Statistical mechanics of strange metals and black holes

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1. Beckenstein-Hawking entropy of a black hole
2. Schwarzian theory of SYK fluctuations and linear-*T* resistivity
3. Critical Fermi surfaces
4. Universal *T*-linear resistivity in two-dimensional quantum-critical metals from spatially random interactions

The Einstein action for gravity in 3+1 dimensions is

$$I_E = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4\right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g\,\exp(-I_E)\,,$$

where $\kappa^2 = 8\pi G_N$ is the gravitational constant, $\mathcal{R}_4$ is the Ricci scalar. The Schwarzschild solution of the saddle-point equations is

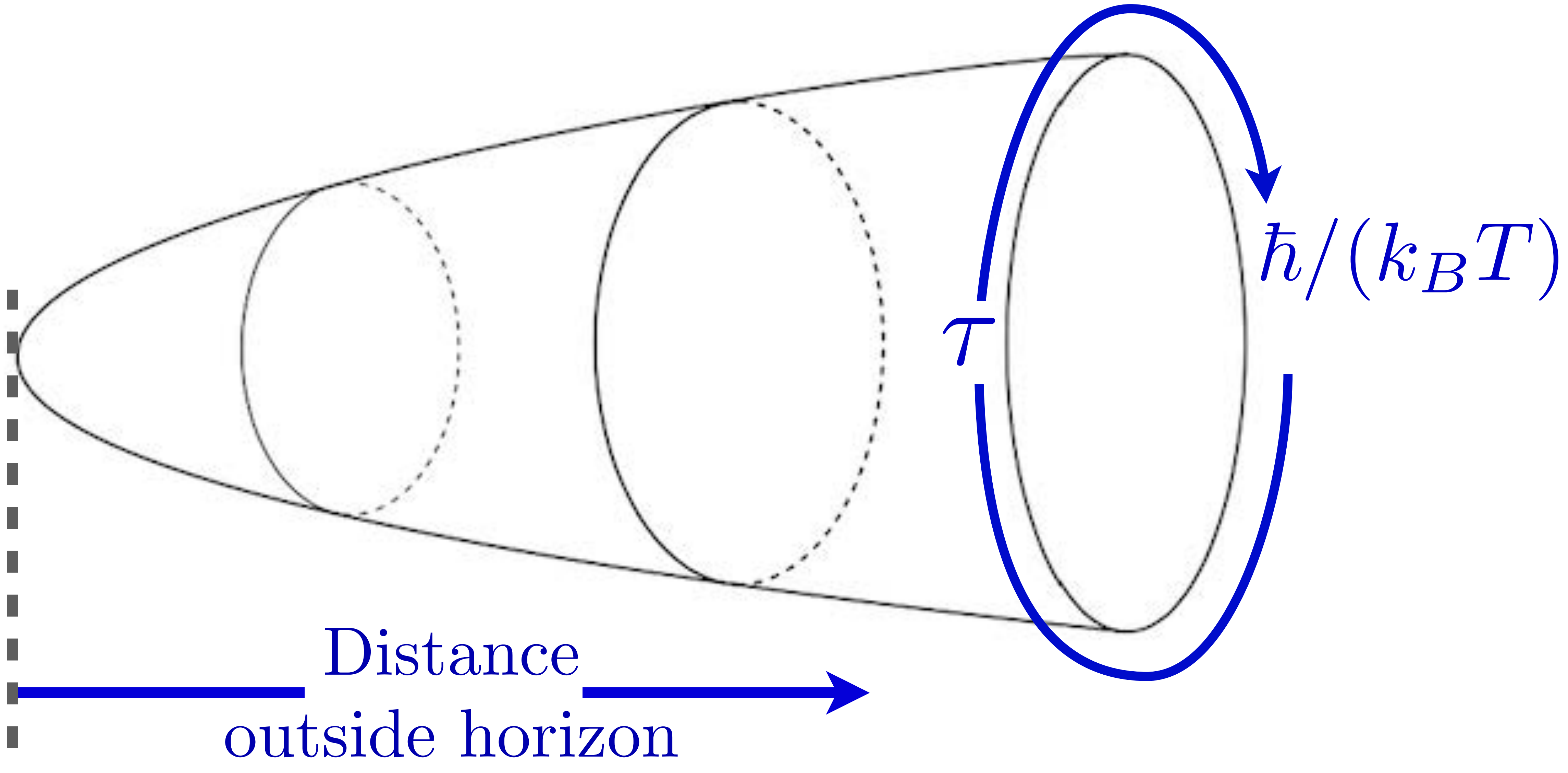
$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_2^2 + \frac{dr^2}{V(r)}$$

where $d\Omega_2^2$ is the metric of the 2-sphere, and

$$V(r) = 1 - \frac{m}{r}.$$

The gravitational mass of the black hole is $M = 2G_N m$. The black hole horizon is at $r = r_0$ where $V(r_0) = 0$; so

$$r_0 = m$$



Quantum mechanics in a spacetime which is periodic as a function of τ with period $1/T$. We have to ensure that there is no singularity at the horizon r_0 where $V(r_0) = 0$. Let us change radial co-ordinates to y, where $r = r_0 + y^2$. Then for small y

$$ds^2 = \frac{4}{V'(r_0)} \left[\frac{(V'(r_0))^2}{4} y^2 d\tau^2 + dy^2 \right] + r_0^2 d\Omega_2^2 = \frac{4}{V'(r_0)} \left[y^2 d\theta^2 + dy^2 \right] + r_0^2 d\Omega_2^2$$

The expression in the square brackets is the metric of the flat plane in polar co-ordinates, with radial co-ordinate y and angular co-ordinate $\theta = V'(r_0)\tau/2$. Smoothness requires periodicity in θ with period 2π, and so

$$4\pi T = V'(r_0) = \frac{1}{m} .$$

The free energy $\beta F = I_E$, where $\beta = 1/T$. So the entropy is

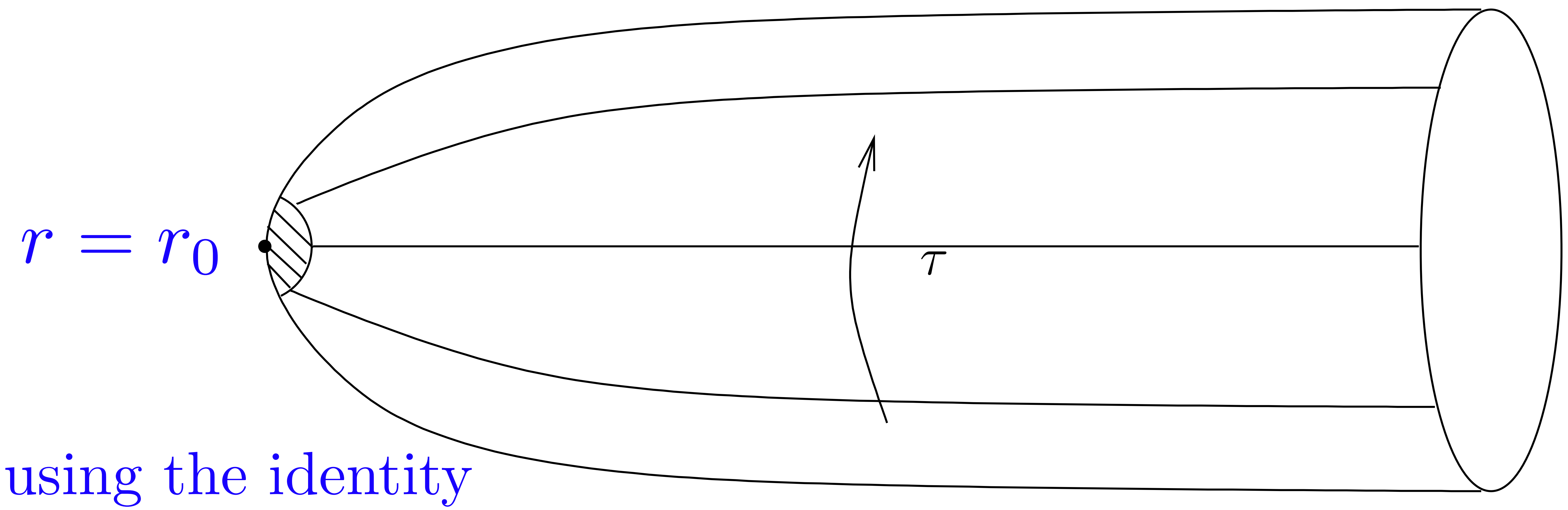
$$S = -\frac{\partial F}{\partial T} = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I_E$$

However, the metric is τ-independent, and the only explicit dependence of the action is via $I_E = \beta H$. Such an action implies $S = 0$.

The entire contribution to the entropy comes from the vicinity of singularity at $r = r_0$. We evaluate the action is the small region around this point

$$I_{\rm grav} = I_E + I_{GH} \quad , \quad I_{GH} = \int_\partial d^3x \sqrt{g_b} \left[-\frac{1}{\kappa^2} \mathcal{K}_3 \right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_{\rm grav}) \, ,$$

where $\mathcal{K}_3$ is the extrinsic scalar curvature of the 3-dimensional boundary of spacetime. I_{GH} is the Gibbons-Hawking boundary term, deduced by the requirement that the Euler-Lagrange equations of $I_{\rm grav}$ co-incide with the Einstein equations, with no additional boundary terms. The entire contribution to the entropy will come from I_{GH}.



We evaluate I_{GH} by using the identity

$$\int_{\partial} d^3x \sqrt{g_b}\, \mathcal{K}_3 = \frac{\partial}{\partial n} \int_{\partial} d^3x \sqrt{g_b}$$

where n is the Gaussian normal co-ordinate of the boundary. Evaluating at $y = \epsilon$, we have

$$\int_{\partial} d^3x \sqrt{g_b} = 2\pi\epsilon\mathcal{A}$$

where $\mathcal{A} = 4\pi r_0^2$ is the area of the horizon. Combining everything, we have the famous result of Hawking

$$S = \frac{2\pi\mathcal{A}}{\kappa^2} = \frac{\mathcal{A}}{4G_N}.$$

Charged black holes

We consider a charged black hole in Einstein-Maxwell theory of g and a U(1) gauge flux $F = dA$

$$I_{EM} = \int d^4x \sqrt{g} \left[-\frac{1}{2\kappa^2} \mathcal{R}_4 + \frac{1}{4g_F^2} F^2 \right] \quad , \quad \mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}g \mathcal{D}A \exp(-I_{EM} - I_{GH}) \, .$$

The saddle-point equations now yield a solution as before with

$$V(r) = 1 + \frac{\Theta^2}{r^2} - \frac{m}{r} \quad ; \quad A_\tau = i\mu \left(1 - \frac{r_0}{r}\right) \quad ; \quad \Theta = \frac{\kappa r_0}{\sqrt{2} g_F} \mu \quad ; \quad \mathcal{Q} = \frac{4\pi \mu r_0}{g_F^2} \quad ; \quad S = \frac{2\pi \mathcal{A}}{\kappa^2}$$

where $\mathcal{Q}$ is the total charge, the chemical potential is μ, and as before the horizon is where $V(r_0) = 0$, the temperature $T = V'(r_0)/(4\pi)$, and $\mathcal{A} = 4\pi r_0^2$.

This defines a two parameter family of charged black hole solutions of I_{EM} determined by T and $\mathcal{Q}$.

Charged black holes

Now we take the limit $T \to 0$ at fixed $\mathcal{Q}$. Then we find the remarkable feature that the horizon radius remains finite

$$R_h \equiv r_0(T \to 0, \mathcal{Q}) = \frac{\mathcal{Q} \kappa g_F}{4\pi}$$

In this limit, entropy becomes

$$S(T \to 0, \mathcal{Q}) = \frac{4\pi R_h^2}{G_N} + \gamma T \quad , \quad \gamma \equiv \frac{4\pi^2 R_h^3}{G_N}$$

For the near-horizon metric, it is useful to introduce the co-ordinate ζ

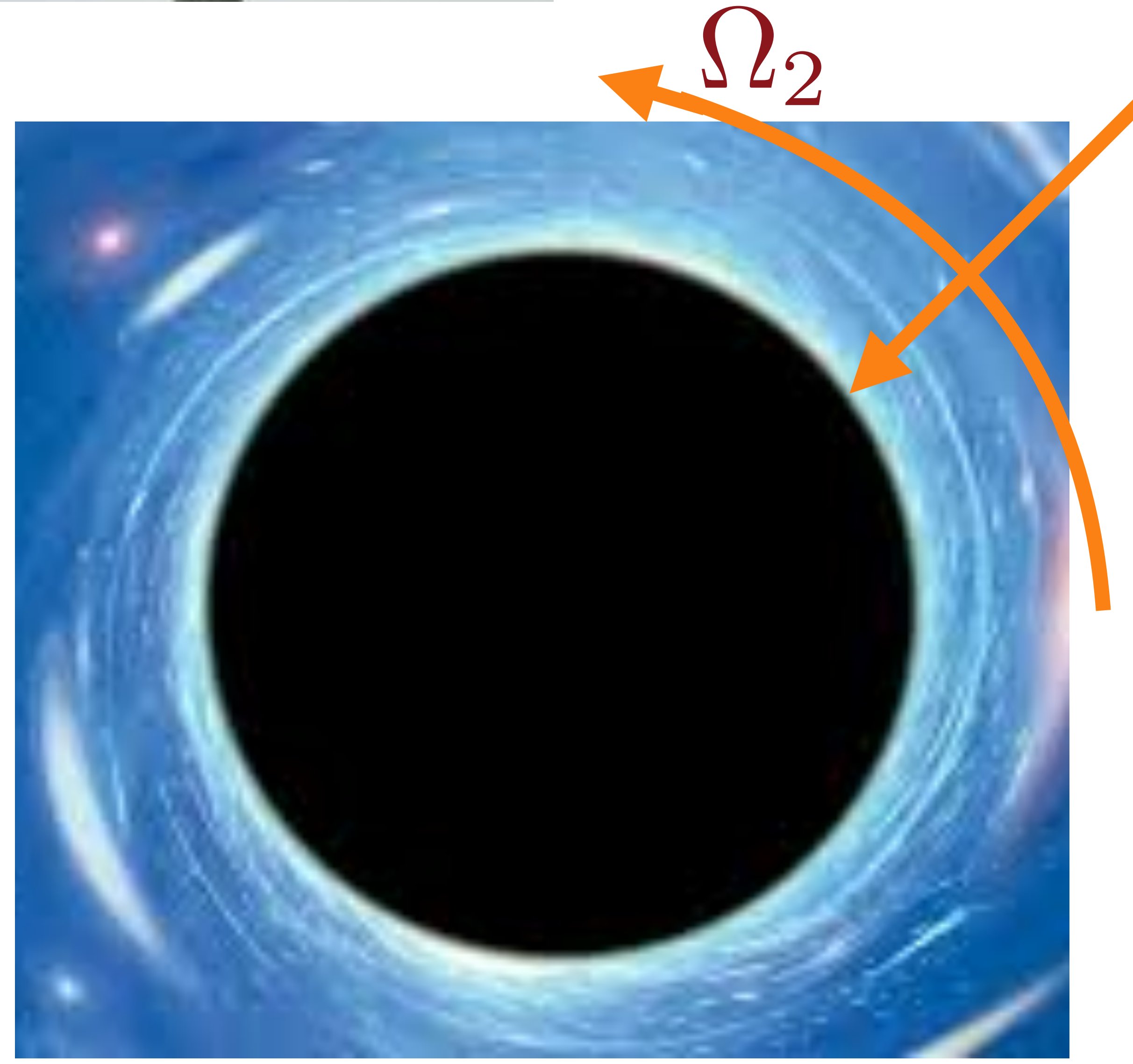
$$r = R_h + \frac{R_h^2}{\zeta}$$

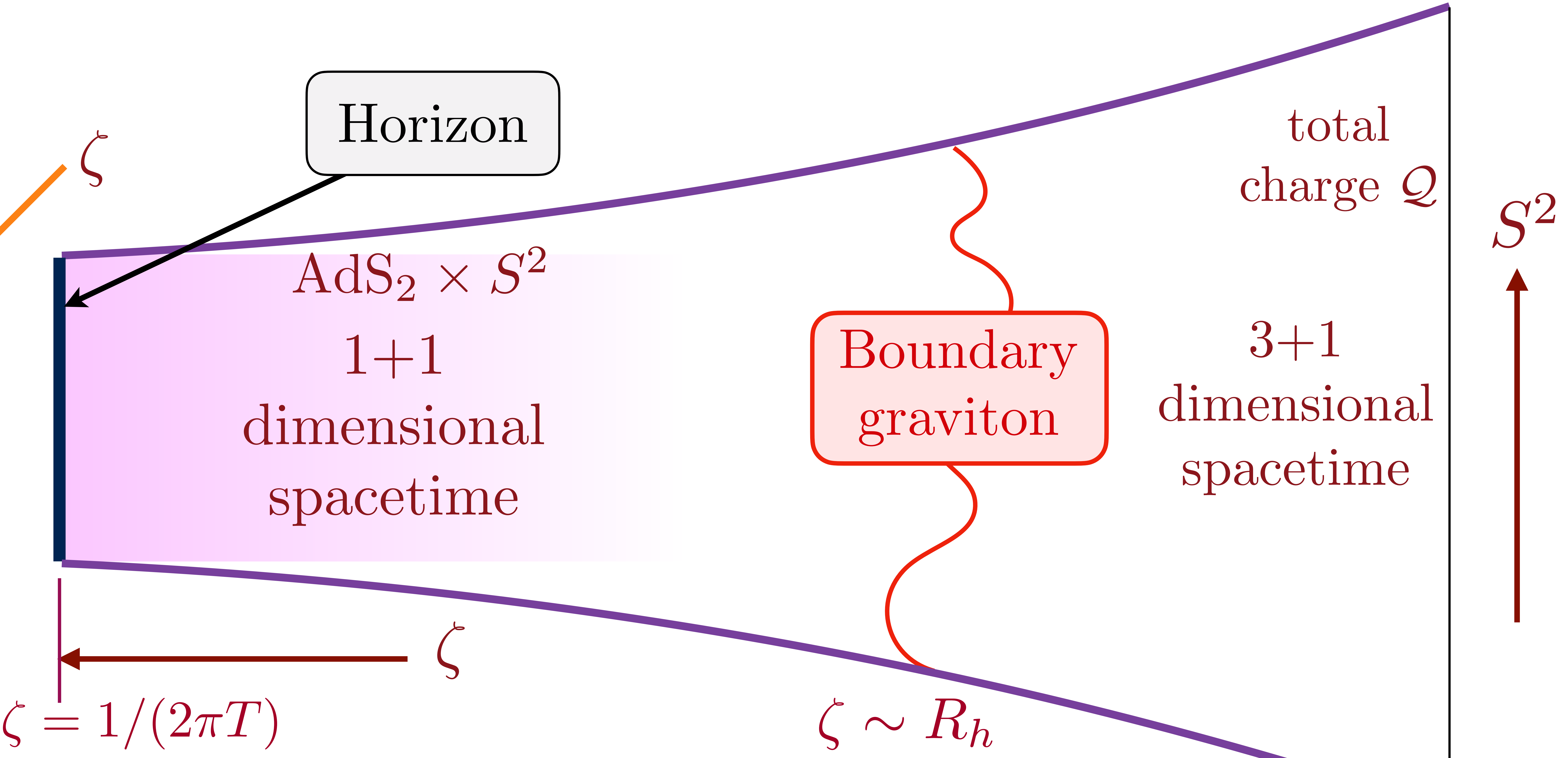
so that the horizon at $T = 0$ is at $\zeta = \infty$. Then in the near-horizon regime $R_h \ll \zeta < \infty$ the $T = 0$ metric is

$$ds^2 = R_h^2 \frac{d\tau^2 + d\zeta^2}{\zeta^2} + R_h^2 d\Omega_2^2$$

This spacetime is $\mathrm{AdS}_2 \times S^2$.

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions
to 1+1 dimensions (AdS_2) at low energies!

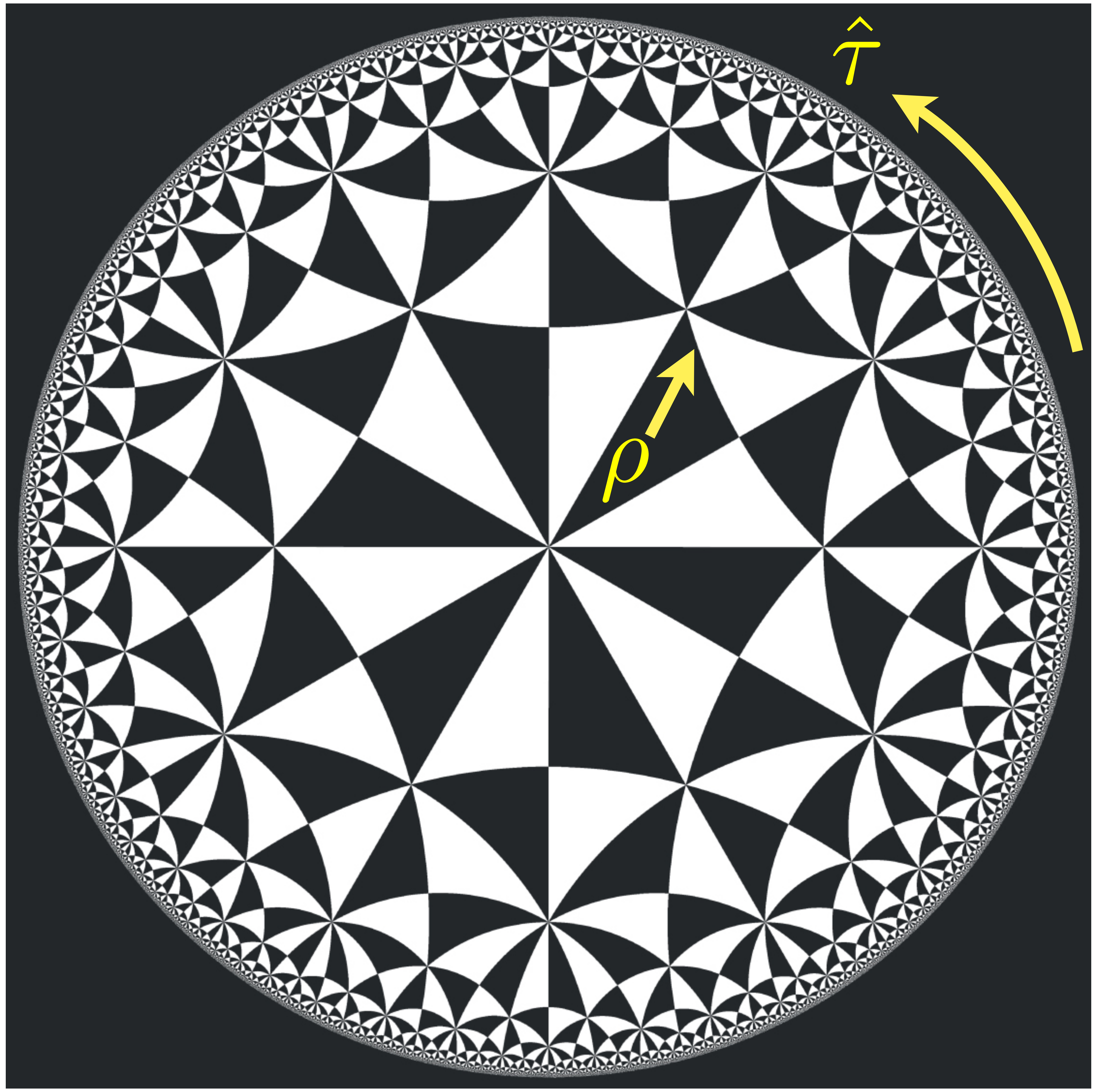
The AdS_2 metric

$$ds^2 = \frac{d\tau^2 + d\zeta^2}{\zeta^2}$$

is invariant under isometries which are SL(2,R) transformations. Verify that the co-ordinate change

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d}\,, \qquad ad - bc = 1\,,$$

with a,b,c,d real, leaves the AdS_2 metric invariant.



The co-ordinate transformation

$$\zeta = \frac{1}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})} \quad , \quad \tau = \frac{\sinh(2\pi T\rho)\sin(2\pi T\hat{\tau})}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})}$$

maps the metric to

$$ds^2 = 4\pi^2 T^2 \left[d\rho^2 + \sinh^2(2\pi T\rho)d\hat{\tau}^2\right]$$

Charged black holes

1. Reduce the 4-spacetime dimensional theory in I_{EM} to a 1+1 dimensional theory $I_{EM,2}$ by taking all fields dependent only upon the radial co-ordinate r and imaginary time τ.

2. Take the low energy limit of $I_{EM,2}$ by mapping it to a near-horizon theory, I_{JT}, in a 1+1 dimensional spacetime with a boundary.

3. Compute fluctuations about the AdS_2 saddle point of I_{JT}. Einstein gravity in 1+1 dimensions has no graviton, and is 'pure gauge'. In the JT-gravity theory with boundary, there is a remnant degree of freedom which is a boundary graviton. The action for this boundary graviton is the Schwarzian theory. The partition function of this Schwarzian theory can be evaluated exactly.

Charged black holes

1. Make the metric ansatz

$$ds^2 = \frac{ds_2^2}{\Phi(\zeta,\tau)} + [\Phi(\zeta,\tau)]^2 \, d\Omega_2^2$$

where ds_2^2 is an arbitrary metric in the (ζ,τ) spacetime, and Φ is a scalar field in the (ζ,τ) spacetime.

2. The low energy theory on the (ζ,τ) spacetime involves a metric h, and a scalar field Φ_1 given by $\lim_{\zeta\to\infty}[\Phi(\zeta,\tau)]^2 = R_h^2 + \Phi_1(\zeta,\tau)$, obeying the action

$$I_{JT} = -\frac{2\pi\mathcal{A}_0}{\kappa^2} + \int d^2x\sqrt{h}\left[-\frac{2\pi}{\kappa^2}\,\Phi_1\left(\mathcal{R}_2 + \frac{2}{R_h^3}\right)\right] - \frac{4\pi}{\kappa^2}\int_\partial dx\sqrt{h_b}\Phi_1\,\mathcal{K}_1$$

where $\mathcal{A}_0 = 4\pi R_h^2$ is the area of the horizon at $T=0$, and $\mathcal{K}_1$ is the extrinsic curvature of the one-dimensional boundary $\zeta\to 0$ where

$$h_{\tau\tau}(\zeta\to 0) = \frac{R_h^3}{\zeta^2} \quad , \quad \Phi_1(\zeta\to 0) = \frac{2R_h^3}{\zeta}$$

Charged black holes

3. Remarkably, the partition function of the $1+1$ dimensional JT gravity theory can be evaluated exactly (here we are ignoring the gauge field path integral, which is subdominant at fixed $\mathcal{Q}$)

$$\mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}h\mathcal{D}\Phi_1 \exp\left(-I_{JT}\right)$$

The action is linear in Φ_1, and the integral over Φ_1 yields a constraint $\mathcal{R}_2 = -2/R_h^3$ *i.e.* the metric h is rigidly AdS_2. The only dynamical degree of freedom in JT gravity is a time reparameterization along the boundary $\tau \to f(\tau)$. To ensure that the bulk metric obeys its boundary condition, we also have to make the spatial co-ordinate ζ a function of τ, so we map $(\tau, \zeta) \to (f(\tau), \zeta(\tau))$. Then the metric obeys its boundary condition provided $\zeta(\tau)$ is related to $f(\tau)$ by (here ζ_b is a small constant whose value cancels in the final result)

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \frac{\left[f''(\tau)\right]^2}{2f'(\tau)} + \mathcal{O}(\zeta_b^4)$$

Finally, we evaluate I_{GH} along this boundary curve. In this manner we obtain the action

$$I_{1,\mathrm{eff}}[f] = -\frac{2\pi\mathcal{A}_0}{\kappa^2} - \frac{\gamma}{4\pi^2}\int d\tau \left\{f(\tau), \tau\right\} \quad , \quad \{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$ is precisely the linear-T co-efficient in the black hole entropy.

Charged black holes

3. After a conformal map to finite temperature (and ignoring the contribution of the gauge field fluctuation), we can write the low energy partition function of a 3+1-dimensional black hole with charge $\mathcal{Q} = 4\pi R_h/(\kappa g_F)$, as a path integral over a single field $f(\tau)$ in one time dimension:

$$\mathcal{Z}_{\mathcal{Q}} = \exp\left(\frac{2\pi \mathcal{A}_0}{\kappa^2}\right) \int \frac{\mathcal{D}f}{||\text{SL(2,R)}||} \exp\left(\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)), \tau\right\}\right)$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$, $\mathcal{A}_0 = 4\pi R_h^2$, and $f(\tau)$ is a monotonic function of τ obeying

$$f(\tau + 1/T) = f(\tau) + 1/T\,.$$

We divide by the (infinite) volume of the SL(2,R) group because

$$\{f, \tau\} = \{\frac{af+b}{cf+d}, \tau\}$$

where a, b, c, d are constants with $ad - bc = 1$.