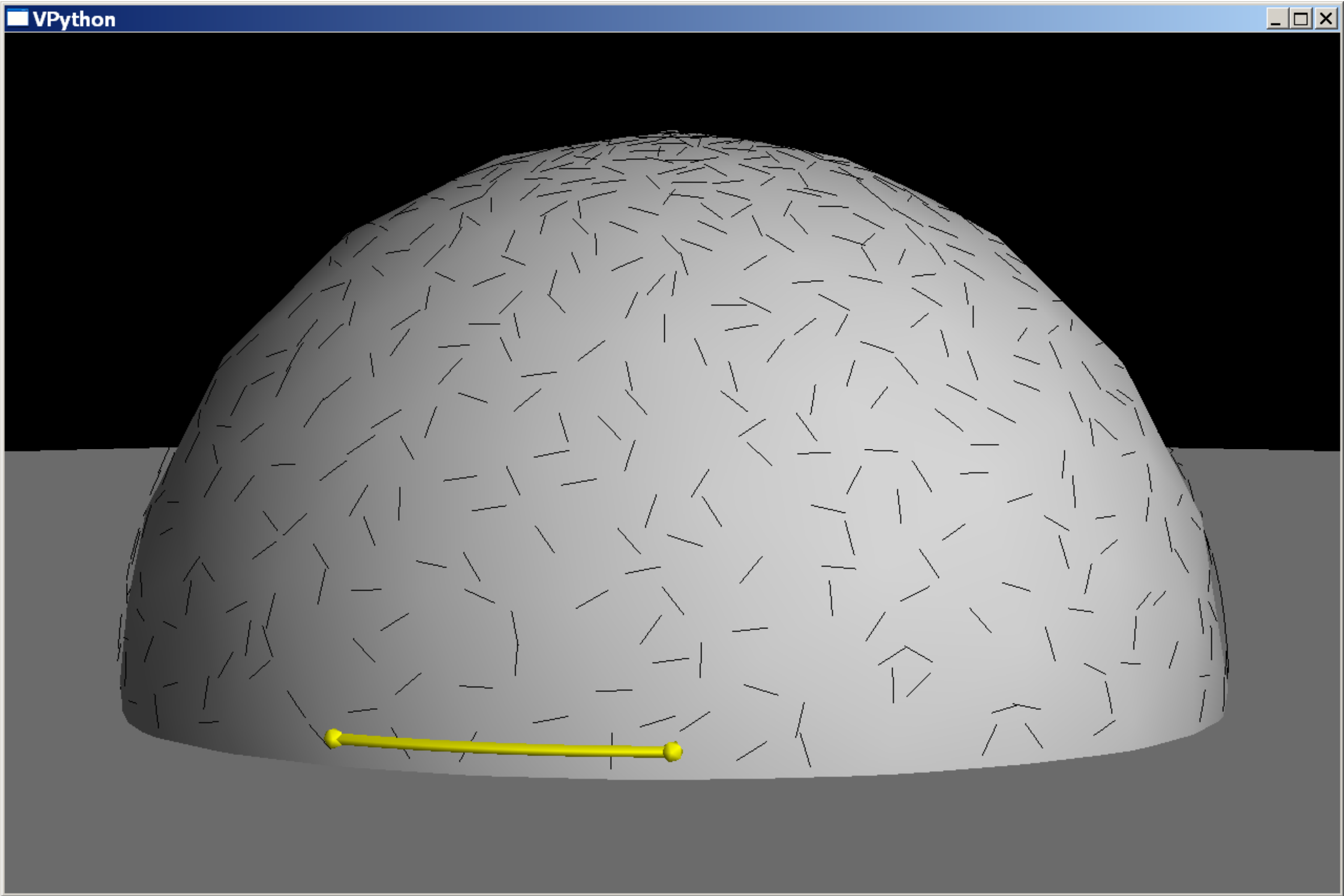
One of the paths leads straight to the other village.



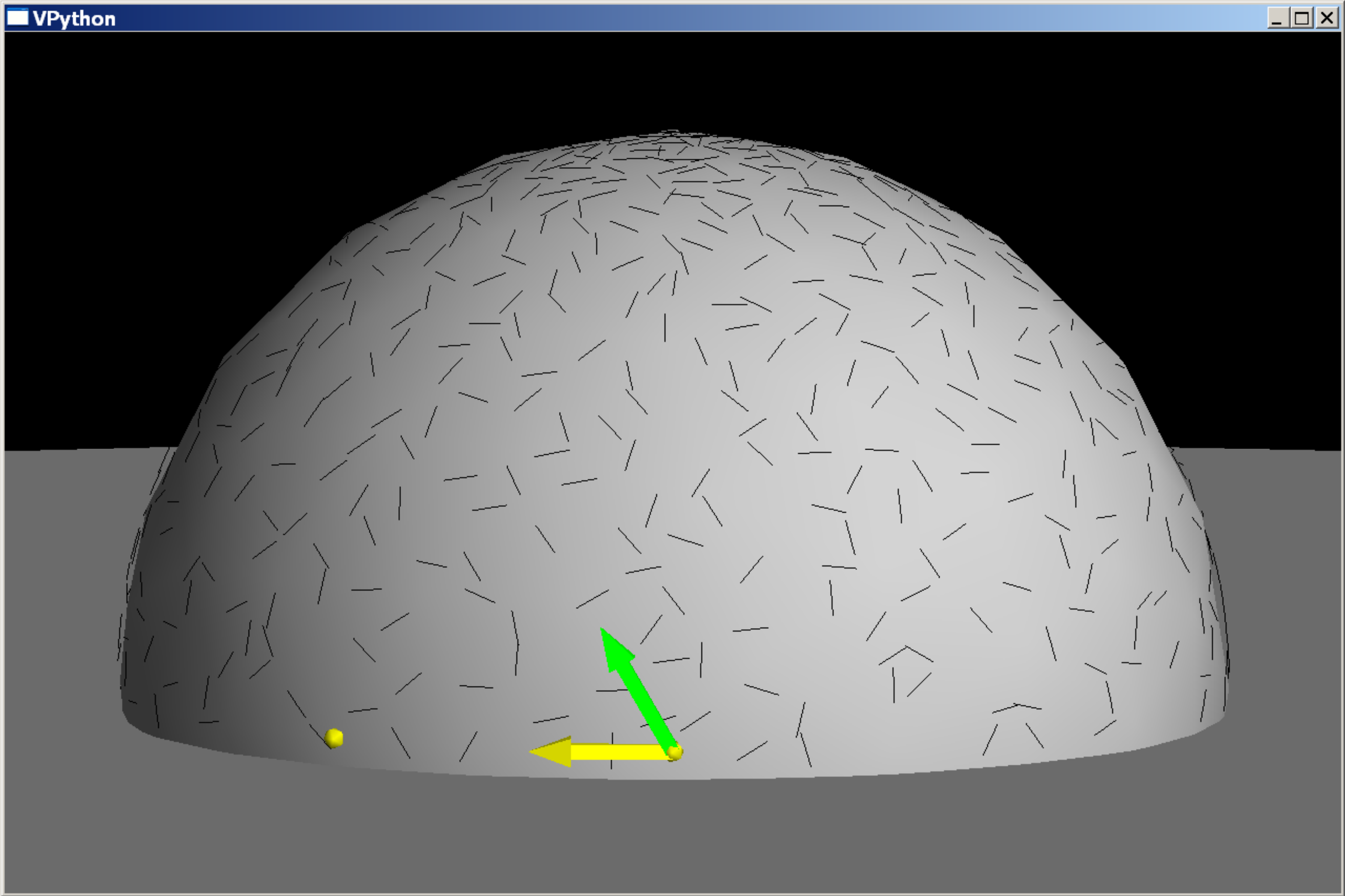
At the other village too, there are two paths. The two green paths go straight to another village. I would like to tell me where those two paths lead. Show me where is the other village. Also show me how the two paths look like at this village.



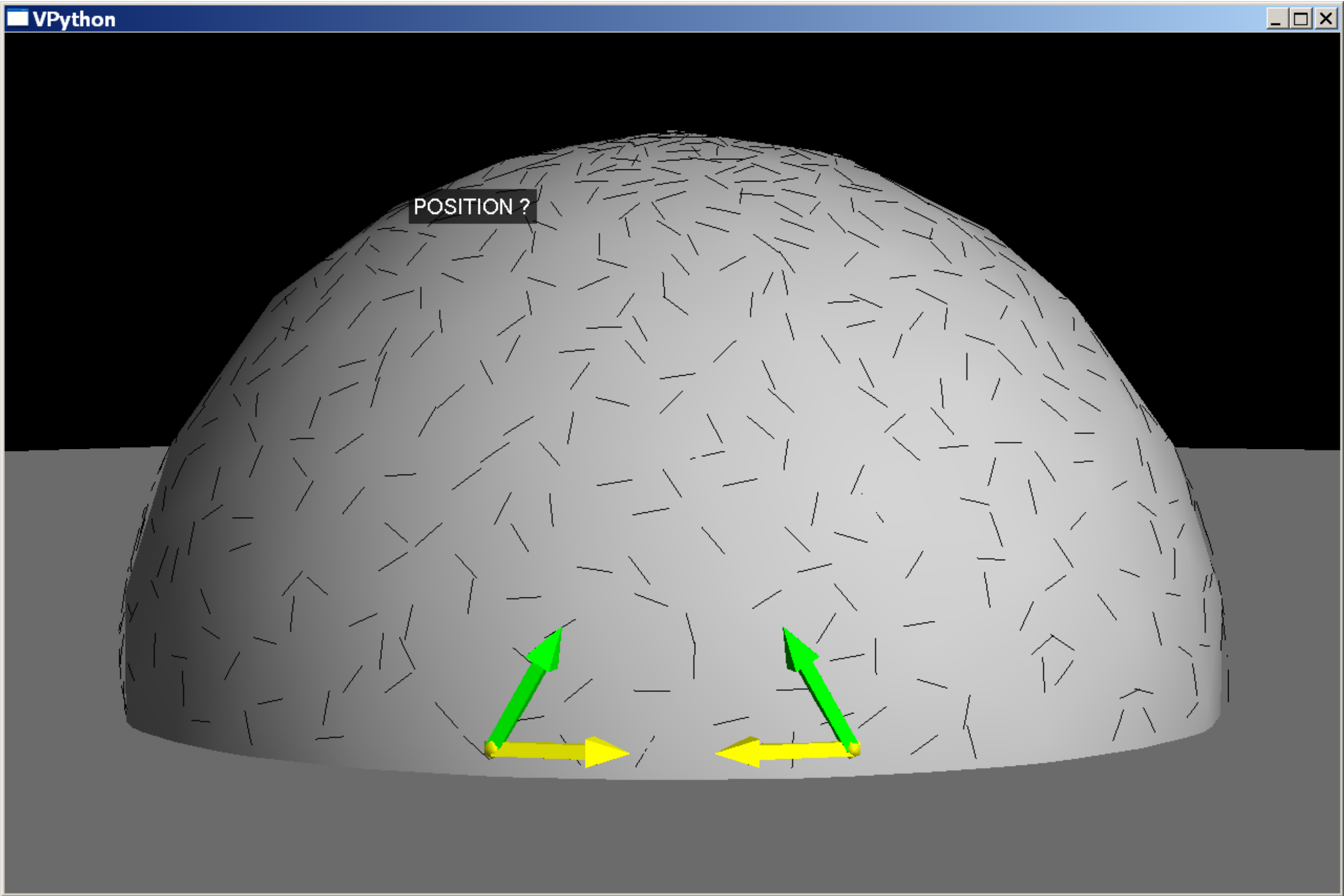
This is a place where the land is very curved and round.
You can see two villages. From this village here, you can see two paths.



One of the paths leads straight to the other village.



At this village too, there are two paths.

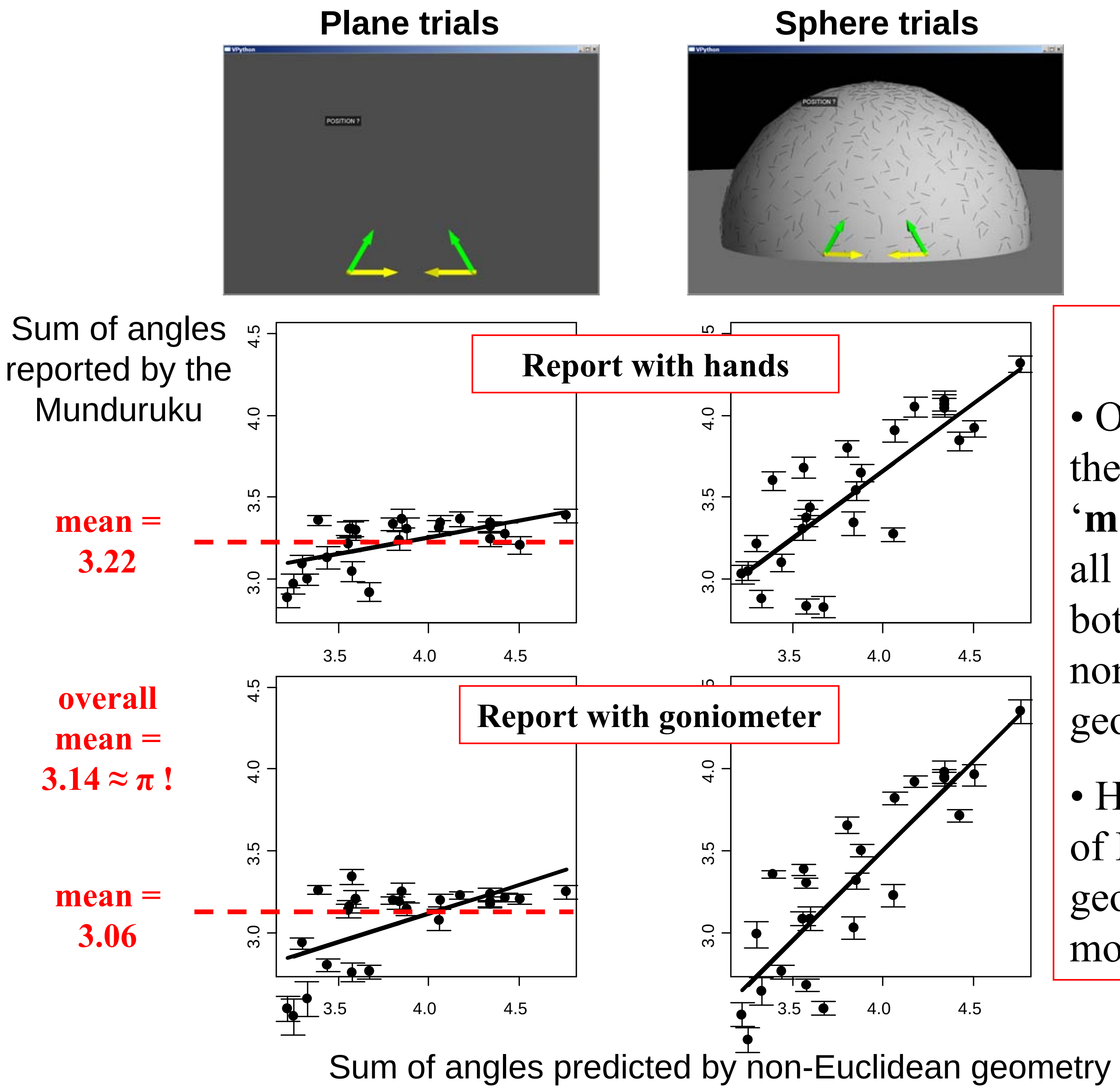


The two green paths go straight to another village. I would like to tell me where those two paths lead. Show me where is the other village. Also show me how the two paths look like at this village.

Two response modes

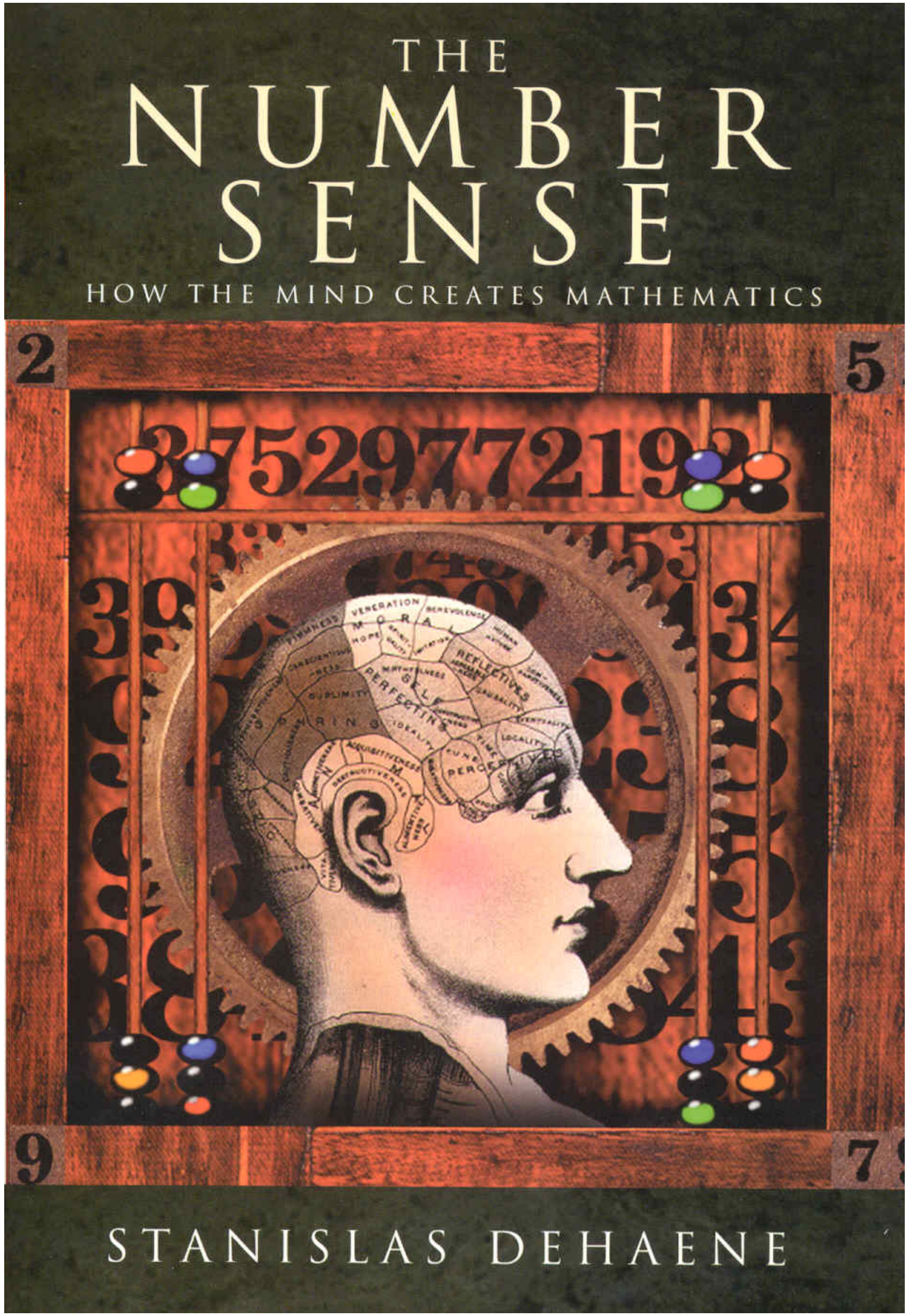
-indicate angle with the two hands (angle measured by the experimenter)

-indicate the angle directly by manipulating the goniometer



Conclusion:

- Once presented with the appropriate ‘**mental model**’, we all have intuitions of both Euclidean and non-Euclidean geometries.
- However, intuitions of Euclidean geometry seem to be more immediate



Conclusion:
Mathematics is a cultural construction based on a universal biological endowment

The foundations of any mathematical construction are grounded on fundamental intuitions such as notions of set, number, space, time or logic, deeply embedded in our brains.

Mathematics can be characterized as the progressive formalization of these intuitions.

Its purpose is to make them more coherent, mutually compatible, and better adapted to our experience of the external world.

Children come to school with strong mathematical intuitions that can be used as a support for learning of more advanced material