# Basic Concepts and some current Directions in Ultracold Gases 

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These are notes on a series of lectures on many-body phenomena in ultracold gases at the Collège de France in the Fall of 2021. Their main focus are Bose systems, for a review of strongly interacting Fermi gases see the Varenna Lectures 2014, accesssible via arXiv:1608.00457. As an introductory comment, I quote from the preface of the two volume book on 'Statistical Field Theory' by C. Itzykson and J.-M. Drouffe who remark:
, A book might give the illusion, especially to students, that some knowledge has become definitive and that the authors understand every part of it. This is a completely false view. No one can really fully master even his own subject, and this is luckily a source of progress.'

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## APPENDIX: SUPERFLUIDITY AND DISSIPATIONLESS CURRENTS



FIG. 1: Schematic setup of out-coupling two atom beams from a trapped BEC via RF-transitions into an untrapped hyperfine state $m_{F}=0$. The visibility of the resulting interference fringes as a function of separation is shown on the right. Below the BEC transition temperature of $T_{c} \simeq 400 \mathrm{nK}$, the visibility approaches a constant for separations exceeding about ten average interparticle distances, thus providing direct evidence for the presence of off-diagonal long range order. The Figures are taken from Bloch et al. [2].

## I. SUPERFLUIDITY IN GASES AND LIQUIDS

Off-diagonal long range order and Widom particle insertion A precise definition of Bose-Einstein condensation (BEC) in an interacting system has been given by Penrose [1]. It is based on the concept of off-diagonal long range order (ODLRO) which states that the off-diagonal elements

$$
\begin{equation*}
\lim _{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right| \rightarrow \infty} \rho_{1}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\lim _{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right| \rightarrow \infty}\left\langle\hat{\psi}^{\dagger}(\boldsymbol{x}) \hat{\psi}\left(\boldsymbol{x}^{\prime}\right)\right\rangle=n_{0} \neq 0 \tag{1}
\end{equation*}
$$

of the one-particle density operator $\hat{\rho}_{1}$ approach a finite constant at arbitrary large separation. The limit defines a condensate density $n_{0}$ which is the square of the order parameter for BEC in the interacting system. Physically, the condition (1) reflects the presence of long range phase coherence: states in which one particle is removed either at $\boldsymbol{x}$ or at a distant position $\boldsymbol{x}^{\prime}$ have a finite overlap for arbitrary large separation. Experimentally, this property has first been observed in the context of ultracold gases by Bloch et al. [2]. As shown in Fig. 1] the visibility in the interference from two beams outcoupled at separate points of a trapped BEC decreases to zero as a function of separation above the critical temperature while it stays finite below the transition.

In the following, we want to ask what are necessary and sufficient conditions in the ground state many-body wave function for the existence of ODLRO. Specifically, we consider a generic non-relativistic Hamiltonian with pure two-body interactions. The associated first quantized Hamiltonian

$$
\begin{equation*}
\hat{H}_{N}=-\frac{\hbar^{2}}{2 m} \sum_{i=1}^{N} \nabla_{i}^{2}+\sum_{1 \leq i<j \leq N} V\left(\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right) \tag{2}
\end{equation*}
$$

gives rise to a proper thermodynamics with an extensive free energy and a positive compressibility provided the interaction obeys

$$
\begin{equation*}
\sum_{1 \leq i<j \leq N} V\left(\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right)>-B \cdot N . \tag{3}
\end{equation*}
$$

Here, $B$ is a positive constant independent of the specific state. As shown by Fisher [3], a sufficient condition for the validity of Eq. (3) is that the two-body potential $V(r) \geq-\epsilon$ has a finite lower bound, decays faster than $1 / r^{3}$ at large distances and increases more rapidly than $1 / r^{3}$ for separations smaller than a short range scale $\sigma$. More specifically, we consider interactions with an asymptotic van der Waals tail $V(r \rightarrow \infty)=-C_{6} / r^{6}$. Apart from $\sigma$, they are characterized by the van der Waals length $\ell_{\mathrm{vdW}}=\left(m C_{6} / \hbar^{2}\right)^{1 / 4} / 2$ as a second length scale which is determined solely by the asymptotic part of the interaction. A standard example is the Lennard-Jones potential $V(r)=4 \epsilon\left[(\sigma / r)^{12}-(\sigma / r)^{6}\right]$ where the short distance scale $\sigma$ and the depth $\epsilon$ are connected with the strength of the van der Waals tail via $C_{6}=4 \epsilon \sigma^{6}$. Independent of the precise form of $V(r)$, the equilibrium free energy $F(N)=f N+\ldots$ for the class of potentials obeying (3) scales linearly with the particle number. At zero temperature, the generic ground state even in the limit of vanishing pressure is a solid, where both the particle statistics and zero point fluctuations play only a minor role. A measure for their strength is provided by the parameter

$$
\begin{equation*}
\Lambda_{\mathrm{dB}}=\frac{\hbar}{\sigma \sqrt{m \epsilon}} \underset{\mathrm{vdW}}{ } \frac{1}{2}\left(\frac{\sigma}{\ell_{\mathrm{vdW}}}\right)^{2} \tag{4}
\end{equation*}
$$

introduced by De Boer [4], which is the square root of the ratio between the zero point energy on the scale $\sigma$ and the depth $\epsilon$ of the attractive part of the potential. From numerical studies, Nosanow et al. [5] found that for Bosons the crystalline solid ground


FIG. 2: The Figure on the left shows the phase diagram of ${ }^{4} \mathrm{He}$, whose ground state is a superfluid liquid below a critical pressure $p_{c} \simeq 25$ bar. On the right, a qualitative phase diagram is shown for a Bose system in the regime $\Lambda_{\mathrm{dB}}>\Lambda_{\mathrm{dB}}^{c}$, where the ground state at low pressure is a superfluid gas. The continuous transition from the superfluid to the normal gas asymptotically exhibits a cubic dependence $p(T) \simeq g / \lambda_{T}^{6} \sim g T^{3}$.
state realized for small values of $\Lambda_{\mathrm{dB}}$ melts into a liquid at a non-universal critical value $\Lambda_{\mathrm{dB}} \simeq 0.37$ via a first order quantum phase transition. Both the solid and the liquid phase have a finite density $\bar{n}$ at vanishing pressure and a negative ground state energy $u(\bar{n})$ per particle. Specifically, for ${ }^{4} \mathrm{He}$, where $\Lambda_{\mathrm{dB}} \simeq 0.42$, precise results for the dimensionless density $\bar{n} \sigma^{3} \simeq 0.364$ or the energy per particle $u(\bar{n}) \simeq-0.7 \epsilon \simeq-k_{B} \cdot 7 \mathrm{~K}$ are available by numerical methods [6, 7]. Upon further increasing the strength of the zero point fluctuations, the liquid eventually unbinds into a gas through a continuous quantum phase transition at $\Lambda_{\mathrm{dB}}^{c} \simeq 0.69$. This transition was first studied numerically by Miller et al. [8] and will be discussed in more detail below. The phase diagram at finite temperature beyond $\Lambda_{\mathrm{dB}}^{c}$, which has neither a triple nor a critical point, is sketched in Fig. 2 . As a true equilibrium configuration it is realized only for spin polarized hydrogen, where $\Lambda_{\mathrm{dB}} \simeq 0.74$ [9]. A gaseous superfluid near vanishing pressure and temperature is also present in ultracold Alkali gases, even though their de Boer parameter is much less than one. This is a result of the fact that in the regime of very low densities $n \ell_{\mathrm{vdW}}^{3} \ll 1$, the short distance length $\sigma$ can effectively be taken to zero and the liquid or solid equilibrium phases are not reached because states with negative energy are inaccessible kinematically with just two-body collisions. As a result, the Hamiltonian can be truncated to one involving only states in the continuum. For positive two-body scattering length $a$, a gaseous state then forms a stable equilibrium configuration. In the following, we will show that the ground state of a Bose system always exhibits BEC or the related phenomenon of superfluidity provided it is a homogeneous fluid, i.e. either a liquid or a gas. For solids with broken translation invariance, superfluidity may still be present, however generically this requires a finite defect density in their ground state.

It was observed by Feynman [10] that the many-body ground state wave function $\psi\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)$ of a Bose system with a permutation symmetric and real Hamiltonian of the form (2) has no nodes. In fact, this is a special case of a more general theorem which states that the lowest energy in an unconstrained minimization of $\langle\psi| \hat{H}|\psi\rangle$ is realized for a positive and symmetric wave function ('minimizers are bosonic') ${ }^{1}$. The theorem relies on the observation that $|\psi|$ gives the same energy as $\psi$ itself and that in a decomposition $\psi=\psi_{s}+\psi_{r}$ into a permutation symmetric part $\psi_{s}$ and a remainder, the cross terms in $\langle\psi| \hat{H}|\psi\rangle$ vanish (for a rigorous proof see Lieb and Seiringer [12], chapter 3.2.4). In order to deal with Bose fluids with strong interactions as in ${ }^{4} \mathrm{He}$, Feynman and later Penrose and Onsager [13] suggested to express the symmetric and positive many-body wave function

$$
\begin{equation*}
\psi\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)=\left[p_{\mathrm{cl}}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)\right]^{1 / 2}=\frac{1}{\sqrt{Q_{N}}} \exp \left\{-\tilde{V}_{N}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right) / 2\right\} \tag{?}
\end{equation*}
$$

in terms of the square root of a $N$-body probability density of a classical fluid at some finite effective temperature. The normalization is provided by the classical configuration integral $Q_{N}=\int d 1 \ldots d N \exp \left\{-\tilde{V}_{N}(1 \ldots N)\right\}$. In principle, such a representation is always possible by defining the dimensionless effective potential $\tilde{V}_{N}(1 \ldots N)$ of the classical reference system such that the square of (5) is obeyed as an identity. This is used e.g. in Laughlin's plasma analogy for incompressible states in the lowest Landau level, connecting the square of Ansatz wavefunctions to a 2d Coulomb gas with logarithmic interactions [14]. In the present context, however, the idea is useful only if $\tilde{V}_{N}$ is similar to the underlying microscopic interaction in the quantum manybody problem. As will be shown below, this is actually impossible for any compressible Bose fluid. An assumption which is often made in addition is that the classical reference system can be described by a sum $\tilde{V}_{N}(1 \ldots N)=\sum_{i<j} \tilde{v}\left(r_{i j}\right)$ involving a

[^1]

FIG. 3: Widom particle insertion: Two particles at $\boldsymbol{x}$ respectively $\boldsymbol{x}^{\prime}$ are added to a classical fluid of $N-1$ particles at positions $\boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}$, represented by full discs. The strength of the interaction of the added particles with those of the fluid is half of that within the fluid itself.
translation and rotation invariant two-body interaction $\tilde{v}(r)$. In this case, the many-body wave function

$$
\begin{equation*}
\psi_{\text {Jastrow }}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)=\frac{1}{\sqrt{Q_{N}}} \exp \left[-\sum_{i<j} \tilde{v}\left(r_{i j}\right) / 2\right] \tag{6}
\end{equation*}
$$

is a product of $N(N-1) / 2$ identical two-body wave functions, as introduced by Bijl [15] and Jastrow [16]. For the following considerations, this form is not necessary, however. Indeed, quite generally, the representation (5) implies that the one-particle density matrix of the quantum system

$$
\begin{equation*}
\rho_{1}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\frac{N}{Q_{N}} \int d 2 \ldots d N \exp \left\{-\left[\tilde{V}_{N}(\boldsymbol{x}, 2 \ldots N)+\tilde{V}_{N}\left(\boldsymbol{x}^{\prime}, 2 \ldots N\right)\right] / 2\right\} \tag{7}
\end{equation*}
$$

can be expressed in terms of a Boltzmann weight of a classical $N$-particle system where one of the particles is either at a position $\boldsymbol{x}$ or at $\boldsymbol{x}^{\prime}$. As indicated schematically in Fig. 3 , the exponent ${ }^{2}$

$$
\begin{equation*}
\frac{1}{2}\left[\tilde{V}_{N}(\boldsymbol{x}, 2 \ldots N)+\tilde{V}_{N}\left(\boldsymbol{x}^{\prime}, 2 \ldots N\right)\right]=\tilde{V}_{N-1}(2 \ldots N)+\Delta_{2} \tilde{W}_{1 / 2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) \tag{8}
\end{equation*}
$$

may be separated into a contribution $\tilde{V}_{N-1}(2 \ldots N)$ which accounts for the full interaction energy of an $N$ - 1-particle system plus an additional term $\Delta_{2} \tilde{W}_{1 / 2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)$ which describes the change in energy associated with adding two particles at positions $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ that do not interact among themselves. The subscript $1 / 2$ indicates that they interact with the $N-1$ particles at positions $\boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}$ with only half the strength of the potential in the $N-1$-particle system. In a similar manner, the configuration integral for $N$ particles

$$
\begin{equation*}
Q_{N}=Q_{N-1} \int d 1\left\langle\exp \left\{-\Delta_{1} \tilde{W}\left(\boldsymbol{x}_{1}\right)\right\}\right\rangle_{N-1} \tag{9}
\end{equation*}
$$

can be expressed in terms of an expectation value of the dimensionless interaction energy $\Delta_{1} \tilde{W}\left(\boldsymbol{x}_{1}\right)$ associated with adding a single particle at position $\boldsymbol{x}_{1}$. Here, the average $\langle\ldots\rangle_{N-1}$ is defined by an integration over the positions $\boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}$ of an $N-1$ particle system with Boltzmann weight $\exp \left\{-\tilde{V}_{N-1}(2 \ldots N)\right\}$ and a normalization through the associated configuration integral $Q_{N-1}$. For a homogeneous system, $\left\langle\exp \left\{-\Delta_{1} \tilde{W}\left(\boldsymbol{x}_{1}\right)\right\}\right\rangle_{N-1}$ does not depend on $\boldsymbol{x}_{1}$, which can be choosen as the reference point for the remaining coordinates $\boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}$. The integral $\int d 1$ then just gives a factor $V$. Moreover, using standard thermodynamic relations, the ratio $Q_{N} / Q_{N-1}=V \cdot \exp \left\{-\tilde{\mu}_{\text {ex }}\right\}$ can be expressed in terms the excess chemical potential $\tilde{\mu}_{\text {ex }}=\tilde{F}_{N}-\tilde{F}_{N-1}-\tilde{\mu}_{\text {id }}$ of the fluid in units of the thermal energy. In the theory of classical fluids, these relations go back to Widom [17] and are called the Widom particle insertion method. In fact, the extraction of $\tilde{\mu}_{\text {ex }}$ in this manner is an example of an equality due to Jarzynski [18], which relates the excess chemical potential in equilibrium to the exponential average of the work $\Delta_{1} \tilde{W}(0)$ needed to add a single particle at fixed total volume $V$. Using the decomposition in Eq. (8), the one-particle density matrix

$$
\begin{equation*}
\rho_{1}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=n \frac{\left\langle\exp \left\{-\Delta_{2} \tilde{W}_{1 / 2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)\right\}\right\rangle_{N-1}}{\left\langle\exp \left\{-\Delta_{1} \tilde{W}(0)\right\}\right\rangle_{N-1}} \underset{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right| \rightarrow \infty}{ } n \frac{\left\langle\exp \left\{-\Delta_{1} \tilde{W}_{1 / 2}(0)\right\}\right\rangle^{2}}{\left\langle\exp \left\{-\Delta_{1} \tilde{W}(0)\right\}\right\rangle}=n_{0} \neq 0 \tag{10}
\end{equation*}
$$

of a homogeneous Bose fluid ground state can be expressed as the ratio of two expectation values in a $N-1$-particle state. For large separation $\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|$, this approaches a finite constant quite generally because inserting two particles at widely separated

[^2]positions in a classical fluid with short range interactions is equivalent to two independent single particle additions. A particularly simple situation arises by assuming that the classical reference system is a fluid of hard spheres with diameter $\sigma$. In this case, $\left.\Delta_{1} \tilde{W}_{1 / 2}(\boldsymbol{x})\right|_{\text {HS }} \equiv \Delta_{1} \tilde{W}(\boldsymbol{x})$ because half of the interaction strength is the same as the full interaction. For the hard sphere fluid, the ratio which determines the condensate fraction in $(10)$ is thus equal to $\exp \left\{-\tilde{\mu}_{\mathrm{ex}}\right\}$. An analytical expression for the associated excess chemical potential, which only depends on the dimensionless filling fraction $\eta$, is obtained within the Carnahan-Starling form of the equation of state of a classical hard sphere fluid which gives ${ }^{3}$
\[

$$
\begin{equation*}
\left.\tilde{\mu}_{\mathrm{ex}}\right|_{\mathrm{HS}}=\eta \frac{8-9 \eta+3 \eta^{2}}{(1-\eta)^{3}} \text { with } \eta=N v_{\sigma} / V=\frac{\pi}{6} n \sigma^{3} . \tag{11}
\end{equation*}
$$

\]

As shown by Penrose and Onsager [13], this approach can be used to provide an estimate for the condensate fraction in ${ }^{4} \mathrm{He}$. Taking the known value $\sigma \simeq 2.5 \AA$ of the short distance scale below which the ${ }^{4} \mathrm{He}-{ }^{4} \mathrm{He}$ interaction becomes strongly repulsive as an effective hard sphere diameter, the density of liquid ${ }^{4} \mathrm{He}$ is about 0.28 times that of close packing, which corresponds to an effective dimensionless filling fraction $\eta_{\text {eff }}\left({ }^{4} \mathrm{He}\right) \simeq 0.2$. Based on Eq. 11 for the associated excess chemical potential, this leads to a condensate fraction $n_{0} / n=\exp -\tilde{\mu}_{\mathrm{ex}}=0.078$. The assumption that the many-body ground state wave function of a Bose fluid has a representation in the form (5) of an effective classical reference system thus leads to two important conclusions:
(a) Any translation invariant ground state of an interacting Bose system is necessarily a superfluid exhibiting ODLRO, because two-particle insertion at widely separated points in a classical fluid with short range interactions factorizes.
(b) Explicit results for the condensate fraction of strongly correlated Bose fluids may be obtained from a generalization of the Widom particle insertion method in classical fluids via Eq. 10. In particular, using the known value $\tilde{\mu}_{\text {ex }}$ for the excess chemical potential of a gas of hard spheres, the prediction of a zero temperature condensate fraction $n_{0} / n \simeq 0.08$ in liquid ${ }^{4} \mathrm{He}$ by Penrose and Onsager is close to the value obtained via path integral Monte Carlo methods [6].

Both conclusions are correct, however their derivation based on the mapping (5) is moot. A simple reason for this becomes evident from the fact that path integral Monte Carlo calculations of the hard sphere Bose fluid by Grüter et al. [20] show that its ground state is a non-superfluid crystal beyond $\eta \simeq 0.12$. The hard sphere system is thus no longer a fluid at ${ }^{4} \mathrm{He}$ densities. A more fundamental problem with Feynman's Ansatz connecting the many-body wave function to the square root of the distribution function of particle positions in a classical fluid is revealed by considering the limit of a dilute gas. The thermodynamic properties of the classical reference fluid may then be obtained from a virial expansion. For the specific case of a hard sphere system, using (11) to leading order in $\eta \ll 1$, this results in $n_{0} / n=1-8 \eta+\ldots$. The deviation of the condensate fraction from the ideal Bose gas limit is thus found to be linear in the density $n$. This contradicts the classic Bogoliubov result [21]

$$
\begin{equation*}
n_{0}=n-\frac{8 n}{3}\left(n a^{3} / \pi\right)^{1 / 2}+\ldots=n-\frac{\sqrt{2}}{12 \pi^{2} \xi^{3}}+\ldots \tag{12}
\end{equation*}
$$

where the correction to $n_{0} / n$ due to interactions scales with the square root of the density, a prediction that was verified experimentally by Lopes et al. [22]. In the limit of a dilute gas, the healing length $\xi=(8 \pi n a)^{-1 / 2}$ only depends on the density and the scattering length $a>0$ as a single parameter characterizing the interaction. The physical origin of the discrepancy between Bogoliubov theory and a naive virial expansion is hidden in the fact that the interactions in the classical reference fluid underlying the representation (5) can not be of short range. Indeed, for any classical compressible fluid, the static structure factor $S_{\mathrm{cl}}(q \rightarrow 0)=(\partial n / \partial \tilde{\mu}) / n$ is finite in the limit of vanishing wave vector. By contrast, a compressible quantum fluid at zero temperature has a static structure factor

$$
S(q \rightarrow 0)=|q| \xi / \sqrt{2}+\ldots \rightarrow g^{(2)}(r \rightarrow \infty)=1- \begin{cases}\frac{\xi}{\pi^{2} \sqrt{2} n r^{4}} & \text { in } d=3  \tag{13}\\ \frac{\xi}{2 \pi \sqrt{2} n_{2} r^{3}} & \text { in } d=2\end{cases}
$$

which vanishes in a non-analytic manner. For a fluid of Bosons, the associated characteristic length $\xi$ is fixed by the sound velocity $c_{s}$ via $\xi=\hbar /\left(\sqrt{2} m c_{s}\right)$, a relation which in fact holds for arbitrary strength of the interactions. This is a consequence of the fact that the Feynman-Bijl single mode result $E_{q}=\varepsilon_{q} / S(q) \rightarrow \hbar c_{s} q$ for the excitation energy becomes exact in the limit of small wave vectors [23], as will be discussed in more detail in Lecture III. Due to $S_{\mathrm{cl}}(q=0) \neq 0$, the Ansatz (5) does not describe correctly the long wavelength physics and therefore fails to reproduce the Bogoliubov result in the dilute limit. The

[^3]excellent agreement of the prediction $n_{0} / n \simeq 0.08$ for the condensate fraction of ${ }^{4} \mathrm{He}$ with precise ab initio results which is obtained by using this mapping must therefore be considered as fortuitous. Formally, the behavior $S(q \rightarrow 0) \rightarrow|\boldsymbol{q}| \xi / \sqrt{2}$ can be enforced in a finite temperature classical fluid by adding long range repulsive two-body interactions $\tilde{v}(r) \rightarrow 1 /\left(\sqrt{2} \pi^{2} n \xi r^{2}\right)$ (or $\tilde{v}(r) \rightarrow 1 /\left(\sqrt{2} \pi n_{2} \xi r\right)$ in two dimensions), as pointed out by Reatto and Chester [24]. Apart from the required knowledge of the effective healing length $\xi$ or the associated sound velocity $c_{s}$, however, this interaction is not only density dependent but decays to zero so slowly that the condition (3) for the existence of a proper thermodynamic limit is violated. Whether the expression (10) for the condensate fraction in strongly interacting Bose fluids can be extended to cover a situation where the short distance scale $\sigma$ can be taken to zero while the two-body scattering length $a$ is of the order of the mean interparticle spacing or even infinite, is an open problem. It is of current interest in view of recent measurements of dimensionless ratios which characterize the unitary Bose gas, whose condensate fraction is estimated to be $n_{0} / n \simeq 0.2$ [25].

Bogoliubov theory as an internal Josephson effect in momentum space At this point, following in part Lectures by Nozières [26], it is instructive to add a few remarks regarding the Bogoliubov approach which are not discussed in standard textbooks. The approach relies on replacing the annihilation operator $\hat{b}_{0} \rightarrow z$ for vanishing momentum by a complex number $z$ (or $\bar{z}$ for $\hat{b}_{0}^{\dagger}$ ) and neglecting contributions to the interaction part of the second quantized form of the Hamiltonian (2) which contain only operators with finite momentum (for a discussion of why a replacement of operators by a c-number still gives the correct thermodynamics see Lieb et al. [27]) As a result, the Hamiltonian is reduced to a quadratic one

$$
\begin{equation*}
\hat{H}_{\mathrm{Bog}}=E_{H}+\sum_{q \neq 0}\left(\varepsilon_{q}+n_{0} V(q)\right) \hat{b}_{q}^{\dagger} \hat{b}_{q}+\frac{1}{2 V} \sum_{q \neq 0} V(q)\left(\bar{z}^{2} \hat{b}_{q} \hat{b}_{-q}+z^{2} \hat{b}_{-q}^{\dagger} \hat{b}_{q}^{\dagger}\right) \rightarrow E_{\mathrm{Bog}}+\sum_{q \neq 0} E_{q} \hat{\alpha}_{q}^{\dagger} \hat{\alpha}_{q} \tag{14}
\end{equation*}
$$

which may be diagonalized by introducing a set of Bosonic quasiparticles. Here, $n_{0}=|z|^{2} / V$ is the condensate density and $V(q)$ is the Fourier transform of the two-particle interaction, which is assumed to be positive. Its value $g^{(0)}=V(q=0)>0$ at vanishing momentum determines the Hartree energy $E_{H}=N \cdot g^{(0)} n / 2$. The Hamiltonian 14 is a bosonic version of the reduced BCS-Hamiltonian for Fermions. Provided that the phase of the complex number $z$ can be choosen to vanish, the associated gap function $\Delta_{q} \equiv n_{0} V(q)$ is real and positive. As will be shown below, this is always possible, however a choice for the phase of $z$ also fixes the phase associated with pairs $(q,-q)$ of particles in the depletion. Following the notation of standard textbooks [28], the operators $\hat{\alpha}_{q}^{\dagger}$ which create the Bosonic quasiparticles with momentum $q$ are connected with the corresponding operators $\hat{b}_{q}^{\dagger}$ of the underlying Bosons by

$$
\begin{equation*}
\hat{\alpha}_{q}^{\dagger}=u_{q} \hat{b}_{q}^{\dagger}+v_{q} \hat{b}_{-q} \quad \leftrightarrow \quad \hat{b}_{q}^{\dagger}=u_{q} \hat{\alpha}_{q}^{\dagger}-v_{q} \hat{\alpha}_{-q} \quad \text { with } u_{q}^{2}-v_{q}^{2}=1 . \tag{15}
\end{equation*}
$$

The amplitudes $u_{q}=\cosh \theta_{q}$ and $v_{q}=\sinh \theta_{q}$ are conveniently parametrized by a real rotation angle $\theta_{q}$, depending only on the magnitude $q=|q|$ of the wavevector. The ground state of the Hamiltonian $\sqrt[14]{ }$ is defined by the condition $\hat{\alpha}_{q}\left|\Psi_{\text {Bog }}\right\rangle \equiv 0$ of being the vacuum state for quasiparticles at all $q \neq 0$. It may be written in the form

$$
\begin{equation*}
\left|\Psi_{\text {Bog }}\right\rangle=\left|z,\left\{\lambda_{q}\right\}\right\rangle=e^{-|z|^{2} / 2} \prod_{q \neq 0}\left(1-\left|\lambda_{q}\right|^{2}\right)^{1 / 2} \exp \left(z \hat{b}_{0}^{\dagger}+\sum_{q \neq 0} \lambda_{q} \hat{b}_{q}^{\dagger} \hat{b}_{-q}^{\dagger}\right)|0\rangle \tag{16}
\end{equation*}
$$

of a product of a coherent state for the condensate with one involving pairs $(q,-q)$ with vanishing total momentum for the depletion. Indeed, choosing $\lambda_{q}=-v_{q} / u_{q}$, the state has a vanishing number of quasiparticles because

$$
\left[\hat{b}_{q}, \exp \left(\lambda_{q} \hat{b}_{q}^{\dagger} \hat{b}_{-q}^{\dagger}\right)\right]=\lambda_{q} \hat{b}_{-q}^{\dagger} \exp \left(\lambda_{q} \hat{b}_{q}^{\dagger} \hat{b}_{-q}^{\dagger}\right) \rightarrow \hat{\alpha}_{q}\left|\Psi_{\text {Воg }}\right\rangle=\left(u_{q} \lambda_{q}+v_{q}\right) \hat{b}_{-q}^{\dagger}\left|\Psi_{\text {Воg }}\right\rangle \equiv 0 \text { if } u_{q} \lambda_{q}+v_{q}=0
$$

Note that the condition $\hat{\alpha}_{q}\left|\Psi_{\text {Bog }}\right\rangle \equiv 0$ only involves finite momenta $q \neq 0$. The precise form choosen for the condensate wavefunction is thus left open. Taking this to be a simple coherent state $|z\rangle$ is just a convenient choice. For a given total number $N$ of Bosons, which is fixed only on average ${ }^{4}$, the associated parameter $z \rightarrow z_{\lambda}$ is eliminated as an independent variable through the constraint $N_{0}=|z|^{2}=N-\sum_{q \neq 0}\left\langle\hat{b}_{q}^{\dagger} \hat{b}_{q}\right\rangle$. The problem is thus reduced to determining the variables $\lambda_{q}$. Now, in order to understand the underlying physics and the generality of Bogoliubov's approach, it is instructive to determine the expectation value of the Bogoliubov Hamiltonian in the normalized state [16, using a parametrization of the - in general complex - variables $\lambda_{q}=\tanh \theta_{q} \exp i \varphi_{q}$ in terms of a real parameter $\theta_{q}$ and a phase $\varphi_{q}$, The necessary expectation values are $\left\langle\hat{b}_{q}^{\dagger} \hat{b}_{q}\right\rangle=\sinh ^{2} \theta_{q}$ for the average occupation number of Bosons with finite momentum and a nonzero 'anomalous' expectation

[^4]value $\left\langle\hat{b}_{q} \hat{b}_{-q}\right\rangle=\sinh \theta_{q} \cosh \theta_{q} \exp i \varphi_{q}$ which depends on the phase of $\lambda_{q}$. Defining a phase $\varphi_{c}$ for pairs of particles in the condensate by $z^{2}=N_{0} \exp i \varphi_{c}$, the expectation value of the Bogoliubov Hamiltonian in the state (16) has the form
\[

$$
\begin{equation*}
\left\langle z_{\lambda},\left\{\lambda_{q}\right\}\right| \hat{H}_{\mathrm{Bog}}\left|z_{\lambda},\left\{\lambda_{q}\right\}\right\rangle=E_{H}+\sum_{q \neq 0}\left[\xi_{q} \sinh ^{2} \theta_{q}+\Delta_{q} \sinh \theta_{q} \cosh \theta_{q} \cdot \cos \left(\varphi_{c}-\varphi_{q}\right)\right] . \tag{17}
\end{equation*}
$$

\]

Here, $\xi_{q}=\varepsilon_{q}+\Delta_{q}$ is the single-particle energy within a Hartree-Fock approximation. It approaches a constant $\Delta_{0}=n_{0} g^{(0)}$ as $q \rightarrow 0$ and thus would lead to a finite excitation gap. This is at variance with the expected gapless nature of the excitations associated with the breaking of the global continuous symmetry $\hat{b}_{q} \rightarrow \hat{b}_{q} \exp (i \varphi)$ which is still present in 14. To see how the actual gapless excitations $E_{q} \rightarrow \hbar c_{s} q+\ldots$ arise within the Bogoliubov approach, it is necessary to include the phase dependent contribution to the energy (17). Apparently, this term is minimized by choosing a fixed relative and momentum independent phase $\Delta \varphi=\varphi_{c}-\varphi_{q}=\pi$. In an interacting BEC, therefore, there is an effective internal $\pi$-Josephson junction in momentum space between pairs of particles in the condensate and those with opposite momentum in the depletion (note that pairs are necessary because the ground state must have zero momentum). The associated phase dependent coupling energy $E_{J} \cos \Delta \varphi=-E_{J}=-\sum_{q \neq 0} \Delta_{q} \sinh \theta_{q} \cosh \theta_{q}$ is negative despite the fact that the underlying interaction is purely repulsive. This is analogous to what happens in the effective $\pi$-Josephson junction at the interface between a d-wave and an s-wave superconductor, where tunneling occurs between gaps which are positive on the s-wave and negative on the d-wave side, a setup, which has been used to determine the non-trivial nature of pairing in high-temperature superconductors by Wollman et al. [33]. On a formal level, the relative phase $\Delta \varphi=\pi$ between the condensate and the depletion just accounts for the minus sign which appears in $\lambda_{q}=-v_{q} / u_{q}$. The underlying physics, however, has a number of important and not widely appreciated consequences:
(a) The internal Josephson coupling between pairs of particles in the condensate and those in the depletion with opposite momentum is both necessary and sufficient for the generic behavior (13) of the static structure factor of a compressible Bose fluid and thus eventually for the gapless nature of the excitation spectrum. It explains, moreover, the fact that the ground state is fully superfluid despite a condensate density which might be below ten percent as in ${ }^{4} \mathrm{He}$.

To see this, consider the static structure factor $S(q)=\left\langle\hat{\rho}_{q}^{\dagger} \hat{\rho}_{q}\right\rangle$ which involves the normalized density fluctuation operator $\hat{\rho}_{q}^{\dagger}=\sum_{k} \hat{b}_{k+q}^{\dagger} \hat{b}_{k} / \sqrt{N}$. Within the Bogoliubov approach, this can be calculated exactly to zeroth order in the small parameter $\left(n a^{3}\right)^{1 / 2}$ by restricting $\hat{\rho}_{q}^{\dagger} \simeq \hat{b}_{q}^{\dagger}+\hat{b}_{-q}$ to those contributions which involve $\hat{b}_{0} \rightarrow z$, which is conveniently choosen to be real. As a result, one obtains

$$
\begin{equation*}
S_{\text {Bog }}(q)=\left\langle\Psi_{\text {Bog }}\right| \hat{\rho}_{q}^{\dagger} \hat{\rho}_{q}\left|\Psi_{\text {Bog }}\right\rangle=\left.\frac{1+2 \tanh \theta_{q} \cos \varphi_{q}+\tanh ^{2} \theta_{q}}{1-\tanh ^{2} \theta_{q}}\right|_{\varphi_{q}=\pi}=\exp -2 \theta_{q} \xrightarrow[\Delta_{q}=\text { const }]{ }\left(1+2 / q^{2} \xi^{2}\right)^{-1 / 2} . \tag{18}
\end{equation*}
$$

Here, in the final form of the expression, we have used that fixing $\Delta \varphi=\pi$ at its optimum value, a minimization of the energy (17) with respect to the remaining variables $\theta_{q}$ leads to $\tanh 2 \theta_{q}=\Delta_{q} / \xi_{q}$. This determines the momentum dependence of the static structure factor $S_{\mathrm{Bog}}(q)=\exp -2 \theta_{q}=\left(1+2 \Delta_{q} / \varepsilon_{q}\right)^{-1 / 2}$. In particular, defining the sound velocity via $\Delta_{0}=n_{0} g^{(0)} \rightarrow m c_{s}^{2}$ and an associated characteristic length $\xi$ via $\xi=\hbar /\left(\sqrt{2} m c_{s}\right)$, its behavior at small momentum is identical with the one given in Eq. (13). The Bogoliubov approach thus provides a proper description of the pair distribution function at long distances of any compressible Bose fluid. Evidently, it is precisely the minus $\operatorname{sign} \cos \varphi_{q}=-1$ associated with the internal Josephson effect which guarantees that the leading contributions at small $q$ in the numerator of the static structure factor 18 precisely cancel.

In order to understand why a BEC is fully superfluid at zero temperature despite the fact that the fraction $f_{0}$ of particles in the condensate may be much less than one, one needs to show that the superfluid fraction $f_{s}=N_{s} / N$ is equal to one at $T=0$. Here, as discussed further below, $N_{s}$ is defined in such a way that $N_{s} \cdot \hbar^{2} \boldsymbol{Q}^{2} / 2 m$ is the increase in the total energy of a state in which the Bose fluid is set into motion with a finite momentum $\boldsymbol{Q}$. Now, as a result of the Josephson coupling between the condensate and the depletion through an extensive energy $E_{J}$, this momentum is carried not only by the particles in the condensate but the complete momentum distribution is translated by $\boldsymbol{Q}$, giving rise to a mass current density $n_{s} \cdot \hbar \boldsymbol{Q}$ with $n_{s}=n$. The particles in the depletion are rigidly dragged along, with pairs now at $\boldsymbol{q}+\boldsymbol{Q},-\boldsymbol{q}+\boldsymbol{Q}$. As a result, the system is a perfect superfluid at zero temperature irrespective of the value of the condensate fraction $f_{0}$ as long as this is finite.
(b) The well defined relative phase between the condensate and the depletion is the origin of anomalously large fluctuations in the respective particle numbers $\hat{N}_{0}$ or $\hat{N}^{\prime}=\Sigma_{q \neq 0} \hat{n}_{q}$ which are enhanced by a factor $L / \xi$ or $L / \lambda_{T}$ at finite temperature compared to the situation in the absence of the coherent coupling (here $L$ is the system size and $\lambda_{T}=\hbar \sqrt{2 \pi / m k_{B} T}$ the thermal wavelength).

Focussing on the zero temperature limit, the fluctuations of the number of particles in the condensate within Bogoliubov

$$
\begin{equation*}
\operatorname{Var} \hat{N}_{0}=\operatorname{Var} \hat{N}^{\prime}=2 \Sigma_{q \neq 0}\left\langle\hat{n}_{q}\right\rangle\left(1+\left\langle\hat{n}_{q}\right\rangle\right)=2 \Sigma_{q \neq 0} u_{q}^{2} v_{q}^{2}=V /\left(8 \pi \sqrt{2} \xi^{3}\right), \tag{19}
\end{equation*}
$$

have been determined by Giorgini et al. [34]. Here, the prefactor two is a direct consequence of pairing in states $(q,-q)$. At first sight, the linear scaling with the volume is the expected behavior for the fluctuations of an extensive variable in thermodynamics. This argument is misleading, however, because at zero temperature the fluctuations of the number of particles enclosed in a volume $V$ are basically a surface effect, obeying an area law $\operatorname{Var} \hat{N} \simeq\left(V / \xi^{3}\right)^{2 / 3} \ln \left(V / \xi^{3}\right)$ which is modified by a logarithmic factor [34]. A behavior of this type is generic for a compressible system in contact with a reservoir, where exchange of particles occurs in an incoherent fashion. By contrast, in the presence of a coherent coupling between system and reservoir, the number fluctuations are enhanced by a factor $\sim V^{1 / 3}$ and are thus of an extensive nature even at zero temperature. The large enhancement of number fluctuations is therefore a consequence of the internal Josephson effect connecting the condensate and the depletion. It also shows up at finite temperature, where $\operatorname{Var} \hat{N}_{0}(T) \simeq\left(L / \lambda_{T}\right)^{4} \sim V^{4 / 3}$ is again a factor $L / \lambda_{T} \sim V^{1 / 3}$ larger than what is expected for a standard extensive variable in thermodynamics [34]. This result is in fact not confined to a Bogoliubov approximation but is a generic feature of BEC's with an arbitrary strength of the interaction [35]. More generally, anomalously large fluctuations of the order parameter appear for all phases with a broken continuous symmetry [36].
(c) With a proper renormalization of the parameters, in particular the replacement $\Delta_{0}=n_{0} g^{(0)} \rightarrow m c_{s}^{2}$ of the bare gap parameter by the square of the exact velocity of sound, Bogoliubov theory provides an asympotically exact description of the low-energy physics of Bose fluids with an arbitrary strength of the interactions.

To appreciate this point, it should be noted first that even for dilute BEC's the parameter $g^{(0)}=V(q=0)=4 \pi \hbar^{2} a^{(0)} / m$ contains the scattering length associated with the two-body interaction $V(\boldsymbol{x})$ only at the Born approximation level $a^{(0)}$. It is standard practice to replace this by the exact value $a$, using e.g. a pseudopotential Hamiltonian as introduced by Huang and Yang [37]. A more general approach which starts with a bare microscopic action and allows to properly account for the low energy constants associated with the two- and three-body and in principle even higher order interactions is provided by the method of effective potentials, as will be used in the context of the gas-liquid transition in Eq. 29p below. In this more modern formulation, the well known LHY-correction $E_{\text {Bog }}=N \cdot g n / 2\left(1+128 \sqrt{n a^{3}} / 15 \pi+\ldots\right)$ [38] to the mean-field ground state energy appears as the properly regularized one-loop contribution $(1 / 2) \sum_{q} E_{q}$ to the Coleman-Weinberg potential which arises from the zero point energy of the Bogoliubov excitations. Concerning the replacement $\Delta_{0} \rightarrow m c_{s}^{2}$ within the Bogoliubov formalism, it is straightforward to see that it accounts properly for the correct linear behavior $E_{q}=\left(\xi_{q}^{2}-\Delta_{q}^{2}\right)^{1 / 2} \rightarrow \hbar c_{s} q$ of the excitation spectrum at low energy as well as the singular nature of the ground state momentum distribution $n_{q}=\sinh ^{2} \theta_{q} \rightarrow\left(m c_{s} / 2 \hbar q\right)$ which - up to a renormalization factor $n_{0} / n$ - has been shown to be an exact result by Gavoret and Nozières [39].

As a final point in this context, we mention a fundamental issue associated with many-body wave functions in general. In fact, their detailed form becomes meaningless in practice for particle numbers beyond $N \simeq 10^{3}$, a problem which has been called the van Vleck catastrophy by Kohn [40]. To understand the origin of this problem, it is instructive to consider the overlap between two many-body wave functions for different interaction strengths specified e.g. by adjacent values $a$ and $a^{\prime}$ of the scattering length. Quite generally, the magnitude of this overlap appears only at second order in the deviation $\delta a=a^{\prime}-a$ but decreases exponentially with the number of particles. The sensitivity of a many-body wavefunction to a small change $\delta a$ in some parameter may thus be characterized by an intensive fidelity susceptibility $\chi_{F}$ which is defined by $\left|\left\langle\Psi(a) \mid \Psi\left(a^{\prime}\right)\right\rangle\right|=\exp \left(-\frac{1}{2} N \chi_{F}(\delta a)^{2}\right)$. By dimensional analysis, the fidelity susceptibility $\chi_{F}=1 / \ell_{F}^{2}$ defines a characteristic scale $\ell_{F}$ for the parameter $a$. As a result, knowledge of the many-body wave function with an accuracy close to one requires to know the microscopic parameter $a$ with an accuracy $|\delta a| \ll \ell_{F} / \sqrt{N}$ which is obviously impossible for large particle numbers. This is one way of expressing the exponential wall encountered in determining many-body wave functions, emphasized by Kohn [40]. The exactly known Bogoliubov wave function (16) serves as a concrete illustration of these ideas. Up to second order in $\delta a$, the overlap of two such states is given by

$$
\begin{equation*}
\left|\left\langle z_{\lambda},\left\{\lambda_{q}\right\} \mid z_{\lambda}^{\prime},\left\{\lambda_{q}^{\prime}\right\}\right\rangle\right|=\exp \left(-\frac{1}{2} N \chi_{F}(\delta a)^{2}\right) \text { with } \chi_{F}=\left(\partial_{a} \sqrt{f_{0}}\right)^{2}+\frac{1}{n} \int_{q} \frac{\left(\partial_{a} \lambda_{q}\right)^{2}}{\left(1-\left|\lambda_{q}\right|^{2}\right)^{2}} . \tag{20}
\end{equation*}
$$

The first term in the fidelity susceptibility arises from the overlap of the coherent states for the condensate. Using the leading order Bogoliubov result 12 for the depletion, it is given by $\chi_{F}^{(0)}=1 /\left(2 \pi^{2} \xi^{2}\right) \simeq n a$. The characteristic scale which determines how an uncertainty in the scattering length affects the accuracy of the many-body state thus appears to be the healing length $\xi$. Surprisingly, this conclusion is changed fundamentally by including the second contribution to $\chi_{F}$ in Eq. (20) which arises from the overlap of the product of two-mode squeezed states. Using that both in the regime $q \xi \ll 1$, where $\lambda_{q} \rightarrow-1+\sqrt{2} q \xi$ and for $q \xi \gg 1$, where $\lambda_{q} \rightarrow-1 /\left(2 q^{2} \xi^{2}\right)$, the derivative $\partial_{a} \lambda_{q}$ with respect to the scattering length can be easily determined, it turns out that in the relevant limit $\left(n a^{3}\right)^{1 / 2} \ll 1$, the fidelity susceptibility $\chi_{F} \simeq(n / a)^{1 / 2}$ is dominated by the second contribution, which diverges for vanishing scattering length. This divergence is a signature of a quantum phase transition from a gaseous to a liquid ground state of Bose fluids at $a=0$ which will be discussed in more detail below. More generally, as shown by Wang et al. [41], the fidelity susceptibility can be calculated efficiently via Quantum Monte Carlo methods in cases where no explicit results for the many-body wave function are available. In particular, it serves as an indicator of putative quantum phase transitions without an a priori knowledge of the order involved.


FIG. 4: A rotating Bose fluid in a ring geometry with non-perfect walls. The right Figure shows a schematic localized many-body wave function $\Phi_{\alpha}(x)$ on a ring with circumference $L$ as a function of one of the coordinates. The Figure is taken from Ref. [42].

Topological nature of many-body wave functions of superfluids As a consequence of the van Vleck catastrophy, the criterion for superfluidity cannot depend on the precise form of the many-body wave function but only on some long distance or topological properties. This point was first elucidated by Kohn [42] in the context of a quite general characterization of insulating ground states of interacting Fermi systems. Kohn's basic idea was to consider the many-body problem in a ring geometry and in the presence of a finite magnetic flux. For electrons with charge $-e$, this is a standard Aharanov-Bohm type setup which had been analyzed earlier by Byers and Yang [43] in their quite general proof of flux quantization in superconducting rings. In the case of neutral particles, an effective flux arises in a situation where the many-body system is enclosed between two concentric cylinders with nearly equal radii $R$, co-rotating with an angular frequency $\omega=\omega e_{z}$. As indicated in Fig. 4 , the walls are assumed to violate perfect cylindrical symmetry to allow for the transfer of angular momentum to the fluid. In the rotating frame, the problem is stationary, however the non-inertial frame gives rise to an effective gauge potential $\boldsymbol{A}(\boldsymbol{x})=m \boldsymbol{\omega} \wedge \boldsymbol{x}$ which appears in the kinetic energy part $\sum_{j}\left(\hat{\boldsymbol{p}}_{j}-\boldsymbol{A}\left(\boldsymbol{x}_{j}\right)\right)^{2} / 2 m$ of the Hamiltonian. Formally, the gauge potential can be eliminated by a gauge transformation $\psi^{(\theta)}\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)=\exp \left[-i(m R \omega / \hbar) \sum_{j} x_{j}\right] \psi\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)$ to a new many-body wave function $\psi^{(\theta)}$ which obeys the Schrödinger equation in the absence of $\boldsymbol{A}$. This function, however, is no longer single-valued. Instead, it changes by a phase factor if any one of the particles is taken around the ring according to

$$
\begin{equation*}
\psi^{(\theta)}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{i}+L, \ldots \boldsymbol{x}_{N}\right)=e^{-i \theta} \psi^{(\theta)}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{i}, \ldots \boldsymbol{x}_{N}\right) \quad \forall \quad i=1 \ldots N \tag{21}
\end{equation*}
$$

with $\theta=2 \pi m R^{2} \omega / \hbar$. Here, $L=2 \pi R$ is the circumference and $x_{i}+L$ means that the $i$-th coordinate is taken around the ring once, with transverse coordinates and possible other degrees of freedom like spin in the case of Fermions held fixed. The twist 21) in the boundary condition leads to a spectrum of eigenvalues $E_{\alpha}(\theta)$ which will in general depend on $\theta$, giving rise to a phase dependent equilibrium free energy $F(\theta)$ in the stationary, rotating system. As realized by Byers and Yang [43], $F(\theta)$ is an even and periodic function $F(\theta+2 \pi)=F(\theta)$, irrespective of the strength of the interactions provided these are time reversal invariant. It can therefore be expanded in a Fourier series

$$
\begin{equation*}
\Delta F(\theta)=F(\theta)-F(\theta=0)=\sum_{l=1}^{\infty} F_{l}[1-\cos (l \theta)] \rightarrow L_{z}^{\mathrm{rot}}(\theta)=-\frac{\partial F(\theta)}{\partial \omega} \underset{\omega \rightarrow 0}{\longrightarrow}-\left(\frac{L m R}{\hbar}\right)^{2} \sum_{l=1}^{\infty} l^{2} F_{l} \cdot \omega=-\left(n_{s} / n\right) L_{z}^{(0)} \tag{22}
\end{equation*}
$$

whose derivative with respect to $\omega$ determines the kinematic angular momentum $L_{z}^{\text {rot }}$ in the rotating frame. The superfluid fraction $n_{s} / n$ in this setup is now defined by expressing $L_{z}^{\text {rot }}=-\left(n_{s} / n\right) L_{z}^{(0)}$ in terms of the characteristic angular momentum $L_{z}^{(0)}=I_{\mathrm{cl}} \omega$ in a situation where a fluid is fully carried along by the walls at angular frequency $\omega$, with $I_{c l}=N m R^{2}$ the associated moment of inertia. Physically, a finite and negative angular momentum $L_{z}^{\text {rot }}=-\left(n_{s} / n\right) L_{z}^{(0)}$ in the rotating frame implies that a fraction $n_{s} / n$ of the superfluid stays at rest in the lab frame for small angular frequencies $\omega \ll \hbar / m R^{2}$. As a result, the apparent moment of inertia is smaller than that of classical rigid body rotation. The property of a non-classical rotational inertia (NCRI) has been introduced as a definition of superfluidity in a paper by Leggett [44] where he discussed the possibility of a finite $n_{s}$ even in a solid, an issue that will be investigated in more detail below. In the context of cold gases, the prediction that a superfluid does not rotate with its walls for small rotation frequencies has been demonstrated in experiments at the ENS [45, 46]: a trapped BEC in the presence of a small, non-symmetric perturbation remains at zero angular momentum below a finite critical rotation frequency. A direct signature for the existence of NCRI is provided by the so-called scissors mode in BEC's with anisotropic confinement $\omega_{x} \neq \omega_{y}$ in the plane perpendicular to the rotation. For superfluid flow, their effective moment of inertia $I_{\mathrm{SF}}=\delta^{2} I_{\mathrm{cl}}$ is smaller than the classical rigid body value $I_{\mathrm{cl}}=N m\left\langle X^{2}+Y^{2}\right\rangle$ by a factor $\delta^{2}<1$ which depends on the deformation parameter $\delta=\left\langle X^{2}-Y^{2}\right\rangle /\left\langle X^{2}+Y^{2}\right\rangle$. As predicted by Guéry-Odelin and Stringari [47], the fact that angular momentum in an anisotropic trap is not conserved gives rise to an oscillation of the gas after a sudden rotation of the trap around the new equilibrium position with frequency $\omega_{\text {scis }}=\left(\omega_{x}^{2}+\omega_{y}^{2}\right)^{1 / 2}$ which is absent in the normal phase, in perfect agreement with experiment [48].

Eq. (22) shows that a finite superfluid fraction requires the existence of a rigidity parameter $\gamma$ with dimension energy per length such that the second moment $\sum_{l} l^{2} F_{l} \simeq \gamma \cdot A_{\perp} / L$ of the Fourier amplitudes scales linearly with the cross section area $A_{\perp}$ and has a slow power law decay $\sim 1 / L$ with the circumference of the ring. In the limit $A_{\perp}, L \rightarrow \infty$, this gives rise to a superfluid density $n_{s}=\gamma m / \hbar^{2}$ which is independent of the sample dimension. To define the underlying rigidity in a more general form and, moreover, to describe states of a superfluid with finite currents, it is useful to introduce a slowly varying local phase $\varphi(\boldsymbol{x})$ on scales much larger than the interparticle spacing which is connected with the total phase difference between two arbitrary points by $\theta=\int d s \nabla \varphi(x)$. The free energy increase due to a finite value of $\nabla \varphi(x)$ can then be expressed in a local form

$$
\begin{equation*}
\Delta F[\varphi(\boldsymbol{x})]=\frac{\gamma}{2} \int_{x}(\nabla \varphi(\boldsymbol{x}))^{2} \quad \text { with } \quad \gamma=\frac{\hbar^{2} n_{s}}{m}=\left.\frac{L^{2}}{V} \frac{\partial^{2} \Delta F(\theta)}{\partial \theta^{2}}\right|_{\theta=0} \quad \overrightarrow{\text { ring }} \frac{L}{A_{\perp}} \sum_{l=1}^{\infty} l^{2} F_{l} \tag{23}
\end{equation*}
$$

which, however, hides the periodic dependence on $\theta$ stated in Eq. $22{ }^{5}$. Physically, a non-vanishing phase gradient corresponds to a finite superfluid velocity $\boldsymbol{v}_{s}=(\hbar / m) \nabla \varphi(\boldsymbol{x})$. The rigidity energy is thus just the kinetic energy of superfluid flow which may be present even in an equilibrium configuration (see the Appendix for a more detailed discussion). In the particular case of a uniform twist $\nabla \varphi(\boldsymbol{x})=\boldsymbol{Q}$, Eq. 23) shows that the total number $N_{s}$ of particles in the superfluid is defined by the increase $N_{s} \cdot \hbar^{2} \boldsymbol{Q}^{2} / 2 m$ in free energy if the whole fluid acquires a finite momentum $\boldsymbol{Q}$, as was used above in the context of the internal Josephson effect in the Bogoliubov approach. It is important to note that the definition (23) for superfluidity is based only on equilibrium properties and it also applies to finite systems. Obviously, however, it is quite different from the definition of BEC via the concept of ODLRO, as stated in Eq. (1). Yet, it turns out, that the two phenomena are intimately connected. In fact, superfluidity in the sense defined in Eq. 23) is the more general phenomenon. On a qualitative level, the connection between a finite value of the superfluid stiffness $\gamma$ and the presence of ODLRO may be understood by using the representation $\hat{\psi}(\boldsymbol{x}) \simeq \sqrt{\tilde{n}_{0}} \exp i \hat{\varphi}(\boldsymbol{x})$ of the Bose field operator in terms of a finite bare condensate density $\tilde{n}_{0}$ and the phase operator $\hat{\varphi}(\boldsymbol{x})$. The asymptotic decay of $\rho^{(1)}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\tilde{n}_{0} \exp \left[-\delta \varphi^{2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) / 2\right]$ is then determined by the mean square fluctuations $\delta \varphi^{2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\left\langle\left(\hat{\varphi}(\boldsymbol{x})-\hat{\varphi}\left(\boldsymbol{x}^{\prime}\right)\right)^{2}\right\rangle$ of the phase difference between points separated by $\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|$. Using the effective Hamiltonian (23) together with the assumption of a finite compressibility it is possible to show (see e.g. the Appendix in Ref. [49]) that the phase fluctuations remain finite in the limit of infinite separation in three dimensions. As a result, $\gamma \neq 0$ implies ODLRO with a condensate density $n_{0}=\tilde{n}_{0} \exp \left[-\delta \varphi^{2}(\infty) / 2\right]$. In two dimensions, this result only holds at $T=0$, while $\delta \varphi^{2}\left(x, x^{\prime}\right) \rightarrow 2 \eta \ln \left|x-x^{\prime}\right|$ diverges logarithmically at finite temperatures below the BKT-transition, where $\eta\left(T_{\mathrm{BKT}}\right)=1 / 4$. This leads to an algebraic decay $\rho^{(1)}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) \sim\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|^{-\eta}$, consistent with the Mermin-Wagner-Hohenberg theorem, which states that no long range order is possible in two dimensions if $T \neq 0$ in the case of a continuous symmetry. A similar behavior, due to quantum rather than thermal phase fluctuations, applies in one dimension at zero temperature.

In the following, it will be shown that the definition of superfluidity based on Eqs. 22, and 23) allows to characterize superfluids in terms of a topological property of the many-body wave function which implies, in particular, that ground states of Bosons are always superfluid provided they have a uniform density. The argument relies on the geometry introduced above, where the many-particle configuration space is an $N$-torus $\mathbb{T}^{N}=S^{1} \otimes \cdots \otimes S^{1}$ with respect to motion around the ring. The dependence of the energy levels $E_{\alpha}(\theta)$ and the associated free energy $F(\theta)$ of the many-body system in the rotating frame or of the charged system in the presence of a finite magnetic flux is determined by the change in energy induced by the twist in Eq. 21) associated with closed paths in configuration space. To single out the dependence on the variable $\theta$, it is useful to consider the representation of the partition function of the many-body system in terms of a Feynman propagator over closed paths $\left\{\boldsymbol{x}_{j}\right\} \rightarrow\left\{\boldsymbol{x}_{j}\right\}$ in imaginary time $\beta \hbar$. Since the configuration space is multiply connected, this propagator is a sum over the different elements of the first homotopy group $\pi_{1}\left(\mathbb{T}^{N}\right)=\mathbb{Z}^{N}$ of the $N$-torus which are labelled by the set of $N$ integer winding numbers $m_{j} \in \mathbb{Z}$. Physically they correspond to taking any of the $j=1 \ldots N$ particles around the ring $m_{j}$ times. As shown by Pollock and Ceperley [50], the change in free energy due to the twist in the boundary condition is determined by the characteristic function

$$
\begin{equation*}
\exp (-\beta \Delta F(\theta))=\sum_{\left\{m_{j} \in \mathbb{Z}\right\}} e^{-i M \theta} p\left(m_{1} \ldots m_{N} ; \beta\right) \text { with } M=\sum_{j} m_{j} \tag{24}
\end{equation*}
$$

of the winding number probability distribution $p\left(m_{1} \ldots m_{N} ; \beta\right)$ in the absence of the twist. Considering in particular the limit where the temperature approaches zero, the question of whether the ground state energy in the rotating frame exhibits a non-trivial dependence on the twist $\theta$ is determined by the connectedness properties of the ground state wave function. In the ground breaking papers on this subject by Kohn [42] and Leggett [44, 51], two limiting cases were considered:

[^5]a) The wave function of the ground state is disconnected in the sense that on all closed paths with $M \neq 0$, there is at least one region where the wave function is exponentially small. In the presence of rotation, the modified boundary condition (21) can then be accomodated by adding the phase shift in precisely these regions. The resulting change in energy $\sim \exp \left(-L / \xi_{\text {loc }}\right)$ vanishes exponentially and thus the free energy $F(\theta) \simeq F(0)$ in the rotating frame becomes independent of the twist as $L \gg \xi_{\text {loc }}$. This is the characterisation given by Kohn for insulators. Specifically, Kohn discussed electrons in a regular lattice with a set $\boldsymbol{R}_{v}$ of sites whose number is commensurate with those of the electrons. As indicated schematically in Fig. 4 , they were described by exponentially localized Wannier functions $w(\boldsymbol{x}-\boldsymbol{R})$ at the single-particle level, leading to a disconnected many-body state.
b) The wave function $\psi_{0}\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)$ of the ground state is connected in the sense that there exist closed paths with non-vanishing total winding number $M \neq 0$ on which the magnitude $\left|\psi_{0}\right|$ is everywhere bounded below by a finite constant independent of both $N$ and $L$. In this case, the system is a superfluid with a reduced moment of inertia because the twist leads to an energy increase $\sim \int\left|\psi_{0}\right|_{\text {min }}^{2}(\nabla \varphi)^{2}$ of order $A_{\perp} / L$. For fluid ground states, the existence of closed paths of this type may be viewed as a consequence of the positivity of the many-body ground state wave function. As pointed out by Leggett [51], ground states of Bosons with a uniform density are therefore always superfluid ${ }^{6}$. For non-uniform ground states like in a crystal, the positivity requirement, however, is not sufficient to infer the existence of a finite superfluid density because the minimum magnitude $\left|\psi_{0}\right|_{\min } \sim \exp \left(-L / \xi_{\text {loc }}\right)$ could be exponentially small as in insulators. In this situation, there is only an upper bound on the superfluid fraction given in Eq. 38) below which will be discussed in detail in Lecture II.

An important point to note in this context is that the magnitude $|M|$ of the relevant total winding numbers are of order one or two and not of order $N$ because the relevant Fourier components $F_{l}$ in Eq. 22 are $l=1$ or $l=2$ for standard Bose superfluids or superfluids of Fermion pairs, respectively. In physical terms, this requires that there are paths in the configuration space where the many-body wave function stays finite upon taking one or maybe two particles around the ring while the coordinates of the remaining $N-1$ particles are held fixed. Obviously, this is the case in the presence of ODLRO as defined in Eq. (11), which thus turns out to be a sufficient criterion for superfluidity. It is not a necessary one, however, and indeed as stated above, superfluidity is the more general phenomenon rather than BEC and the equivalent existence of ODLRO.

In the case of charged systems, the dependence of the eigenvalues in the presence of a non-trivial boundary condition (21) leads to a characterization of insulators or superconductors in terms of the so-called Drude weight [42]

$$
D_{s}=\pi \lim _{\omega \rightarrow 0} \omega \operatorname{Im} \sigma(\omega)=\pi \frac{n_{s} e^{2}}{m}=\left.\frac{e^{2}}{\hbar^{2}} \frac{\pi L^{2}}{V} \frac{\partial^{2} \Delta F(\theta)}{\partial \theta^{2}}\right|_{\theta=0} \underset{L \rightarrow \infty}{ }\left\{\begin{array}{c}
\sim \exp \left(-L / \xi_{\text {loc }}\right)  \tag{25}\\
D_{s} \neq 0
\end{array} \text { insulator }\right. \text { superconductor }
$$

For superconductors, this implies a $1 / \omega$-singularity of strength $D_{s} / \pi$ in the imaginary part of the frequency dependent conductivity which is precisely the content of the phenomenological first London equation (175). In the case of insulators, such a contribution is absent and the odd function $\operatorname{Im} \sigma(\omega)$ therefore vanishes linearly at low frequencies. However, this is also true in metals with a finite amount of disorder. The relevant distinction between metals and insulators shows up in the behavior of $\operatorname{Re} \sigma(\omega)$ as $\omega \rightarrow 0$ : for any non-perfect metal, the real part of the conductivity has a finite value while $\operatorname{Re} \sigma(\omega) \sim \omega^{2} \ln ^{d+1}(\bar{\omega} / \omega)$ vanishes essentially quadratically in insulators. A discussion of how the empirical description of the different ground states in terms of the complex conductivity $\sigma(\omega)$ is reflected at the level of the Drude weight has been given by Scalapino et al. [52]. According to Eq. (23), the Drude weight at $T=0$ is obtained from the curvature of the many-body ground state. This requires to follow the ground state adiabatically as a function of the twist $\theta$. Now, it turns out that the characteristic magnitude $\theta_{c}$ of the twist at which another many-body level crosses or drops below the ground state varies like $\theta_{c} \sim 1 / L^{d-1}$. In dimension $d>1$, therefore, the order of limits $\theta \rightarrow 0$ and $L \rightarrow \infty$ matters: taking the second derivative of $E_{0}(\theta)$ with respect to $\theta$ first, and then sending $L \rightarrow \infty$ gives a Drude weight $D$. It differs from the $D_{s}$ defined above, which involves the curvature of the envelope of the $E_{\alpha}(\theta)$ curves of individual many-body states $\psi_{\alpha}$. Both $D$ and $D_{s}$ approach zero for an insulator and they are both finite in a superconductor. In the case of a metal with no disorder, however, $D$ is finite while $D_{s}=0$ [52]. A different way to see that there is no topological characterization of metallic or normal fluid states is revealed by the fact that the second moment $\left.\sum_{l} l^{2} F_{l \mid}\right|_{\text {normal }} \simeq\left(\hbar^{2} / m L\right) n \xi_{t}^{2}$ of the Fourier amplitudes in Eq. 22 still scales with $1 / L$. The linear increase with the transverse area $A_{\perp}$ in the superfluid phase, however, is replaced by the square of a characteristic length $\xi_{t}$ which appears in the momentum dependence $\chi_{t}(q)=\rho\left[1-\left(q \xi_{t}\right)^{2} \ldots\right]$ of the transverse current response, associated with diamagnetism in the charged case. The periodic dependence of $L_{z}^{\text {rot }}$ on $\theta$ is still present and it describes the persistent currents in a normal metal ring predicted by Büttiker, Imry and Landauer [53]. The observed magnitude of the associated Fourier coefficients $F_{l}$ agrees well with a model of non-interacting electrons [54], however the role of interactions in this context has remained controversial.

[^6]

FIG. 5: The Figure on the left shows the phase diagram of particles with a Lennard-Jones interaction in the classical limit of a vanishing de Boer parameter $\Lambda_{\mathrm{dB}}=0$ as determined by Travesset [55]. The dimensionless pressure and temperature are defined by $\hat{P}=p \sigma^{3} / \epsilon$ and $\hat{T}=k_{B} T / \epsilon$. The Figure on the right shows the dependence of the dimensionless temperature $\hat{T} \rightarrow t^{*}$ of the critical and the triple point as a function of the square $\eta=\Lambda_{\mathrm{dB}}^{2}$ of the de Boer parameter. It is taken from Lectures given by P. Nozières at the Collège de France in 1983.

Quantum-unbinding at a zero temperature liquid-gas transition Following recent work [56, 57], we will discuss the liquid-to-gas quantum unbinding transition in Bose fluids induced by an increasing strength of the zero point fluctuations. The existence of such a transition is indicated in Fig. 5. where the dimensionless temperature of both the triple and the critical point are shown as a function of the square of the de Boer parameter. The transition from a solid to a liquid ground state occurs when the triple point vanishes. It is first order and the associated critical de Boer parameter $\Lambda_{\mathrm{dB}}^{\mathrm{c} \text {,solid }} \simeq 0.37$ for Bosons can only be determined numerically by a genuine many-body calculation [5] ${ }^{7}$. Remarkably, the transition from a liquid to a gaseous ground state at $\Lambda_{\mathrm{dB}}^{c} \simeq 0.69$ [8], where also the critical point for a system of Bosons disappears, is continuous. Moreover, its location is fixed by a vanishing scattering length, i.e. by two-body physics. Indeed, as noted by Lieb [58], a necessary condition for a gaseous ground state is that the two-body interaction $V(x)$ in Eq. (2) has no bound state and a positive scattering length. In the following, we will argue that for interactions considered here in connection with Eq. (3), this condition is also sufficient. Moreover, the liquid and gaseous ground states are separated by a quantum tricritical point. Specifically, we follow an approach due to Sachdev [59] and consider the transition out of the vacuum state into one with a finite particle density $n$ as a function of the chemical potential $\mu$. In the case where the ground state is a gas, the associated effective field theory is the well known $\psi^{4}$-theory for a complex scalar field. In a formal manner, this can be derived by starting from the microscopic action of a Bose system with pure two-body interactions as described by Eq. (2). The associated generating functional $Z[J]=\int D \psi \exp \left(-S[\psi] / \hbar+\int J \psi\right)$ for the correlation functions of the complex scalar field $\psi(\tau, x)$ can be written as a functional integral with action

$$
\begin{equation*}
S[\psi]=\int_{\tau} \int_{\boldsymbol{x}}\left\{\psi^{*}(\tau, \boldsymbol{x})\left(\hbar \partial_{\tau}-\frac{\hbar^{2}}{2 m} \nabla^{2}-\mu\right) \psi(\tau, \boldsymbol{x})+\frac{1}{2}|\psi(\tau, \boldsymbol{x})|^{2} \int_{\boldsymbol{x}^{\prime}} V\left(\boldsymbol{x}-\boldsymbol{x}^{\prime}\right)\left|\psi\left(\tau, \boldsymbol{x}^{\prime}\right)\right|^{2}\right\} . \tag{26}
\end{equation*}
$$

At the mean-field level, the effective potential for field configurations with no dependence on the time and spatial variables $\tau$ and $\boldsymbol{x}$, where $|\psi|^{2}=n$ can be identified with the particle density, has the form $V_{\text {eff }}^{(0)}=-\mu n+(g / 2) n^{2}$. The coefficient $g=4 \pi \hbar^{2} a / m>0$ is fixed by the two-body scattering length in vacuum. More precisely, as mentioned above in the context of point c) on the exactness of the Bogoliubov approach at low energies, in the naive mean-field approach $g \rightarrow g^{(0)}$ contains the scattering length only in the Born approximation, which is ill-defined for potentials which increase more strongly than $1 / r^{3}$ at short distances. This problem is eliminated in the formulation based on an effective potential in Eq. 29) below. Provided that $g>0$, the onset transition from the vacuum to a superfluid gas to lowest order in the density is properly accounted for in terms of a mean-field description. In particular, the density of bosons $n(\mu)=\mu / g+\ldots$ rises linearly for $\mu \rightarrow 0^{+}$while $n(\mu) \equiv 0$ vanishes for negative values of the chemical potential. Thus, $\mu=0, g>0$ is a line of quantum critical points which separates the vacuum state from a superfluid gas at finite density [59]. Despite the finite jump in the compressibility from $\tilde{\kappa}=\partial n / \partial \mu=0$ to $\tilde{\kappa}=1 / g>0$, the vacuum to superfluid transition is a continuous one. Indeed, approaching the line $\mu=0$ from above, the correlation length is the well known healing length $\xi=\hbar / \sqrt{2 m \mu}=(8 \pi n a)^{-1 / 2}$ of a weakly interacting BEC which diverges as $\mu \rightarrow 0^{+}$. Moreover, using the zero temperature Gibbs-Duhem relation $\mu=u+p / n$ which connects the chemical potential and the pressure to the energy $u$ per particle, both $u(n) \rightarrow g n / 2=\sqrt{g p / 2}$ and the density $n(p) \rightarrow \sqrt{2 p / g}$ vanish in the zero pressure limit, as required for a gas.

[^7]

FIG. 6: Qualitative dependence of the scattering length in units of the van der Waals length $\ell_{\mathrm{vdW}}$ as a function of the de Boer parameter defined in Eq. (4). The last two-body bound state disappears beyond the pole of the scattering length at $\Lambda_{\mathrm{dB}}^{*}(N=2)$ indicated by the dashed vertical line. The scattering length reaches zero at a critical value $\Lambda_{\mathrm{dB}}^{c} \simeq 0.68$, beyond which it stays positive. The value $\Lambda_{\mathrm{dB}}^{*}(N=3)$ for the disappearance of three-body bound states is also indicated.

The range of de Boer parameters where a given microscopic interaction gives rise to a positive scattering length and thus a gaseous ground state is determined by the solution of the two-body problem. In the regime $\Lambda_{\mathrm{dB}} \ll 1$, there is a large number $N_{b} \simeq 1 /\left(\pi \Lambda_{\mathrm{dB}}\right) \gg 1$ of s-wave bound states. Upon reduction of the strength of the attractive interaction, their number decreases and eventually reaches zero at a critical value of the de Boer parameter. In physical terms, this happens when the van der Waals length $\ell_{\mathrm{vdW}}=\left(m C_{6} / \hbar^{2}\right)^{1 / 4} / 2$ has decreased to a value of the order of the short distance scale $\sigma$. For the specific case of a Lennard-Jones potential, the limit beyond which the two-body Hamiltonian $\hat{H}_{2}$ no longer has a bound state is reached at $\Lambda_{\mathrm{dB}}^{*}(N=2)=0.423 \ldots$ or $\ell_{\mathrm{vdW}}=1.09 \sigma$. At this point, the scattering length jumps form $+\infty$ to $-\infty$, as sketched in Fig. 6 . In fact, this is close to the situation present in ${ }^{4} \mathrm{He}$, where $\Lambda_{\mathrm{dB}} \simeq 0.42$ and the attractive part of the two-body interaction is just barely sufficient to give rise to a bound state with a binding energy $B_{2} \simeq k_{B} \cdot 1.7 \mathrm{mK}$. Upon further increasing the de Boer parameter, the scattering length increases monotonically from $-\infty$ towards zero, which is reached at some critical value $\Lambda_{\mathrm{dB}}^{c}$. Specifically, one finds $\Lambda_{\mathrm{dB}}^{c}=0.679 \ldots$ for a Lennard-Jones potential, corresponding to a van der Waals length $\left.\ell_{\mathrm{vdW}}\right|_{c} \simeq 0.86 \sigma$. Increasing $\Lambda_{\mathrm{dB}}$ beyond its critical value, the scattering length stays positive. In particular, near $\Lambda_{\mathrm{dB}}^{c}$, the scattering length

$$
\begin{equation*}
a\left(\Lambda_{\mathrm{dB}}\right)=a_{\Lambda} \ell_{\mathrm{vdW}}\left(\Lambda_{\mathrm{dB}}-\Lambda_{\mathrm{dB}}^{c}\right)+\ldots \tag{27}
\end{equation*}
$$

vanishes linearly with a positive numerical constant $a_{\Lambda}$ of order one. The regime $g>0$ of a gaseous ground state is realized for $\Lambda_{\mathrm{dB}}>\Lambda_{\mathrm{dB}}^{c}$. As mentioned above, the same situation applies for ultracold gases despite $\Lambda_{\mathrm{dB}} \ll 1$ provided the scattering length is positive and the many two-body bound states are inaccessible on relevant time scales.

For negative scattering lengths, the ground state of a uniform Bose fluid is obviously not a gas. As will be shown below, there is a finite range of the Boer parameters below $\Lambda_{\mathrm{dB}}^{c}$, where the ground state is a liquid which is stabilized by repulsive three-body interactions. Its properties near the first-order transition to the vacuum state are determined by a solution of the three-body problem. Now, as predicted by Efimov [60] in a nuclear physics context, identical bosons support three-body bound states in a regime where the scattering length is negative and no two-body bound state exists. As indicated in Fig. 6, where the critical value $\Lambda_{\mathrm{dB}}^{*}(N=3) \simeq 0.45$ for the disappearance of the last three-body bound state is shown, this requires a minimum value of the magnitude $\left|a_{-}(3)\right|$ of the associated two-body scattering length which is $a_{-}(3)=-9.6 \ell_{\mathrm{vdW}}$ for a Lennard-Jones interaction 61]. The three-body bound states predicted by Efimov were first observed in an ultracold gas of ${ }^{133} \mathrm{Cs}$ by Kraemer et al. [62]. Surprisingly, the ratio $\left|a_{-}(3)\right| / \ell_{\mathrm{vdW}} \simeq 8-10$ turned out to vary in an only narrow range for many different atoms [63]. An explanation for this so called van der Waals universality has been given independently by Wang et al. [64] and by Schmidt et al. [65]. Wang et al. consider direct two-body interactions with different single channel potentials at short distance but identical van der Waals tails. The solution of the associated three-body problem then shows that the ratio $\left.\left(a_{-}(3) / \ell_{\mathrm{vdw}}\right)\right|_{N_{b} \gg 1}=-9.45$ approaches a universal value in the limit of a large number $N_{b} \gg 1$ of bound states [64]. In practice, a change in the scattering length relies on the use of Feshbach resonances. As shown by Schmidt et al. [65] within a standard two-channel model, a nearly universal value of the ratio $a_{-}(3) / \ell_{\mathrm{vdW}} \simeq-9$ then appears only in the open-channel dominated limit $s_{\text {res }} \gg 1$ [66]. Moreover, considerable deviations towards more negative numbers were predicted for Feshbach resonances with intermediate strength $s_{\mathrm{res}} \simeq 1$. They have recently been observed in ${ }^{39} \mathrm{~K}$ by the JILA group, see Chapurin et al. [67] and Xie et al. [68].

For the many-body problem at finite density, the endpoint at $g=0$ of the line $\mu \equiv 0$ turns out to be a quantum tricritical point (see Fig. 77). It separates the continuous onset transition from the vacuum to a gaseous state in the regime $g>0$ from a first-order transition at $\mu_{c}<0$ between the vacuum and a finite density liquid for negative values of the scattering length. In order to properly deal with the regime $g<0$, it is necessary to include the quantum fluctuations of the field $\psi(\tau, \boldsymbol{x})$ to all orders. On a formal level, this can be expressed in terms of an effective potential

$$
\begin{equation*}
\Gamma[\psi]=\sum_{N=1}^{\infty} \frac{1}{N!} \int_{p_{1} \ldots q_{N}} \Gamma_{N}\left(p_{1} \ldots p_{N} q_{1} \ldots q_{N}\right) \psi^{*}\left(p_{1}\right) \ldots \psi^{*}\left(p_{N}\right) \psi\left(q_{1}\right) \ldots \psi\left(q_{N}\right)=\int_{\tau, \boldsymbol{x}}\left\{V_{\mathrm{eff}}[\psi]+\psi^{*} \tilde{D} \psi+\ldots\right\} \tag{28}
\end{equation*}
$$

which is defined via a Legendre transform $\Gamma[\psi]=\ln \{Z[J] / Z[0]\}-\int J \psi$ of the generating functional $Z[J]$ associated with the action (26) ${ }^{8}$. In practice, the Legendre transform can only be performed if one is able to determine the expectation value of the field for an arbitrary form of external source $J(\tau, \boldsymbol{x})$ and then invert this relation to determine $J(\tau, \boldsymbol{x})$ as a functional of the associated configuration $\psi(\tau, \boldsymbol{x})$. The resulting exact vertex functions $\Gamma_{N}$ are essentially the amplitudes for scattering processes with $N$ incoming and $N$ outgoing particles. Knowledge of the $\Gamma_{N}$, including their dependence on the $2 N$ momentum variables $p_{1} \ldots q_{N}$ which are constrained only by translation invariance in space and time $p_{1}+\ldots+p_{N}=q_{1}+\cdots+q_{N}$, therefore requires a complete solution of the $N$-body problem. This is clearly impossible. Fortunately, however, for the discussion of the behavior near the quantum tricritical point, which is a zero density fixed point, we need only the leading non-vanishing terms in the expansion of the effective potential

$$
\begin{equation*}
V_{\mathrm{eff}}[\psi]=-\mu|\psi|^{2}+\frac{g}{2}|\psi|^{4}+\frac{\lambda_{3}}{3}|\psi|^{6}+\ldots \tag{29}
\end{equation*}
$$

associated with a time and space independent 'classical' field $\psi$. Here, as mentioned above, the prefactor $g=\Gamma_{2}(0)=4 \pi \hbar^{2} a / m$ of the quartic term is fixed by the exact value $a$ of the two-body scattering length which may be defined through the asymptotic behavior $\psi_{E=0}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}\right)=1-a / r_{12}$ of the two-body wave function at zero energy. If $g$ is positive, the transition out of the vacuum state is completely fixed by the first two terms in Eq. 29], recovering the scenario for a gaseous ground state discussed above. For negative $g$, in turn, one needs the next-to-leading contribution $\sim|\psi|^{6}$. Its prefactor $\lambda_{3}=\hbar^{2} D / 2 m$ arises from the zero momentum limit $\Gamma_{3}(0)=\hbar^{2} D / m$ of the vertex function which is associated with effective three-body interactions. The corresponding parameter $D$ has been called the three-body scattering hypervolume by Tan [70]. It has dimension (length) ${ }^{4}$ and may be defined by the asymptotic behavior

$$
\begin{equation*}
\left.\psi_{E=0}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \boldsymbol{x}_{3}\right)\right|_{a=0}=1-\frac{\sqrt{3} D}{2 \pi^{3}\left(r_{12}^{2}+r_{13}^{2}+r_{23}^{2}\right)^{2}}+\ldots \tag{30}
\end{equation*}
$$

of the three-body wave function at zero energy and vanishing scattering length [70]. Similar to the standard connection between two-body bound states and poles of the scattering length, the occurence of three particle bound states is determined by poles of the hypervolume $D$. Now, as indicated in Fig. 6, the last three-body bound state disappears at a finite negative scattering length $a_{-}(3) \simeq-9 \ell_{\mathrm{vdW}}$. Near $\Lambda_{\mathrm{dB}}^{c}$, therefore, the vertex function $\Gamma_{3}$ has no poles. Moreover, the associated hypervolume $D(a=0)>0$ is positive near the zero of the scattering length at $\Lambda_{\mathrm{dB}}^{c}$ according to a numerical solution of the three-body problem with a LennardJones interaction [71]. This implies a repulsive effective three-body force and an energy per particle $\left.u(n)\right|_{a=0}=\left(\hbar^{2} D / 6 m\right) \cdot n^{2}$ which scales quadratically with density $n[70]$. At vanishing scattering length, therefore, the many-body Bose fluid is stabilized by repulsive three-body interactions, a behavior quite different from that of the naively expected ideal Bose gas. In particular, the finite density fluid at $a=0$ is characterized by a non-trivial relation between pressure and chemical potential of the form

$$
\begin{equation*}
\left.p(\mu)\right|_{a=0}=\left.\left(\frac{8 m}{9 \hbar^{2} D}\right)^{1 / 2} \cdot \mu^{3 / 2} \quad \rightarrow \quad \mu(n)\right|_{a=0}=\frac{\hbar^{2} D}{2 m} \cdot n^{2} . \tag{31}
\end{equation*}
$$

As a result, the density $n(\mu)=\partial p / \partial \mu$ scales with the square root of the chemical potential rather than the linear behavior found for positive scattering lengths. This is a consequence of the non-standard critical exponent $\beta=1 / 4$ associated with the appearance of a finite order parameter $|\psi|(\mu) \sim \mu^{\beta}$ right at the quantum tricritical point which separates the gaseous from the liquid ground state in the zero density limit.

[^8]

FIG. 7: Zero temperature phase diagram as a function of the chemical potential $\mu$ and the deviation $g \sim \Lambda_{\mathrm{dB}}-\Lambda_{\mathrm{dB}}^{c}$ of the de Boer parameter from its critical value. The gaseous ground state in the regime $g>0$ arises from the vacuum at $\mu<0$ via a continuous transition. For $g<0$, the ground state is a liquid. It is separated from the vacuum by a first-order transition at $\mu_{c}<0$. The point $\mu=g=0$ is a quantum tricritical point. The finite temperature phase diagram for $\Lambda_{\mathrm{dB}}<\Lambda_{\mathrm{dB}}^{c}$ on the right is adapted from Son et al. [57]. Beyond a tricritical point at $T^{*} \simeq \hbar^{2} \bar{n}^{2 / 3} / \mathrm{m}$, the transition from a superfluid liquid to a non-superfluid gas changes from first order to a continuous one.

In the regime $g<0$, the symmetry broken phase with a finite density $n(\mu)=|\bar{\psi}|^{2} \neq 0$ appears already beyond a negative value

$$
\begin{equation*}
\mu_{c}=-3 g^{2} /\left(16 \lambda_{3}\right)=-6 \pi^{2} \hbar^{2} a^{2} /(m D) \tag{32}
\end{equation*}
$$

of the chemical potential, which vanishes with the square of the distance from the quantum tricritical point as indicated in Fig. 7 . By the Gibbs-Duhem relation, the critical chemical potential $\mu_{c}=u(p=0)$ coincides with the energy per particle since the pressure vanishes along the line separating the vacuum from the finite density liquid. Right on the line $\mu=\mu_{c}$, the density jumps from zero in the vacuum state $\mu<\mu_{c}$ to a finite value

$$
\begin{equation*}
\bar{n}=n\left(\mu_{c}\right)=3|g| /\left(4 \lambda_{3}\right)=6 \pi|a| / D \quad \rightarrow \quad \bar{n} \sigma^{3}=6 \pi|a| \sigma^{3} / D \underset{\mathrm{LJ}}{\longrightarrow} 1.32\left(\Lambda_{\mathrm{dB}}^{c}-\Lambda_{\mathrm{dB}}\right)+\ldots \tag{33}
\end{equation*}
$$

The dimensionless product $\bar{n} \sigma^{3}$ therefore approaches zero linearly with the deviation from the quantum tricritical point. The numerical prefactor in the final expression is specific for a Lennard-Jones interaction, where the factor $a_{\Lambda}=3.828$ in Eq (27) and the three-body hypervolume $D(a=0)=(86 \pm 2) \ell_{\mathrm{vdW}}^{4}$ near the last zero crossing of the scattering length have been determined by Mestrom et al. [71]. Despite the considerable deviation $\Lambda_{\mathrm{dB}}^{c}-\Lambda_{\mathrm{dB}} \simeq 0.26$ of the de Boer parameter of ${ }^{4} \mathrm{He}$ from the critical value for a liquid-gas transition, a naive application of Eq 33 predicts a dimensionless density $\bar{n} \sigma^{3} \simeq 0.34$ for ${ }^{4} \mathrm{He}$ at zero pressure which is close to the observed value. This agreement is again a fortuitous coincidence, however, because two ${ }^{4} \mathrm{He}$ atoms form a weakly bound dimer and thus the relation 27) does not apply. A system rather close to the quantum tricritical point, still on the liquid side, would be ${ }^{2} \mathrm{He}$. Its de Boer parameter is expected to be $\Lambda_{\mathrm{dB}} \simeq \sqrt{2} \cdot 0.42=0.59$ due to the factor two in mass. Unfortunately, this extremely dilute superfluid liquid does not exist in nature because the di-proton is not bound ${ }^{9}$. The evolution of the finite temperature phase diagram in the regime of de Boer parameters between $\Lambda_{\mathrm{db}} \simeq 0.37$ and $\Lambda_{\mathrm{dB}}^{c}$, where the ground state at vanishing pressure is a liquid, has been discussed by Son et al [57]. Surprisingly, this diagram is of the familiar form observed in ${ }^{4} \mathrm{He}$ (see Fig. 2] only in a finite range of $\Lambda_{\mathrm{dB}}$ above 0.37 . For values that correspond to the hypothetical ${ }^{2} \mathrm{He}$ fluid and up to $\Lambda_{\mathrm{dB}}^{c}$, in turn, the critical endpoint of the $\lambda$-line on the liquid-gas boundary has disappeared. Instead, as shown in Fig. 7, there is a tricritical point along the coexistence line between a superfluid liquid and the normal gas above which the transition changes from being first order to a continuous one. Its temperature $T^{*} \simeq \hbar^{2} \bar{n}^{2 / 3} / m$ is set by the finite density $\bar{n}$ of the liquid ground state at zero pressure given in $\mathrm{Eq} \sqrt{33}$ which also determines the jump in density below $T^{*}$ by the simple relation $\Delta n=\left[1-\left(T / T^{*}\right)^{3 / 2}\right] \bar{n}$ [57]. Since $\bar{n} \rightarrow 0$ in the limit of vanishing scattering length, the tricritical point shifts to zero temperature and then coincides with the quantum tricritical point $\mu=g=0$ shown in Fig. 7 on the left.

Regarding a possible realization of a liquid state in ultracold Bose gases near vanishing scattering length which is stabilized by repulsive three-body interactions, it is necessary to account for the finite imginary part of the three-body scattering hypervolume that is present at generic zero crossings of $a$ in the standard regime where the de Boer parameter $\Lambda_{\mathrm{dB}}$ is much less than one. As will be discussed in Lecture III, this leads to a corresponding loss rate $\Gamma_{3}=-\hbar \operatorname{Im}(D) n^{2} / m$ [72]. Experimentally, these losses

[^9]have been studied by Shotan et al. [73], who measured the recombination length $L_{m}$ defined by $\operatorname{Im} D \simeq L_{m}^{4}$ near a zero crossing of the scattering length at $B \simeq 850 \mathrm{G}$ in ${ }^{7} \mathrm{Li}$. Remarkably, the observed value $L_{m} \simeq 4 \ell_{\mathrm{vdW}}$ is close to that quoted above for the fourth root $D^{1 / 4} \simeq 3.1 \ell_{\mathrm{vdW}}$ of the purely real three-body scattering hypervolume near the zero crossing of the scattering length at $\Lambda_{\mathrm{dB}}^{c}$, where no two-body bound state exists. Now, according to Eq. (33), the density of a liquid state stabilized by three-body repulsion is of order $\bar{n} \ell_{\mathrm{vdW}}^{3} \simeq|a| / \ell_{\mathrm{vdW}}$ for typical values $\operatorname{Re} D \simeq\left(\ell_{\mathrm{vdW}}\right)^{4}$. In practice, such high densities are not accessible with ultracold gases. However, as suggested by Petrov [74], a dilute liquid phase of bosons at negative scattering length which is stabilized by repulsive three-body interactions might be realized in a situation where two internal states $|\uparrow\rangle$ and $|\downarrow\rangle$ are coupled by an rf-field. By varying the effective Rabi coupling, the scattering length in the symmetric configuration $(|\uparrow\rangle+|\downarrow\rangle) / \sqrt{2}$ can be tuned to zero. The associated three-body scattering hypervolume $D(a=0) \simeq a_{\uparrow \uparrow}^{4} / \xi$ is large and positive provided $\xi=\left(a_{\uparrow \downarrow}+a_{\uparrow \uparrow}\right) /\left(a_{\uparrow \downarrow}-a_{\uparrow \uparrow}\right) \ll 1$. In particular, it is a factor $1 / \xi \gg 1$ larger than the characteristic magnitude $\operatorname{Im} D \simeq a_{\uparrow \uparrow}^{4}$ of its imaginary part, as determined by the standard scaling of the three-body loss rate. Neglecting losses, the resulting effective potential 29 , gives rise to a dilute Bose liquid in the regime where $a<0$. Its dimensionless density $\bar{n} a_{\uparrow \uparrow}^{3} \simeq \xi|a| / a_{\uparrow \uparrow}$ vanishes linearly with the scattering length as in Eq. 33 and - moreover - is small enough to be accessible with dilute ultracold gases. The state is a three-body interaction analog of self-bound droplets in two - component Bose gases which are stabilized by the Lee-Huang-Yang contribution to the interaction energy. They were predicted by Petrov [75] and have been realized experimentally by Cabrera et al. [76]. In fact, liquid-like droplets of Bosons have been observed earlier by Ferrier-Barbut et al. [77] in dipolar gases, where the mean-field instability due to the attractive part of the dipolar interaction is eliminated by the repulsive LHY correction $\left.e(n)\right|_{\text {LHY }} \sim g n^{2}\left(n a^{3}\right)^{1 / 2}$ to the ground state energy density. This stabilizes droplets at densities of order $10^{14} \mathrm{~cm}^{-3}$ [77].

Self-bound droplets and $N$-body bound states near vanishing scattering length In the regime $\Lambda_{\mathrm{dB}}<\Lambda_{\mathrm{dB}}^{c}$ of negative scattering lengths, the ground state at vanishing pressure is a superfluid liquid. By the Gibbs-Duhem relation, the energy per particle $u(p=0)=\mu_{c}<0$ is negative. A given number $N$ of particles thus has an extensive binding energy $B_{N}=|u(p=0)| N$. Moreover, since the liquid has a finite density $\bar{n}$ at zero pressure, the radius of an $N$-cluster scales like $R_{N} \simeq(N / \bar{n})^{1 / 3}$. In the limit where the scattering length approaches zero, both $u(p=0)$ and $\bar{n}$ vanish. The zero pressure liquid thus evaporates into a gas precisely at the quantum tricritical point $\mu=g=0$. This is true, however, only in the thermodynamic limit. For finite particle numbers, the binding energy $B_{N}$ is reduced because particles on the surface are less bound than those in the bulk. For the specific case of a Lennard-Jones interaction, this has been studied numerically for small clusters by Meierovich et al. [78] and by Sevryuk et al. [79]. In particular, it has been found that, at finite $N$, quantum unbinding appears at values $\Lambda_{\mathrm{dB}}^{*}(N)<\Lambda_{\mathrm{dB}}^{c}=0.679 \ldots$ of the de Boer parameter which are considerably lower than what is expected in the thermodynamic limit. This observation can be understood by including a finite, positive surface energy $f_{s}$ per particle in the liquid phase, which also accounts for the essentially flat radial density distributions found numerically near $\Lambda_{\mathrm{dB}}^{c}$ [79]. The surface energy is defined by the subleading term in the expansion

$$
\begin{equation*}
E_{0}(N)=u N+f_{s} N^{2 / 3}+\ldots \tag{34}
\end{equation*}
$$

of the $N$-body ground state energy for $N \gg 1$. Taking into account the surface contribution, the condition $E_{0}(N+1)=E_{0}(N)$ for the unbinding of an $N$-cluster, which is equivalent to a vanishing single particle addition energy $\mu(N)=E_{0}(N+1)-E_{0}(N)=0$, can be written in the form

$$
\begin{equation*}
\frac{-3 u}{2 f_{s}}\left[\Lambda_{\mathrm{dB}}^{*}(N)\right]=N^{-1 / 3} \tag{35}
\end{equation*}
$$

The finite size scaling of the deviation $\Lambda_{\mathrm{dB}}^{c}-\Lambda_{\mathrm{dB}}^{*}(N)$ for $N \gg 1$ is thus determined by the dependence of the bulk energy $u$ and the surface energy $f_{s}$ per particle on the de Boer parameter. Now, Eq. (32) shows that the energy per particle $u(p=0)=\mu_{c}$ on the zero pressure line separating the vacuum from the finite density liquid vanishes quadratically with the distance from the quantum tricritical point. To determine how the surface energy $f_{s}$ per particle vanishes near $\Lambda_{\mathrm{dB}}^{c}$, we use the result for the underlying surface tension ${ }^{10}$

$$
\begin{equation*}
\bar{\sigma}=\frac{\lambda_{3} \bar{n}^{3}}{6 \kappa_{0}} \simeq \frac{\hbar^{2} a^{2}}{m D^{3 / 2}} \sim\left(\Lambda_{\mathrm{dB}}^{c}-\Lambda_{\mathrm{dB}}\right)^{2} \tag{36}
\end{equation*}
$$

derived by Bulgac [81] on the basis of the exact domain wall solution $n(z)=\bar{n} /\left(1+\exp \left(2 \kappa_{0} z\right)\right)$ for the liquid-to-vacuum boundary with an effective potential of the form 29 right at the critical value 32 of the chemical potential. The associated healing

[^10]length $1 / \kappa_{0}=\hbar / \sqrt{2 m\left|\mu_{c}\right|} \simeq \sqrt{D} /|a|$ diverges linearly with the distance from the quantum tricritical point, implying that the surface tension vanishes quadratically. This is consistent with a scaling relation due to Widom [82] which connects the exponent of the surface tension $\bar{\sigma} \sim 1 / \xi^{d-1}$ with that of the correlation length. More precisely, the scaling argument by Widom states that $\bar{\sigma} \simeq k_{B} T_{c} / \xi^{d-1}$ vanishes like the characteristic energy $k_{B} T_{c}$ at a finite temperature phase transition divided by the surface area $\xi^{d-1}$ of a domain with size $\xi$. For the phase transition at the quantum tricritical point studied here, the role of $k_{B} T_{c}$ is apparently played by $\hbar^{2} /(m \sqrt{D})$. Combining the results 33) for the average interparticle spacing $\bar{n}^{-1 / 3}$ and Eq. 36) for the surface tension shows that the surface energy $f_{s} \simeq 4 \pi \bar{n}^{-2 / 3} \cdot \bar{\sigma} \sim\left|\Lambda_{\mathrm{dB}}-\Lambda_{\mathrm{dB}}^{c}\right|^{4 / 3}$ vanishes with a non-trivial exponent $4 / 3$ near the quantum tricritical point. Based on Eq. 35 , the threshold values $\Lambda_{\mathrm{dB}}^{*}(N)$ of the de Boer parameter beyond which $N$-body bound states disappear therefore approach the critical value $\Lambda_{\mathrm{dB}}^{c}$ of the bulk liquid-gas transition according to $\Lambda_{\mathrm{dB}}^{c}-\Lambda_{\mathrm{dB}}^{*}(N) \sim N^{-1 / 2}$. Moreover, in view of Eq. [27), this leads immediately to a power law behavior
\[

$$
\begin{equation*}
-a_{-}(N \gg 1) \simeq(\sqrt{D} / N)^{1 / 2} \text { or } N^{*}(a) \simeq \sqrt{D} / a^{2} \tag{37}
\end{equation*}
$$

\]

of the associated scattering lengths $a_{-}(N)$ or the critical number $N^{*}(a)$ where self-bound droplets of $N^{*}(a)$ Bosons unbind at a given negative scattering length $a$. It has the remarkable feature that the three-body scattering hypervolume $D(a=0)$ at vanishing scattering length sets the scale for the unbinding of $N$-body bound states in the asymptotic limit $N \gg 1$. This is a consequence of the fact that $D$ appears in the leading term $\sim D|\psi|^{6}$ in Eq. 29 which stabilizes the superfluid at both vanishing and small negative scattering lengths, while higher order contributions are negligible near the quantum tricritical point, where $\bar{n} \rightarrow 0$.

The result 37 provides a solution to a long standing problem on how to connect well known results in few-body physics to the many-body limit $N \gg 1$. As mentioned above, the existence of three-body bound states for identical Bosons had been predicted in the early seventies by Efimov [60]. Many-body bound states exist also for larger particle numbers. This has been studied in detail for $N=4$, where theory predicts an infinite sequence of two tetramer states per Efimov trimer [83-86]. Experimentally, the lowest tetramer state has been observed by Ferlaino et al. [87] at $a_{-}(4) \simeq 0.47 a_{-}(3)$ and even signatures of a five-body bound state have been inferred from a characteristic feature in the recombination rate of Cesium near a scattering length $a_{-}(5) \simeq 0.64 a_{-}(4)$ [88]. More generally, the energetically lowest $N$-body bound states, which are the true ground states of the $N$-particle system in the regime $\Lambda_{\mathrm{dB}}^{*}(N=2) \leq \Lambda_{\mathrm{dB}}<\Lambda_{\mathrm{dB}}^{c}$, detach from the continuum at a sequence $a_{-}(N)<0$ of scattering lengths which apparently approaches zero in a monotonic manner. This has been investigated by von Stecher via numerical solutions of the Schrödinger equation up to $N=13$ [89]. In particular, it turns out that the consecutive ratios $a_{-}(4) / a_{-}(3) \simeq 0.44, a_{-}(5) / a_{-}(4) \simeq 0.64$ and $a_{-}(6) / a_{-}(5) \simeq 0.73$ are not very sensitive to the detailed form of the two-body interactions [90]. An obvious question is then whether the sequence of $N$-body bound states continues up to $N=\infty$ and - if so - what is the asymptotic scaling of the scattering lengths $a_{-}(N)$ where they first appear, starting from $a=0^{-}$. The finite size scaling theory for self-bound liquid droplets near the quantum tricritical point developed above provides an explicit answer to this in the limit $N \gg 1$. In particular, it shows that the effective binding energy of $N$-clusters of identical Bosons vanishes at a sequence of negative scattering lengths $a_{-}(N)$ which approach zero in an algebraic fashion as described by Eq. 37). The existence of an infinite sequence of $N$-body bound states with an accumulation point at $a=0$ is consistent with a theorem due to Seiringer [91], which states that some $N$-body bound state must exist for arbitrary small negative scattering lengths. It is also consistent with an earlier theorem by Amado and Greenwood [92] which shows that the number of $N$-body bound states is finite for any $N \geq 4$ precisely at the position $a_{-}(N-1)$ where a zero-energy $N-1$-body bound state appears. An experimental verification of the prediction (37) is an open challenge and requires to determine the size dependence in the unbinding of self-bound droplets near the limit $a \rightarrow 0^{-}$of their stability. Remarkably, a related problem appears in nuclear physics where the binding energy of nuclei with an equal and even number of protons and neutrons depends on the strength of the effective interaction between two alpha particles. Similar to a change of the de Boer parameter discussed above, this interaction may be tuned to a quantum tricritical point which separates a nuclear liquid and an unbound gas of alpha particles [93].


FIG. 8: The Figure on the left shows the triangular vortex lattice in rotating Fermi gases near unitarity observed by Zwierlein et al. [94]. The right Figure displays the phase diagram within a BCS-description of a Fermi gas with weak attractive interactions in the presence of a finite difference $2 h=\mu_{\uparrow}-\mu_{\downarrow}$ of the chemical potential between the two spin-components. It exhibits a Lifshitz-point (LP) below which the superfluid exhibits a periodic modulation of the FFLO-type beyond a Clogston-Chandrasekhar field $h_{c}$ (in blue) due to pairs with finite momentum [95].

## II. SUPERFLUID LIQUID CRYSTALS AND SUPERSOLIDS

Spatially modulated superfluids and the Leggett-bound In the following, we discuss the question whether superfluidity can still persist in a regime where the ground state is a solid, for instance above the critical pressure $p_{c}$ where the superfluid liquid or gaseous phases shown in Fig. 2 crystallize or, more generally, in the presence of some periodic modulation of the density. This question was first adressed by Penrose and Onsager [13], who argued that superfluidity cannot be present in a solid because mobile defects in the form of either vacancies or interstitials that might be delocalized over the whole sample would always be frozen out at zero temperature. Indeed, as will be discussed below, in a commensurate solid, with each unit cell containing an integer number of particles, superfluidity is impossible. In order to understand under which circumstances superfluidity may coexist with broken translation invariance, we start by defining supersolids in a rather wide sense by

A supersolid is a system where off-diagonal long range order (i.e. $\rho_{1}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) \rightarrow n_{0} \neq 0$ for $\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right| \rightarrow \infty$ ), appears together with an interaction-induced periodic modulation of the density $\left(\langle\hat{n}(\boldsymbol{x})\rangle=\rho_{1}(\boldsymbol{x}, \boldsymbol{x})=\bar{n}+\sum_{\boldsymbol{G} \neq 0} n_{G} \exp (i \boldsymbol{G} \boldsymbol{x})\right.$ with $\left.n_{G} \neq 0\right)$.

This definition includes a number of phases that have been known for quite some time in ultracold gases, among which the triangular array of vortices observed both in rotating BEC's [96] and in Fermi gases near infinite scattering length [94] shown in Fig. 8 is certainly the most familiar one. Now, the periodic order in the density of the vortex lattice reflects that of the underlying superfluid. Thus, there is only a single order parameter and both ODLRO and the associated lattice structure disappear simultaneously at the superfluid transition temperature. A vortex lattice should therefore more properly be called a spatially modulated superfluid. Quite generally, such phases are characterized by the fact that they exhibit BEC where the eigenfunction $\phi_{0}(\boldsymbol{x})$ associated with the extensive eigenvalue $\lambda_{0}^{(1)}=N_{0}$ of the one-particle density operator exhibits a regular lattice structure. Provided that the periodic order in $\phi_{0}(\boldsymbol{x})$ is generated by interactions and not imposed externally, as in the case of ultracold atoms in an optical lattice, these phases are superfluid analogs of liquid crystals. Generically, they are separated from a spatially uniform superfluid by a Lifshitz point [97]. An example is shown in Fig. 8, where the phase diagram of a two-component Fermi gas with a finite Zeeman-field $2 h=\mu_{\uparrow}-\mu_{\downarrow}$ which does not affect the orbital motion is displayed in dimensionless units within a BCS - description. The homogeneous superfluid is separated from the normal gas by a continuous transition at a critical temperature which reaches the universal BCS limit $T_{c} / \Delta_{0}=(1.76)^{-1}=0.57$ at zero field $h=0$. At zero temperature, the homogeneous and spin-balanced superfluid disappears at a Clogston-Chandrasekhar field $h_{c}=\Delta_{0} / \sqrt{2}$. Even beyond $h_{c}$, however, superfluidity still persists in a range $h_{c}<h<0.754 \Delta_{0}{ }^{11}$, where pairs are formed with finite momentum $q_{0}=k_{F \uparrow}-k_{F \downarrow}$. The non-vanishing spin-density $s=n_{\uparrow}-n_{\downarrow} \neq 0$ in this regime may give rise to a complex order parameter $\phi_{0}(\boldsymbol{x}) \sim \exp i \boldsymbol{q}_{0} \boldsymbol{x}$ in a Fulde-Ferrell state [99] or the generically more favorable Larkin-Ovchinnikov phase where $\phi_{0}(\boldsymbol{x}) \sim \cos \boldsymbol{q}_{0} \boldsymbol{x}$ is real [100].

[^11]According to the rather broad definition above, spatially modulated superfluids like the vortex lattice, where the density modulation is due to interactions and not externally imposed, are indeed supersolids. However, they must be distinguished carefully from a genuine supersolid that has been searched for in solid ${ }^{4} \mathrm{He}$, where crystalline order and superfluidity might be present simultaneously as two independent order parameters. In contrast to the case of spatially modulated superfluids, the regular lattice is then still there even after superfluidity in the crystalline phase is lost. A subtle issue that will be discussed further in the section on hydrodynamic modes of a superfluid smectic phase below is whether spatially modulated superfluids and genuine supersolids may be distinguished according to the number of Goldstone modes. Since gauge and translation invariance as two continuous symmetries are spontaneously broken by the very definition of supersolids above, a separate set of Goldstone modes are expected. For spatially modulated superfluids with a single order parameter, they would disappear simultaneously above the superfluid transition. By contrast, for genuine supersolids those connected with the broken translation invariance would still be present in the normal phase. Now, it turns that this simple scenario is only valid provided that:
(a) the generators of translations and gauge transformations are independent
(b) the modulation of the superfluid order parameter is accompanied by a corresponding one in the particle density.

As discussed by Watanabe and Murayama [101], condition (a) is violated in a vortex lattice, whose two-component deformation field $\boldsymbol{u}(x, y)$ is not an independent variable but is rigidly coupled to the superfluid velocity $\boldsymbol{v}_{s}$ by $\boldsymbol{u}=\hat{z} \wedge \boldsymbol{v}_{s} /(2 \Omega)$. Here $\hat{z}$ is the unit vector perpendicular to the ( $x, y$ )-plane and $\Omega$ the rotation frequency. Physically, the rigid coupling between the deformation field and the superfluid velocity arises from the fact that the equilibrium vortex positions are precisely those at which the superfluid velocity due to all other vortices precisely cancel. The Tkachenko modes with their quadratic dispersion $\omega=\sqrt{\hbar / m \Omega} c_{s} q^{2} / 4$ are therefore the single, so-called type B Goldstone mode in a vortex lattice despite the fact that both gauge and translational invariance are broken ${ }^{12}$.

As a hermitean operator, the one-particle density operator $\hat{\rho}_{1}$ has a complete set $\left|\phi_{\alpha}\right\rangle$ of eigenstates which are the so-called natural orbitals. The very definition (1) of ODLRO shows that long range phase coherence may be present in a crystalline configuration only if the natural orbital $\phi_{0}(\boldsymbol{x})=\left\langle\boldsymbol{x} \mid \phi_{0}\right\rangle$ associated with the dominant extensive eigenvalue is extended over the whole crystal. As will be discussed in the following section, a trivial example for this is provided by a weakly interacting BEC in an optical lattice, where $\phi_{0}(\boldsymbol{x})$ coincides with the one-particle eigenstate for crystal momentum $\boldsymbol{q}=0$ which is delocalized by Bloch's theorem. Now, an important result derived by Leggett [44] is that independent of how the periodic modulation of the density arises, for any ground state which breaks both translation and gauge invariance there is an upper bound on the magnitude of the superfluid fraction $f_{s}$. In particular, the bound implies that in the presence of a non-vanishing modulation of the density, there must be a finite normal fraction $f_{n}$ even in the ground state. Specifically, based on the topological criterion for superfluidity discussed in Lecture I, Leggett has shown that the superfluid fraction

$$
f_{s} \leq\left[\frac{\bar{n}}{d} \int_{0}^{d} \frac{d x}{n_{1}(x)}\right]^{-1} \text { where } n_{1}(x)=\int_{\text {unitcell }} d \tilde{y} d \tilde{z}\langle\hat{n}(\boldsymbol{x})\rangle \rightarrow f_{s} \leq \begin{gather*}
m / m_{B} \sim \exp -2 \sqrt{V_{0} / E_{r}} \quad \text { optical lattice }  \tag{38}\\
\left(1-\delta^{2}\right)^{3 / 2} /\left(1+\delta^{2} / 2\right) \rightarrow 1-2 \delta^{2} \text { superfluid smectic }
\end{gather*}
$$

is bounded above by an expression where the inverse of the density $n_{1}(x)$ is integrated over a unit cell of the lattice. Here, $n_{1}(x)$ is the average of the microscopic density $\langle\hat{n}(\boldsymbol{x})\rangle$ over the transverse directions of the unit cell, using dimensionless coordinates $\tilde{y}=y / d_{y}$ and $\tilde{z}=z / d_{z}$. Moreover, without loss of generality, the $x$-direction has been singled out, with $d$ the associated length of the unit cell. Apparently, the bound becomes increasingly tight for a large contrast with respect to the spatially averaged density $\bar{n}=N / V$ but is always finite unless the density vanishes identically in some region. For a crystal, where the particles are concentrated close to a discrete set $\{\boldsymbol{R}\}$ of lattice sites with a rather small density at interstitial positions, the upper bound on $f_{s}$ is much smaller than one. In the limit of a fluid with uniform density, in turn, Eq. (38) reduces to the trivial identity $f_{s} \leq 1$. Two important points should be noted: First, the bound (38) does not provide a sufficient criterion for superfluidity in a state with broken translation invariance: a finite value of the bound is still compatible with no superfluidity at all. As will be discussed below, this happens e.g. for the Mott-insulator state (40) of the Bose-Hubbard model. A second point is that the bound (38) makes no assumption about the physical origin of the density modulation, which may be induced by interactions or externally imposed, nor about the question of commensurability, i.e. whether the average number of particles within a unit cell happens to be an integer or not. From the analysis of superfluid ground states in the Bose-Hubbard-model in the next section it will become clear, however, that the latter issue plays a crucial role for the existence of supersolids.

[^12]An explicit evaluation of the bound (38) requires detailed knowledge of the microscopic ground state density, which is hardly available in practice. The two cases where an explicit result has been given above refer to a gas of non-interacting Bosons in an optical lattice, where the density $\langle\hat{n}(\boldsymbol{x})\rangle_{\text {SF }}^{(0)}=N\left|\psi_{q=0}(\boldsymbol{x})\right|^{2}$ is given by the square of the one-particle Bloch state with crystal momentum $\boldsymbol{q}=0$. Indeed, in the limit of small repulsive interactions, the natural orbital $\phi_{0}(\boldsymbol{x}) \rightarrow \psi_{\boldsymbol{q}=0}(\boldsymbol{x})$ associated with the dominant eigenvalue of $\hat{\rho}_{1}$ approaches the lowest one-particle eigenstate in the lattice ${ }^{13}$. As pointed out by Leggett [51], in this case the bound is equal to the ratio between the bare and the band mass $m_{B}$. In a deep optical lattice with $V_{0} \gg E_{r}$, the latter becomes exponentially small of order $m / m_{B}=\pi^{2} J / E_{r} \sim \exp -2 \sqrt{V_{0} / E_{r}}$. In the limit of weak interactions, the ground state of the Bose-Hubbard model thus has a condensate fraction $f_{0} \rightarrow 1$ which approaches one, while the superfluid fraction is exponentially small. A second example where an explicit result for the Leggett bound can be given is a BEC with a uni-directional, weak density wave. As will be discussed below, this example is of relevance for dipolar gases, which exhibit a superfluid phase with a finite density modulation along the weakly confined direction of a cigar-shaped trap if the dipolar and the short range scattering length become comparable. If one assumes that even in this strong coupling regime the one-particle density operator $\hat{\rho}_{1}=\sum_{\alpha} \lambda_{\alpha}^{(1)}\left|\phi_{\alpha}\right\rangle\left\langle\phi_{\alpha}\right|$ is dominated by a single extensive eigenvalue $\lambda_{0}^{(1)} \simeq N$, the eigenfunction $\phi_{0}(\boldsymbol{x}) \sim 1+\delta \cos \left(q_{0} x\right)+\ldots$ in the supersolid phase near the transition involves a small admixture $|\delta| \ll 1$ which breaks translation symmetry along the $x$-direction. The resulting ground state density

$$
\begin{equation*}
n(\boldsymbol{x})=\langle\boldsymbol{x}| \hat{\rho}_{1}|\boldsymbol{x}\rangle \rightarrow \frac{\bar{n}}{\left(1+\delta^{2} / 2\right)}\left[1+\delta \cos \left(q_{0} x\right)\right]^{2} \tag{39}
\end{equation*}
$$

then has a rather simple form for which the Leggett bound can be evaluated easily. For $|\delta| \geq 1$, the density (39) vanishes quadratically at either one (for $|\delta|=1$ ) or two different points within the unit cell, which leads to a divergent integral in the denominator of Eq. (38). This is revealed in the expression for the Leggett bound of the superfluid smectic in Eq. (38) which is ill-defined for $|\delta|>1$. We emphasize that the use of the Leggett bound to extract a finite normal fluid density from a Gross-Pitaevskii Ansatz for the supersolid ground state requires to take into account excitations beyond a Gross-Pitaevskii description, which in itself does not have a proper notion of a normal fluid fraction. Indeed, the bound (38) relies on a trial Ansatz $\psi^{(\theta)}=\exp \left[i \sum_{j} \varphi\left(\boldsymbol{x}_{j}\right)\right] \psi_{0}\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)$ for the many-body wave function where the twist in the boundary condition 21$)$ is accounted for by a phase $\varphi(\boldsymbol{x})$ whose detailed form is determined by minimizing the increase $\hbar^{2} /(2 m) \int_{\boldsymbol{x}}(\nabla \varphi)^{2} n(\boldsymbol{x})$ in energy (here, $n(\boldsymbol{x})=N \int d 2 \ldots d N\left|\psi_{0}\left(\boldsymbol{x}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)\right|^{2}$ is the exact ground state density). This is achieved by concentrating the imposed overall phase change $\theta=\varphi(0)-\varphi(L)$ in regions of small density. In addition, note that a Gross-Pitaevskii Ansatz is restricted to the phase where off-diagonal long range order coexists with broken translation invariance but it is obviously not adequate for describing the full phase diagram of a genuine supersolid, where the periodic density is still present after superfluidity is lost.

In order to see that Eq. (38) gives a finite upper bound for the superfluid fraction even in states which definitely exhibit no superfluidity, consider the simple Ansatz for the ground state of a commensurate crystal

$$
\begin{equation*}
\psi_{\text {solid }}^{(0)}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}\right)=\frac{1}{\sqrt{N!}} \sum_{P\left[\boldsymbol{x}_{i}\right]} w\left(\boldsymbol{x}_{1}-\boldsymbol{R}_{1}\right) w\left(\boldsymbol{x}_{2}-\boldsymbol{R}_{2}\right) \ldots w\left(\boldsymbol{x}_{N}-\boldsymbol{R}_{N}\right) \text { i.e. }\left|\psi_{\text {solid }}^{(0)}\right\rangle=\prod_{\boldsymbol{R}} \hat{b}_{\boldsymbol{R}}^{\dagger}|0\rangle \tag{40}
\end{equation*}
$$

in which each of the $N_{L}=N$ lattice sites $\{\boldsymbol{R}\}$ is occupied with a single particle with probability one. Similar to the Wannier states of the lowest band in an optical lattice, the functions $w(\boldsymbol{x}-\boldsymbol{R})$ are assumed to be centered around $\boldsymbol{R}$ and they form an orthonormal basis according to $\int_{\boldsymbol{x}} w^{*}(\boldsymbol{x}-\boldsymbol{R}) w\left(\boldsymbol{x}-\boldsymbol{R}^{\prime}\right)=\delta_{\boldsymbol{R}, \boldsymbol{R}^{\prime}}$. To account for the indistinguishability of the particles, there is a sum $P\left[x_{i}\right]$ over all $N$ ! permutations of the coordinates $\left\{\boldsymbol{x}_{i}\right\}$, a point which is often ignored in the discussion of regular solids. As mentioned in Lecture I, however, statistics matters even in a solid: the critical de Boer parameter $\Lambda_{\mathrm{dB}}^{\mathrm{c} \text { s.lid }}$ for melting into a liquid ground state due to zero-point motion is larger for Fermions than for Bosons. In practice, symmetrization only plays a role if there is an appreciable overlap between the one-particle wave functions $w(\boldsymbol{x}-\boldsymbol{R})$ at different sites $\boldsymbol{R} \neq \boldsymbol{R}^{\prime}$ such that particles are able to exchange their places in the lattice. The one-particle density matrix associated with the simple quantum solid described by Eq. (40) is equal to

$$
\begin{equation*}
\rho_{1, \text { solid }}^{(0)}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\sum_{\boldsymbol{R}} w^{*}(\boldsymbol{x}-\boldsymbol{R}) w\left(\boldsymbol{x}^{\prime}-\boldsymbol{R}\right) \rightarrow\langle\hat{n}(\boldsymbol{x})\rangle_{\text {solid }}^{(0)}=\sum_{\boldsymbol{R}}|w(\boldsymbol{x}-\boldsymbol{R})|^{2} . \tag{41}
\end{equation*}
$$

It vanishes exponentially with the separation $\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|$ provided the products $w^{*}(\boldsymbol{x}-\boldsymbol{R}) w\left(\boldsymbol{x}-\boldsymbol{R}^{\prime}\right)$ decay accordingly with distance $\left|\boldsymbol{R}-\boldsymbol{R}^{\prime}\right|$. The fact that $\rho_{1, \text { solid }}^{(0)}$ shows no long range phase coherence even though $w(\boldsymbol{x}-\boldsymbol{R})$ may have appreciable amplitude at sites other than $\boldsymbol{R}$ is not surprising: indeed this state is a product of local Fock states and it describes the trivial

[^13]limit of a Mott-insulator at vanishing hopping in the Bose-Hubbard model discussed below. Nevertheless, the Leggett bound for the associated density $\langle\hat{n}(\boldsymbol{x})\rangle_{\text {solid }}^{(0)}$ is finite. While a quantitative result depends on the detailed form of the Wannier functions, the expression (38) shows that the upper bound on $f_{s}$ essentially measures the ratio between the minimum and the maximum value of $|\omega(\boldsymbol{x}-\boldsymbol{R})|^{2}$ within the unit cell centered around $\boldsymbol{R}$. In fact, a simple estimate for the bound can be given by noting that the density of the trivial quantum solid $\sqrt{40}$ differs from the density $\langle\hat{n}(\boldsymbol{x})\rangle_{\mathrm{SF}}^{(0)}=\left|\sum_{\boldsymbol{R}} w(\boldsymbol{x}-\boldsymbol{R})\right|^{2}$ of a weakly interacting superfluid at the same value of $\bar{n}$ only by small corrections arising from the overlap of Wannier states at neighboring or more distant sites (see also Eq. 45 below). Since the superfluid fraction associated with the density $\langle\hat{n}(\boldsymbol{x})\rangle_{\mathrm{SF}}^{(0)}$ is just the ratio $m / m_{B}$ between the bare and the band mass, this shows that $\left(f_{s}\right)_{\text {solid }}^{(0)}<m / m_{B}$. In practice, this is tiny for a crystal with well localized functions $w(\boldsymbol{x}-\boldsymbol{R})$, yet it is always finite despite the fact that the true superfluid fraction of the state (40) is zero. Now, as will be discussed in the following section, a straightforward way in which the commensurate Mott-insulator (40) may be converted into a supersolid is when the number of particles is either less or larger than those of the lattice sites. With a finite concentration of vacancies e.g. there is a $\binom{N_{L}}{N}$ - fold degeneracy due to the different possibilities of distributing $N$ atoms on $N_{L}$ sites. In the presence of a finite density of delocalized defects, the ground state of Bosons even with a strong periodic modulation of the density turns out to be a supersolid. This generic scenario for a supersolid state has originally been suggested by Andreev and Lifshitz [103], who predicted a new hydrodynamic mode associated with wave like propagation of defects. It was also formulated as a conjecture by Chester [104]. Note that the value for the superfluid fraction obtained from the Leggett bound is hardly affected by the presence of defects. In fact, the density distribution is not sensitive to the occurence or not of off-diagonal long range order, which involves the phase rather than the magnitude of the many-body wave function. A proof that delocalized defects are indeed necessary for a supersolid has been given by Prokof'ev and Svistunov [105].

In the brief discussion above of Goldstone modes in supersolids, a nontrivial issue is whether a periodic modulation of the order parameter is also reflected in the particle density. This question already arises for the Abrikosov vortex lattice in type II superconductors as the first example of a spatially modulated superfluid but it is hardly raised in this context. Indeed, it turns out that for superfluids in the BCS limit, the modulation of the particle density associated with a corresponding one in the order parameter is completely negligible for a neutral Fermi gas and thus even more so in the presence of the Coulomb interaction in real superconductors ${ }^{14}$. A quantitative estimate of the coupling between the order parameter and the density can be given for neutral systems in the vicinity of the superfluid transition, where the change $\delta n$ in particle density is related to the finite order parameter $|\psi|^{2}$ by a linear relation $\delta n=\alpha \tilde{\kappa}|\psi|^{2}$ to lowest order. Here, $\tilde{\kappa}=\partial n / \partial \mu$ is the compressibility and $\alpha$ is the coupling constant between density and the order parameter, which is generically of the form $-\alpha \delta n|\psi|^{2}$ [57]. For weak-coupling BEC's, where a modulation of the superfluid order parameter is fully reflected in the particle density, one has $\alpha \tilde{\kappa} \equiv 1$. More generally, the coupling between density and the superfluid order parameter follows by noting that for uniform fields the Lagrange density $\mathcal{L} \rightarrow-p$ coincides with the negative of the pressure $p(\mu, T)$ in equilibrium [107]. Since the derivative $\partial p /\left.\partial n\right|_{S}=m c_{1}^{2}$ of the pressure with respect to the density determines the velocity $c_{1}$ of first sound, this allows to infer the parameter $\alpha$ from the change of $c_{1}^{2}$ due to the appearance of a finite order parameter $|\psi|^{2}$. Formally, this follows from first order perturbation theory in $\rho_{s}=m|\psi|^{2} \rightarrow 0$ in Eq. (63) below for the special case of a homogeneous superfluid, where the layer compression modulus $B$ vanishes. The dimensionless parameter $\alpha \tilde{\kappa}$ may thus be expressed in terms of the ratio $S / C_{p}$ between entropy and specific heat $C_{p}=\left.T \partial_{T} S\right|_{p}$ together with the thermal expansion coefficient $\alpha_{p}=\left.(1 / V) \partial_{T} V\right|_{p}$, both taken at the superfluid transition, by

$$
\begin{equation*}
\alpha=\frac{\partial}{\partial|\psi|^{2}}\left(\left.\frac{\partial p}{\partial n}\right|_{T}\right) \rightarrow \alpha \tilde{\kappa}=\left(T \alpha_{p} S / C_{p}\right)^{2} . \tag{42}
\end{equation*}
$$

In the BCS-limit, where the superfluid transition is effectively an instability of a Fermi gas at $T_{c} \ll T_{F}$, one finds that $\alpha \tilde{\kappa} \simeq\left(T_{c} / T_{F}\right)^{4}$ is exponentially small. The fact that the superfluid order parameter has a negligible coupling to the particle density in this limit is well known. Specifically, it has been shown by Leggett [108] that even in the presence of strong Fermi liquid corrections in the normal state, the compressibility of a neutral Fermi liquid is unchanged by a transition to superfluidity to leading, linear order in $T_{c} / T_{F}$. For the strongly interacting unitary Fermi gas, scale invariance implies that the product $T \alpha_{p}=3 \mathrm{LP} / 2$ coincides with the Landau-Placzek ratio $\mathrm{LP}=\left(C_{p} / C_{V}\right)-1$ up to a universal factor. The experimental data for the thermodynamic parameters which enter Eq. (42) may then be extracted from the measurements by Ku et al. [109], giving rise to a dimensionless coupling constant $\alpha \tilde{\kappa}$ around 0.05 . This small number is consistent with the fact that an appreciable density contrast in the vortex lattice of Fig. 8 near unitarity and on the BCS-side has only been achieved by ramping the magnetic field towards the BEC-side of the Feshbach resonance before the vortex lattice is released from the trap [94].

[^14]Defects and the Andreev-Lifshitz scenario for supersolids in the Bose-Hubbard-model Even though it does not realize a supersolid in a proper sense because the periodicity is imposed externally by the optical lattice, the Bose-Hubbard model (BHM) provides an instructive toy model to understand the coexistence of superfluidity and a periodic modulation of the density. In particular, it allows to understand the role of defects and commensurability emphasized in the groundbreaking work on supersolids by Andreev and Lifshitz [103]. The underlying Hamiltonian

$$
\begin{equation*}
\hat{H}-\mu \hat{N}=\sum_{\boldsymbol{q}} \varepsilon(\boldsymbol{q}) \hat{b}_{\boldsymbol{q}}^{\dagger} \hat{b}_{\boldsymbol{q}}+\frac{U}{2} \sum_{\boldsymbol{R}} \hat{n}_{\boldsymbol{R}}\left(\hat{n}_{\boldsymbol{R}}-1\right)+\sum_{\boldsymbol{R}}\left(\epsilon_{\boldsymbol{R}}-\mu\right) \hat{n}_{\boldsymbol{R}} \tag{43}
\end{equation*}
$$

for lattice Bosons with on-site repulsive interactions $U>0$ was originally introduced by Fisher et al. [110] as a model for the study of the interplay between interactions and disorder described by random on-site energies $\epsilon_{\boldsymbol{R}}$ with zero mean and finite variance. Its relevance for ultracold atoms in optical lattices was recognized by Jaksch et al. [111]. In this context, the last term with a variable on-site energy $\epsilon_{\boldsymbol{R}} \rightarrow V_{\text {trap }}(\boldsymbol{R})$ describes the effect of the smooth trapping potential which acts like a spatially varying chemical potential. In the following, we consider a homogeneous situation with no disorder, where only two different ground states are possible: a superfluid (SF) and a Mott-insulating (MI) state. In fact, as will be discussed below, the homogeneous ground state for a fixed total number of particles is always a superfluid unless this number $N=\langle\hat{n}\rangle \cdot N_{L}$ happens to be precisely an integer $\langle\hat{n}\rangle=1,2 \ldots$ times the number $N_{L}$ of lattice sites, i.e. the 'solid' is commensurate. In this special case, a quantum phase transition between a SF and a MI appears at fixed density. It is driven by the competition between the on-site repulsion $U$ and the kinetic energy due to motion in the lowest band of the optical lattice with dispersion $\varepsilon(\boldsymbol{q})$. The band minimum is assumed to be located at zero energy, thus defining a band mass $m_{B}$ by $\varepsilon(\boldsymbol{q})=\hbar^{2} \boldsymbol{q}^{2} /\left(2 m_{B}\right)+\ldots$. In order to eliminate the singular structure that a phase transition only appears at precisely commensurate filling, it is convenient to use a grand canonical description, where the particle density is regulated by a chemical potential $\mu$. Mott-insulating states, which are incompressible by definition with $\tilde{\kappa}=\partial n / \partial \mu \equiv 0$, then cover a finite width $\Delta \mu \neq 0$ over which the density is pinned to an integer $\langle\hat{n}\rangle=1,2 \ldots$. They can be reached from the compressible superfluid via a density-driven transition at a fixed ratio $U / B$ of the on-site repulsion and the total bandwidth $B$. The phase diagram at zero temperature is shown in Fig. 9 for the specific case of a three-dimensional cubic lattice, where the bandwidth is connected with the nearest neighbor tunneling matrix element $J$ by $B=12 J$. The associated critical value $(U / B)_{c}=2.445$ for the SF-MI transition at the tip of the Mott lobe with density $\langle\hat{n}\rangle=1$ has been determined rather precisely by Quantum Monte Carlo [112]. In the realization of the BHM with ultracold atoms, the ratio $U / J$ can be changed easily by varying the dimensionless depth $V_{0} / E_{r}$ of the optical lattice. In particular, for deep lattices $V_{0} \gg E_{r}$, the nearest neighbor tunneling matrix element $J \sim \exp -2 \sqrt{V_{0} / E_{r}}$ vanishes exponentially. The ratio between the repulsive interaction and the kinetic energy then determines the critical value of the dimensionless lattice depth $V_{0} / E_{r}$ by

$$
\begin{equation*}
\left.\frac{U}{J}\right|_{V_{0} \gg E_{r}}=\frac{\pi a}{\sqrt{2} d} \exp \left(2 \sqrt{V_{0} / E_{r}}\right) \rightarrow\left(V_{0} / E_{r}\right)_{c}=\frac{1}{4} \ln ^{2}\left(\frac{\sqrt{2} d}{\pi a} \cdot(U / J)_{c}\right) . \tag{44}
\end{equation*}
$$

With the experimental parameters $d=426 \mathrm{~nm}$ and $a=5.7 \mathrm{~nm}$ for the lattice constant and the scattering length, respectively, the precise result for $(U / J)_{c}$ predicts a critical value $\left(V_{0} / E_{r}\right)_{c}=11.9$ for the SF-MI transition with $\langle\hat{n}\rangle=1$. This agrees quite well with the lattice depth $V_{0}=12-13 E_{r}$ where the transition has been observed by Greiner et al. [113], based on the disappearance of sharp peaks at reciprocal lattice vectors $\boldsymbol{k}=\boldsymbol{G}$ in the time-of-flight images. They provide a direct measure of the momentum distribution of the original many-body state by mapping a given momentum to a point $\boldsymbol{x}=\hbar \boldsymbol{k} t / m$ in the absorption image after a ballistic expansion time $t$. For an idealized homogeneous situation, the momentum distribution

$$
\begin{equation*}
\langle\hat{n}(\boldsymbol{k})\rangle=\left\langle\hat{\psi}^{\dagger}(\boldsymbol{k}) \hat{\psi}(\boldsymbol{k})\right\rangle=N_{L}|\tilde{w}(\boldsymbol{k})|^{2} \sum_{\boldsymbol{R}} e^{i \boldsymbol{k} \cdot \boldsymbol{R}}\left\langle\hat{b}_{\boldsymbol{R}}^{\dagger} \hat{b}_{0}\right\rangle \quad \text { while } \quad\langle\hat{n}(\boldsymbol{x})\rangle=\left\langle\hat{\psi}^{\dagger}(\boldsymbol{x}) \hat{\psi}(\boldsymbol{x})\right\rangle=\sum_{\boldsymbol{R}} \rho_{1, \text { solid }}^{(0)}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}=\boldsymbol{x}+\boldsymbol{R}\right)\left\langle\hat{b}_{\boldsymbol{R}}^{\dagger} \hat{b}_{\mathbf{0}}\right\rangle \tag{45}
\end{equation*}
$$

is proportional to the Fourier transform of the one-particle density matrix $\left\langle\hat{b}_{R}^{\dagger} \hat{b}_{0}\right\rangle$ on the lattice. In the superfluid, the presence of long range phase coherence $\left\langle\hat{b}_{\boldsymbol{R}}^{\dagger} \hat{b}_{0}\right\rangle \rightarrow f_{0} \neq 0$, gives rise to pronounced peaks $\langle\hat{n}(\boldsymbol{k})\rangle \sim f_{0} N_{L}^{2}$ at reciprocal lattice vectors, where $\exp i \boldsymbol{G} \cdot \boldsymbol{R}=1$. A quite different behavior is found for the density $\langle\hat{n}(\boldsymbol{x})\rangle$, where the factor $\exp i \boldsymbol{k} \cdot \boldsymbol{R}$ is effectively replaced by the one-particle density matrix of the trivial MI state introduced in Eq. 41 . Due to the exponential decay of $\rho_{1, \text { solid }}^{(0)}$ with separation $\boldsymbol{R}$, the density on a microscopic scale is hardly affected by the presence or not of ODLRO. In a harmonic trap, however, superfluid and Mott-insulating phases may be distinguished in the wedding cake structure of the density $\bar{n}(\boldsymbol{X})$ spatially averaged over the size of a unit cell. Indeed, since the local chemical potential $\mu(\boldsymbol{X})=\mu(\mathbf{0})-m \omega^{2} \boldsymbol{X}^{2} / 2$ decreases monotonically towards zero at the edge of the atomic cloud, the incompressible nature of Mott-insulators gives rise to rings of constant density $\bar{n}(\boldsymbol{X})$ separated by compressible superfluids, as observed in single-site resolution images [114, 115], see Fig. 9 .

As stated above and evident from the phase diagram of Fig. 9 , the ground state of the BHM remains a superfluid down to arbitrary small hopping amplitudes $B \neq 0$ unless the number of particles is precisely an integer $\langle\hat{n}\rangle=n_{0}=1,2 \ldots$ times the number of lattice sites. The underlying physics behind this initially surprising observation is that the ground state of Bose crystals with a finite density of defects is always a supersolid, as first pointed out by Andreev and Lifshitz [103]. In order to


FIG. 9: Zero temperature phase diagram of the Bose-Hubbard model. The average density $\langle\hat{n}\rangle$ varies continuously with the chemical potential in the superfluid, while it is pinned to integer values $\langle\hat{n}\rangle=1,2 \ldots$ in the different Mott-insulators. In the SF, the lines of integer density end at the tip of the Mott-lobes and they form a set of zero measure. For incommensurate densities $\langle\hat{n}\rangle \neq 1,2 \ldots$ superfluidity persists down to arbitrary small bandwidth $B$ due to the Andreev-Lifshitz mechanism of delocalized defects. The Figure on the right shows a single-site resolution image of the Mott-insulators with filling $n_{0}=2$ (in the center) and $n_{0}=1$ (in green) in a trap obtained by Bakr et al. [114].
understand the role which defects play for establishing superfluidity in a situation with a strong periodic modulation of the density, consider the transition at a critical lower value $\mu_{-}\left(n_{0}\right)$ of the chemical potential from a compressible superfluid with density $\langle\hat{n}\rangle<n_{0}$ to the Mott-insulator with integer filling $n_{0}$ in the limit of small but finite hopping $B \ll U$. Similar to the discussion of the gas-liquid transition in Lecture I , it is convenient to start with the vacuum state $|0\rangle=\left|\mathrm{MI}_{0}\right\rangle$ at negative chemical potential, which may formally be viewed as a Mott-insulator with filling $n_{0}=0$ because the density stays fixed at zero upon increasing $\mu \rightarrow 0^{-}$. Moreover, the compressible superfluid with filling $0<\langle\hat{n}\rangle<1$ immediately above the vacuum state is a supersolid due to a finite density $n_{\Delta}=\langle\hat{n}\rangle$ of delocalized particles $=$ interstitials on top of this trivial Mott-insulator. For a single particle, the exact defect state is simply $\left|\psi_{\text {def }}\right\rangle=\hat{b}_{q=0}^{\dagger}\left|\mathrm{MI}_{0}\right\rangle$. Due to the rather strong on-site repulsion $U \gg B$, the state with a finite concentration $n_{\Delta}=N_{\Delta} / N_{L}$ of the interstitials is, however, quite different from the superfluid in the limit $U \ll B$, where $|\mathrm{SF}\rangle_{N_{\Delta}} \rightarrow\left(\hat{b}_{\boldsymbol{q}=0}^{\dagger}\right)^{N_{\Delta}}|0\rangle$. Effectively, the interstitials behave like hard core Bosons and reach the value of unit filling $n_{\Delta}=\langle\hat{n}\rangle=1$ already at a small critical value $\mu_{-}\left(n_{0}=1\right)=\mu\left(n_{\Delta}=1\right) \rightarrow B$ for $B \ll U$. Physically, this may be understood by noting that hard core Bosons effectively behave like non-interacting Fermions ${ }^{15}$. Since the lowest band contains exactly $N_{L}$ single-particle states, the configuration at filling $n_{\Delta}=1$ is therefore equivalent to a Fermionic band insulator. A formal derivation of the fact that the Mott-insulator state $\left|\mathrm{MI}_{1}\right\rangle$ with unit filling is reached already at $\mu_{-}\left(n_{0}=1\right) \rightarrow B$ can be given by using the systematic expansion around the limit of vanishing hopping by Freericks and Monien [116]. To lowest order, it is sufficient to consider a state $\left|\psi_{\text {hole }}\right\rangle=\hat{b}_{\boldsymbol{q}=0}\left|\mathrm{MI}_{1}\right\rangle=\left(1 / \sqrt{N_{L}}\right) \sum_{\boldsymbol{R}}\left|11110_{\boldsymbol{R}} 11111\right\rangle$ with a single delocalized hole on top of the Mott-insulator. The associated hole creation energy $\Delta_{h}=\langle\hat{H}-\mu \hat{N}\rangle_{\text {hole }}-\langle\hat{H}-\mu \hat{N}\rangle_{\mathrm{MI}}=\mu-B+\ldots$ vanishes precisely at $\mu=\mu_{-}\left(n_{0}=1\right) \rightarrow B$.

Quite generally, a Mott-insulator is characterized by finite, positive values of both the particle addition energy $\Delta_{p}=\mu_{+}-\mu$ and the hole creation energy $\Delta_{h}=\mu-\mu_{-}$. Their sum $\Delta \mu=\Delta_{p}+\Delta_{h}=\mu_{+}-\mu_{-}$determines the Mott gap i.e. the range of $\mu$ where the density remains pinned at the integer $n_{0}$. In the regime $B / U<(B / U)_{c}$, non-integer values of the filling $\langle\hat{n}\rangle \neq 1,2 \ldots$ can therefore only be realized by adding either vacancies or interstitials on top of the commensurate Mott-insulators. The resulting superfluid ground state has a finite density $n_{\Delta}=\langle\hat{n}\rangle-n_{0}$ of delocalized defects and is compressible, with a vanishing value of both $\Delta_{h}$ and $\Delta_{p}$. A similar scenario for supersolids arises in real Bose crystals, whose periodic density results from interactions instead of being externally imposed as in the BHM. Again, the ground state of a commensurate crystal like ${ }^{4} \mathrm{He}$ above the critical pressure $p_{c}$ is a Mott-insulator. It may turn into a supersolid only if the gap $\Delta_{I}$ for interstitials or $\Delta_{V}$ for vacancies may be tuned to vanish. For solid ${ }^{4} \mathrm{He}$, the numerical values $\Delta_{I} \simeq 23 \mathrm{~K}$ and $\Delta_{V} \simeq 13 \mathrm{~K}$ for these energies have been determined numerically within a fully microscopic model by Boninsegni et al. [117]. They are quite large, thus excluding supersolid behavior of the Andreev-Lifshitz type in ${ }^{4} \mathrm{He}$. In principle, it is also possible to have a supersolid whose average defect density $n_{\Delta}=0$ vanishes. As emphasized by Prokof'ev and Svistunov [105] however, this requires a fine-tuning of the interactions and thus the probability for a commensurate supersolid is zero. An example of this exceptional situation within the Bose-Hubbard model are the lines $\langle\hat{n}\rangle \equiv n_{0}$ of integer density emanating from the tip of the Mott lobes into the superfluid phase, shown in Fig. 9.

[^15]Supersolid phases in dipolar gases and the roton instability In recent years, supersolid phases with an interaction-generated density modulation along the axial direction of a cigar-shaped trap have been realized with dipolar gases [118-120]. In practice, these are highly inhomogeneous systems with only a few times $10^{4}$ atoms. The modulation of the density in the supersolid phase, which has a typical length of the unit cell of around $0.3 \mu \mathrm{~m}$, splits the BEC into a small number of coherently coupled droplets. The supersolid phase is thus generically incommensurate, with many atoms per unit cell. In the following, we discuss a simple model for dipolar gases in two dimensions [121] which are not confined in the $(x, y)$ - plane and whose motion in the $z$ - direction, along which the dipoles are oriented, is restricted to the lowest single-particle eigenstate. The necessary condition $\mu \ll \hbar \omega_{z}$ is always reached at low densities and it is compatible with the fact that typical values for the transverse confinement lengths $\ell_{z} \simeq 0.5 \mu \mathrm{~m}$ are much larger than the dipolar length $\ell_{d}=m d^{2} / \hbar^{2}$ of order $\ell_{d} \simeq 100 a_{B}$ in present experiments. In practice, the condition $\mu \ll \hbar \omega_{z}$ is not obeyed for dipolar gases in cigar-shaped traps, which are also far from uniform in the plane perpendicular to the orientation of the dipoles. Nevertheless, a number of features like the absence of a homogeneous superfluid with a well-defined roton minimum are common with those of the simplified model system studied in the following. Due to $\ell_{z} \gg\left(\ell_{d}, a_{s}\right)$, the two-body scattering problem requires a solution of the full three-dimensional Schrödinger equation in the presence of both a short-range and the dipolar potential

$$
\begin{equation*}
V(\boldsymbol{x})=V_{\mathrm{sr}}(\boldsymbol{x})+\frac{d^{2}}{r^{3}}\left(1-\frac{3 z^{2}}{r^{2}}\right)-\frac{8 \pi d^{2}}{3} \delta^{(3)}(\boldsymbol{x}) . \tag{46}
\end{equation*}
$$

Here, the attractive delta function term is a dipolar contribution to the short-range interaction which - in contrast to the contribution $V_{\mathrm{sr}}(\boldsymbol{x})$ - is known explicitely in analytical terms. It is specific to the case of magnetic point dipoles and is the analog of the effective contact interaction $\hat{H}^{\prime}=a_{\mathrm{hf}} \hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{I}}$ between the electron and the nuclear spin which gives rise to the hyperfine splitting of s-states in Hydrogen or the Alkalis, see e.g. Ref. [122]. The contribution to the effective two-body interaction $V_{\mathrm{dd}}(\boldsymbol{q})$ that results from projecting the dipolar part of the interaction (46) onto the lowest transverse oscillator level is [123, 124]

$$
\begin{equation*}
V_{\mathrm{dd}}(\boldsymbol{q})=-g_{2}^{\mathrm{dd}} \sqrt{\frac{\pi}{2}}\left(q \ell_{z}\right) \exp \left(q^{2} \ell_{z}^{2} / 2\right) \operatorname{erfc}\left(q \ell_{z} / \sqrt{2}\right) \tag{47}
\end{equation*}
$$

Here, following the notation in Ref. [49], we have introduced a coupling constant $g_{2}^{\mathrm{dd}}=\left(\hbar^{2} / m\right) \tilde{g}_{2}^{\mathrm{dd}}$ with a dimensionless factor $\tilde{g}_{2}^{\mathrm{dd}}=\sqrt{8 \pi} \ell_{d} / \ell_{z}$ for dipolar interactions, which is much less than one in practice. The effective interaction $V_{\mathrm{dd}}(\boldsymbol{q})$ is always negative and approaches the constant value $-g_{2}^{\mathrm{dd}}$ in the limit $q \ell_{z} \gg 1$. In physical terms, this describes attractive head-to-tail collisions between aligned dipoles with an effective 3D scattering length $-\ell_{d}$. In the opposite limit $q \ell_{z} \ll 1$, the projected dipolar interaction $V_{\mathrm{dd}}(\boldsymbol{q}) \rightarrow-2 \pi d^{2}|\boldsymbol{q}|$ vanishes with momentum in a non-analytic manner, reflecting the repulsive $d^{2} / r^{3}$ - potential at distances much larger than $\ell_{z}$. The total momentum-dependent interaction $V_{\mathrm{tot}}(q)=g_{2}+V_{\mathrm{dd}}(q)$, which arises from the combination of a short-range part described by an associated scattering length $a_{s}$ via $g_{2}=\left(\hbar^{2} / m\right) \sqrt{8 \pi} a_{s} / \ell_{z}$ [49] and the magnetic dipolar potential gives rise to a thermodynamically stable low-density gas provided that $a_{s}>0$. Here, stability is understood in the minimal sense that the density response function $\chi(q)$ which describes the change in energy

$$
\begin{equation*}
E\left[\left\{\delta n_{q}\right\}\right]=E_{0}+\frac{1}{2} \int_{q} \chi^{-1}(q)\left|\delta n_{q}\right|^{2}+\ldots \tag{48}
\end{equation*}
$$

associated with small fluctuations $\delta n_{q}$ around a homogeneous fluid state is positive in the limit $q \rightarrow 0$, where $\chi(q) \rightarrow \tilde{\kappa}=\partial n / \partial \mu$. At low densities, $\tilde{\kappa} \rightarrow 1 / g_{2}$ and thus stability is guaranteed by a positive value of the Fourier transform $V_{\text {tot }}(q)$ of the total interaction at $q=0$. In the limit $q \ell_{z} \gg 1$, the interaction approaches a constant proportional to the effective scattering length $a_{s}^{\mathrm{eff}}=a_{s}-\ell_{d}$ for head-to-tail collisions. It becomes negative when the ratio $\varepsilon_{\mathrm{dd}}=\ell_{d} / a_{s}$ between the dipolar and the short-range scattering length is larger than one. The presence of attractively interacting dipoles in a weakly confined configuration with $\ell_{z} \gg \ell_{d}$ is the basic origin for the eventual instability of the homogeneous superfluid into phases with spatial order.

It has been noted in early work on dipolar gases by Santos et al. [125] and O'Dell et al. [126], that the negative Fourier transform of the interaction in a finite range of momenta leads to a roton minimum in the excitation spectrum, superficially similar to the one observed in superfluid ${ }^{4} \mathrm{He}$. Within a Bogoliubov approximation, the gas becomes dynamically unstable towards a spatially ordered phase with a characteristic wave vector $q_{c}$ when the excitation energy $E_{\mathrm{Bog}}(q)=\varepsilon_{q} / S(q)$ vanishes due to a diverging static structure factor $S_{\text {Bog }}\left(q_{c}\right) \rightarrow \infty$. This is the roton instability, which clearly does not occur in ${ }^{4} \mathrm{He}$, where - at the critical pressure $p_{c}$ - the superfluid freezes into a commensurate solid with finite values of the interstitial and vacancy energies by a first order SF-MI transition, Both the dominant peak $S_{c}\left(q_{0}\right) \simeq 1.4$ in the static structure factor at $q_{0} \sigma \simeq 5$ and the roton minimum $\Delta_{\text {rot }}$ are finite at the transition. A vanishing value of $\Delta_{\text {rot }}$ is expected only when the superfluid no longer exists even as a metastable phase, which is estimated to happen at pressures beyond 200 bar [127]. The first order nature of the transition from liquid to solid ${ }^{4} \mathrm{He}$ at the critical pressure $p_{c}$ is in fact a rather general feature of crystallization transitions, as was shown by Landau [128] and, in extended form, by Brazovskii et al. [129]. An empirical criterion which determines the point where the transition from a homogeneous fluid to a state with broken translation invariance occurs has been found by



FIG. 10: Static strucure factor (left) and stability diagram of a weakly confined dipolar Bose gas within a Bogoliubov approximation. The dominant peak $S\left(q_{0}\right)=1.7$ appears at $q_{0} \ell_{z} \simeq 1.3$. In the Figure on the right, the gas is unstable below the blue line. The orange line shows the onset of the roton minimum, which essentially coincides with the Hansen-Verlet criterion marked in green. From Ref. [121].

Hansen and Verlet [130] for classical liquids. It states that freezing appears when the dominant first peak of the static structure factor reaches a critical value $S_{c}\left(q_{0}\right)=2.85$. As discussed by Babadi et al. [131], a modified version of this criterion turns out to determine the limit for the existence of a spatially homogeneous phase also for many quantum fluids at zero temperature. Since configurations with a strongly inhomogeneous density are suppressed in quantum mechanics, the associated critical value $S_{c}\left(q_{0}\right)$ is substantially lower than the classical Hansen-Verlet value. Surprisingly, it does not change much with particle statistics or the specific form of the repulsive interactions. In the particular case of Bose fluids in 2 D with dipolar interactions, the value is $S_{c}\left(q_{0}\right) \simeq 1.7$ [131] while an even smaller value $S_{c}\left(q_{0}\right) \simeq 1.4$ applies to ${ }^{4} \mathrm{He}$ at the critical pressure $p_{c} \simeq 25$ bar [132].

Regarding the transition to the supersolid phase observed in dipolar gases, a fully microscopic description at the level available for ${ }^{4} \mathrm{He}$ [6] or the examples discussed in Ref. [131] does not exist at present ${ }^{16}$. It has been realized in Ref. [121], however, that an approach based on the Hansen-Verlet criterion for freezing of quantum fluids provides some insight even for the transition to the supersolid phase of dipolar gases, which is driven by attractive rather than repulsive interactions and where the emergent phase which breaks translation invariance is an incommensurate supersolid rather than a Mott-insulator. Consistent with the first order nature of crystallization in general, the Hansen-Verlet criterion states that a transition to a phase with an inhomogeneous density appears when the dominant peak in the static structure factor of the fluid has reached a critical value of order one. For the specific case of purely repulsive dipolar interactions in 2 D , the associated value is $S_{c}\left(q_{0}\right) \simeq 1.7$. The relevance of this criterion for weakly confined dipolar gases in the limit $\mu \ll \hbar \omega_{z}$ can be tested easily at the level of a Bogoliubov approximation by noting that the associated static structure factor

$$
\begin{equation*}
S_{\text {Bog }}(q)=\left[1+2 n_{0} V_{\mathrm{tot}}(q) / \varepsilon_{q}\right]^{-1 / 2} \underset{q l_{\gtrless} \gg 1}{\longrightarrow} 1-\frac{4 \sqrt{2 \pi} n_{0} a_{s}^{\mathrm{eff}}}{\ell_{z} q^{2}}+\ldots \tag{49}
\end{equation*}
$$

is completely determined by the effective interaction and the condensate density $n_{0}$. Based on the expression (47) for the momentum dependent interaction, Fig. 10 shows the static structure factor for a dimensionless dipolar interaction strength $\varepsilon_{\mathrm{dd}}=2$ at a density where $n_{0} \ell_{z}^{2} \tilde{g}_{2}^{\text {dd }}=1$. In this parameter regime, the excitation spectrum which - within Bogoliubov theory is given by the single-mode expression $E_{q}=\varepsilon_{q} / S(q)$ for all momenta, is just about to develop a roton minimum. In addition, we show the associated stability diagram. Here, the blue line marks the roton instability, where the excitation energy reaches zero at finite momentum $q_{c}$, determined by a divergent value of the static structure factor 49). Surprisingly, the orange line, where the excitation spectrum $E_{q}$ just starts to develop a roton minimum, essentially coincides with the Hansen-Verlet criterion $S_{c}\left(q_{0}\right) \simeq 1.7$ (green line) for a purely repulsive dipolar gas in 2D. This suggests that - in contrast to the case of ${ }^{4} \mathrm{He}$ - dipolar gases have no homogeneous superfluid phase with a well developed roton minimum: near the point where such a minimum starts to appear, there is a first-order transition to a spatially modulated superfluid where a density wave coexists with long range phase coherence. The roton instability predicted by the Bogoliubov approximation, where the static structure factor diverges at some finite momentum $q_{c}$, is thus preempted by the spatially ordered supersolid phase.

[^16]

FIG. 11: Bragg scattering spectra of a dipolar gas of ${ }^{166} \mathrm{Er}$ in a cigar-shaped trap for different values $\epsilon_{\mathrm{dd}}=\ell_{d} / a_{s}$ of the ratio between the dipolar and the short range scattering length. The upper panel (a-g) shows the variance $\left\langle q_{y}^{2}\right\rangle$ of the momentum along the weakly confined direction which results from the transfer of atoms to finite momenta by the Bragg pulse at the resonance condition $\hbar \omega=E_{q}$. With increasing values of $\epsilon_{\mathrm{dd}}$ the excitation spectrum in panels (b-h) extracted from $\left\langle q_{y}^{2}\right\rangle$ develops a roton minimum near $q \ell_{z} \simeq 1.3$. The Figure is taken from Ref. [134].

Now, of course, the Bogoliubov approximation, where the spectrum of excitations is exhausted by a set of non-interacting quasiparticles at all momenta, is not quantitatively reliable in a regime $\ell_{d} \simeq a_{s}$ where the homogeneous fluid is unstable towards the formation of a density wave, despite the fact that the diluteness parameter $n a_{s}^{3}$ is tiny. Nevertheless, it seems to capture the qualitative behavior up to the point where the supersolid phase appears. This applies, in particular, to the generic form of the static structure factor near the transition shown in Fig. 10 which approaches its asymptotic limit from above in the relevant regime where $a_{s}^{\text {eff }}$ is negative, see Ref. [121] for a further discussion. Moreover, the prediction that the homogeneous superfluid does not exhibit a pronounced roton minimum is consistent with experiments performed with dipolar gases in a cigar-shaped trap, even though the expression (47) for the momentum-dependent interaction does not apply quantitatively there. The evolution of the excitation spectrum as a function of the dimensionless strength $\varepsilon_{\mathrm{dd}}$ of the dipolar interactions has been extracted from Bragg scattering data by Petter et al. [134]. As shown in Fig. 11] the spectrum develops only a rather shallow minimum near $q \ell_{z} \simeq 1.27$ for the largest values $\varepsilon_{\mathrm{dd}} \simeq 1.3$ displayed, which are close to the critical strength $\varepsilon_{\mathrm{dd}}^{(c)}$ for the transition to a supersolid phase with a finite density modulation along the axial direction ${ }^{17}$. The validity and significance of the Hansen-Verlet criterion in this context has been probed in recent measurements of the static structure factor by Hertkorn et al. [135]. By averaging over many in-situ images of the atomic density, the finite-temperature static structure factor $\left.S(\boldsymbol{q}, T)=\left.\langle | \delta n_{q}\right|^{2}\right\rangle(T) / N$ is inferred from the shot-to-shot density fluctuations $\delta n_{q}$. At the relevant temperature $T \simeq 20 \mathrm{nK}$ of the experiment, the peak value near the transition is around $S\left(q_{0}, T\right) \simeq 260$ [135]. To compare with the Hansen-Verlet criterion, an estimate of the critical peak height at zero temperature may be obtained within the Bogoliubov approximation, where the structure factor $S_{\text {Bog }}(q, T)=S_{\text {Bog }}(q) \operatorname{coth}\left(\beta E_{q} / 2\right)$ is related to its zero temperature value by a simple thermal factor $\operatorname{coth}\left(\beta E_{q} / 2\right) \simeq 2 k_{B} T / E_{q}$, which is estimated to be in the range $50-100$ near $q_{0}$. The extrapolated critical peak height $S_{c}\left(q_{0}\right) \simeq 2.6-5.2$ at zero temperature is considerably larger than the value $S_{c}\left(q_{0}\right)=1.7$ [131] which determines the location of the SF-MI transition at high densitites $\left.\sqrt{n} \ell_{d}\right|_{c} \simeq 20$ due to purely repulsive dipolar interactions in 2 D [131, 136]. Given that a number of assumptions enter into the extrapolation of the experimental data to zero temperature, the estimated range $S_{c}\left(q_{0}\right) \simeq 2.6-5.2$ is, however, still compatible with the general concept that an inhomogeneous phase appears beyond a finite critical value $S_{c}\left(q_{0}\right)$ of order one. In particular, the precise value for $S_{c}\left(q_{0}\right)$ in the quantum version of the Hansen-Verlet criterion is expected to deviate from $S_{c}\left(q_{0}\right)=1.7$ since the instability to the supersolid phase studied here is caused by the attractive part of the dipolar interaction. Clearly, a quantitative prediction for $S_{c}\left(q_{0}\right)$ or the critical coupling $\varepsilon_{\mathrm{dd}}^{(c)}$ requires a proper microscopic theory of the transition from the homogeneous superfluid to the supersolid phase. In this context, two major questions are open even at a qualitative level: First of all, one needs to show that the transition to the state with broken translation invariance is indeed of first order as e.g. in the case of the vortex lattice in type-II superconductors [137]. Second, it is open whether the supersolids observed in dipolar gases are genuine supersolids rather than spatially modulated superfluids. Recent experiments by Sohmen et al. [138] provide support for the first option since the periodic order in the density is still present above the superfluid transition.

[^17]Goldstone and hydrodynamic modes of a superfluid smectic In the following, we will discuss the low energy modes of a supersolid within a simple model where the density order appears along a single direction only [139]. This phase may be considered as a superfluid analog of a classic smectic-A liquid crystal. It is an incommensurate solid whose density fluctuations contain two distinct contributions associated with the divergence of the displacement field and the density of defects. Within a hydrodynamic description of the supersolid phase, originally developed by Andreev and Lifshitz [103], defect diffusion as one of the hydrodynamic modes of normal, compressible solids turns turns into a propagating defect density mode. Compared with the translation invariant superfluid, the associated Goldstone mode may be viewed as a generalization of second sound for the propagation of entropy, which now remains well defined down to zero temperature.

In order to elucidate the similarities and differences between standard liquid crystals and the superfluid version of the smectic phase, we start with the case where no superfluidity is present. Specifically, we consider a two-dimensional situation where the smectic order shows up as a periodic modulation

$$
\begin{equation*}
n_{\mathrm{eq}}(\boldsymbol{x})=\bar{n}+\sum_{l=1}^{\infty} n_{l} \cos \left(l q_{0} x\right) \approx \bar{n}+n_{1} \cos \left(q_{0} x\right)+\ldots \tag{50}
\end{equation*}
$$

of the density along the $x$-direction with a fundamental reciprocal lattice vector $q_{0}$, analogous to the form assumed in Eq. (39) with $n_{1} / \bar{n} \simeq 2 \delta$ to linear order in $\delta$. For non-vanishing Fourier components $n_{l} \neq 0$ in Eq. (50), translation invariance along $x$ is broken. The associated new hydrodynamic variable is a scalar field $u(x, y)$ that is called the layer phase [97]. It is defined by considering deviations from the equilibrium density (50) of the form

$$
\begin{equation*}
n(x, y)=\bar{n}+\sum_{l=1}^{\infty} n_{l} \cos \left(l q_{0}[x-u(x, y)]\right) . \tag{51}
\end{equation*}
$$

Within a hydrodynamic description, there are four conserved quantities, which are particle number, the two-dimensional momentum as well as energy. Combined with the single symmetry-breaking variable $u$, there must be five hydrodynamic modes [140]. Only one of them is a Goldstone mode, which counts twice in a hydrodynamic count because it is necessarily a propagating mode. As found by Martin et al. [140], the Goldstone mode of a smectic-A liquid crystal is a transverse sound mode with a frequency $\omega_{t}(\boldsymbol{q}) \simeq \sqrt{B / \rho q^{2}} q_{x} q_{y} \sim \sin \psi \cos \psi$ that depends on the angle $\psi$ between the wave vector $\boldsymbol{q}$ and the direction of density order. Here, $\rho$ is the equilibrium mass density and $B$ the layer compression modulus. It is defined by the elastic contribution $f_{\mathrm{el}}=B\left(u^{\prime}\right)^{2} / 2+\ldots$ to the free energy density associated with small longitudinal distortions $u^{\prime}=\partial_{x} u$ of the smectic order [97]. The second propagating mode is a bulk sound mode $\omega= \pm c_{l} q$ whose velocity $c_{l}(\psi)$ has only a weak dependence on the angle $\psi$. In particular, for longitudinal propagation, its velocity $c_{l}^{2}(\psi=0)=(K+B) / \rho$ is determined by the sum of the isentropic bulk modulus $K=\rho \partial p /\left.\partial \rho\right|_{s, u^{\prime}}$ and the layer compression modulus $B$ [140]. The last remaining mode in addition to the Goldstone mode and the sound mode describes heat diffusion. Now, a special situation arises for a wave vector $\boldsymbol{q}$ which is directed either along or perpendicular to the $x$-direction. Here, due to the peculiar angular dependence $c_{t}(\psi) \sim \sin \psi \cos \psi$ of the transverse sound velocity, the Goldstone mode is absent. By mode counting, there must then be three diffusive modes in addition to the propagating bulk sound mode. The first one is the heat diffusion mode that is present at arbitrary values of the angle $\psi$. The second one is a transverse momentum diffusion mode with frequency $\omega=-i v q^{2}$, where $v$ is a kinematic viscosity [97]. The third mode with frequency $\omega=-i D_{p} q^{2}$ is special to smectic-A liquid crystals and is called the permeation mode [97]. It describes a diffusive process in which particles are exchanged between adjacent layers without changing the average periodic structure. The associated diffusion constant $D_{p}=\zeta B$ is determined by the layer compression modulus $B$ and a dissipative coefficient $\zeta$. The permeation mode may be viewed as an analog of defect diffusion, a process that gives rise to an independent hydrodynamic mode in any crystal [140]. As will be shown below, it is precisely the permeation mode in combination with the heat diffusion mode that turns into the Goldstone mode of the superfluid smectic phase, where exchange between the layers occurs in a reversible manner by non-dissipative, propagating mass currents.

For a description of the hydrodynamics of a superfluid smectic phase, the presence of superfluidity needs to be accounted for on a thermodynamic level by expressing the differential of the entropy density $s$

$$
\begin{equation*}
T d s=d \varepsilon-(\mu / m) d \rho-\boldsymbol{v}_{n} d \boldsymbol{g}-\boldsymbol{h} d(\nabla u)-\boldsymbol{j}_{s} d \boldsymbol{v}_{s} \tag{52}
\end{equation*}
$$

as a function of the conserved variables energy density $\varepsilon$, mass density $\rho$, and momentum density $\boldsymbol{g}$, together with the gradient $\nabla u$ of the layer phase and the superfluid velocity $\boldsymbol{v}_{s}$ which characterize the two broken symmetries. In the superfluid smectic, both are $U(1)$ symmetries and may therefore be derived from compact angular variables. One of them is the standard phase $\varphi(\boldsymbol{x})$ of the complex order parameter whose gradient determines the superfluid velocity $\boldsymbol{v}_{s}=(\hbar / m) \nabla \varphi$. Regarding smectic order, Eq. (51) shows that $q_{0} u(x)$ and $q_{0} u(x)+2 \pi$ give rise to identical distortions. Density fluctuations in the smectic are therefore
described by an angle $\varphi_{\mathrm{sm}}(\boldsymbol{x})=q_{0} u(\boldsymbol{x})$ which coincides with the layer phase up to a trivial factor ${ }^{18}$. The role which is analogous to the superfluid current density $\boldsymbol{j}_{s}$ associated with $\boldsymbol{v}_{s}$ is played by the thermodynamic field

$$
\begin{equation*}
\boldsymbol{h}=\left.\frac{\partial f_{\mathrm{el}}}{\partial(\nabla u)}\right|_{T, A, N, v_{n}, v_{s}}=B u^{\prime} \boldsymbol{e}_{x}-K_{1} \partial_{y}^{3} u \boldsymbol{e}_{y}+\ldots \tag{53}
\end{equation*}
$$

conjugate to the gradient $\nabla u$ of the layer phase. It determines the elastic free energy of the smectic. Here and in the following, we use the notation $u^{\prime}=\partial_{x} u$ for the derivative of the layer phase in the direction of the periodic modulation. Note that for longitudinal modes, only the layer compression modulus $B$ plays a role. A second Gaussian-curvature type elasticity appears in Eq. 123 for excitations with a finite component of the wave vector parallel to the layers. It involves the splay elastic constant $K_{1}$ [97], which becomes relevant for the dispersion of the Goldstone and hydrodynamic modes with wave vectors at finite angles to the direction of ordering [139]. The conjugate variable to the momentum density $\boldsymbol{g}$ is the normal velocity $\boldsymbol{v}_{n}$, which also appears in the superfluid mass current density $\boldsymbol{j}_{s}=\rho_{s}\left(\boldsymbol{v}_{s}-\boldsymbol{v}_{n}\right)$. Quite generally, for superfluids with an underlying periodic structure, the normal velocity $\boldsymbol{v}_{n}=\partial_{t} \boldsymbol{u}$ is determined by the time derivative of the displacement field $\boldsymbol{u}$. This relation, which is valid at the linearized level around equilibrium and is thus sufficient for the derivation of the hydrodynamic modes, has been derived in general form by Son [142] as a consequence of Galilean invariance. In particular, for the superfluid smectic, the deformation field is a scalar. The normal velocity $\left(\boldsymbol{v}_{n}\right)_{x}=v_{n}=\partial_{t} u$ thus has only a component along the $x$-direction which is just the time derivative of the layer phase variable $u$.

The complete set of hydrodynamic modes in a superfluid smectic phase follows from the equations of motion for the conserved densities $\rho, \boldsymbol{g}$ and $\varepsilon$ together with the two effectively scalar variables $\nabla u$ and $\boldsymbol{v}_{s}$ that describe the underlying broken symmetries. The six resulting equations of motion are given by

$$
\begin{align*}
& \partial_{t} \rho+\nabla \cdot \boldsymbol{g}=0  \tag{54}\\
& \partial_{t} g_{i}+\partial_{j} \Pi_{i j}=0  \tag{55}\\
& \partial_{t} \varepsilon+\nabla \cdot \boldsymbol{j}^{\varepsilon}=0  \tag{56}\\
& \partial_{t}(\nabla u)-\nabla v_{n}=0  \tag{57}\\
& \partial_{t} \boldsymbol{v}_{s}+\nabla \mu / m=0 . \tag{58}
\end{align*}
$$

The first three equations (54)-(56) are the standard continuity equations that link the time derivatives of the densities to the divergences of the momentum density $\boldsymbol{g}$, the stress tensor $\Pi_{i j}$, and the energy current $\boldsymbol{j}^{\varepsilon}$, respectively. Eq. (57) expresses the fact that a constant shift along the direction of smectic order changes the layer phase by a constant. More generally, if irreversible effects are included, the right-hand side in this equation no longer vanishes and contains a contribution $\zeta \nabla^{2} h$, where $\zeta$ is the dissipative coefficient that enters the diffusion constant $D_{p}=\zeta B$ of the permeation mode [97]. Finally, Eq. (58] is the Josephson equation (neglecting a quadratic term in the velocities) that describes the dynamics of the superfluid phase. From the differential of the entropy $(52)$ and the dynamic equations $(54]-(58)$, one obtains an expression for the material derivative $T\left(\partial_{t} s+\boldsymbol{v}_{n} \cdot \nabla s\right)$ of the entropy density that depends on spatial gradients $\nabla T, \nabla \mu, \partial_{i} v_{n, j}$ and $\nabla \cdot \boldsymbol{j}_{s}$ of the thermodynamic forces. For the inviscid fluid considered here there is no entropy production, which implies a series of constitutive relations that link the currents and the thermodynamic forces. To leading order in the velocities, these constitutive relations read:

$$
\begin{align*}
\boldsymbol{g} & =\rho \boldsymbol{v}_{n}+\boldsymbol{j}_{s} \rightarrow \boldsymbol{g}=\underline{\rho_{n}} \boldsymbol{v}_{n}+\underline{\rho_{s}} \boldsymbol{v}_{s} \text { since } \boldsymbol{j}_{s}=\underline{\rho_{s}}\left(\boldsymbol{v}_{s}-\boldsymbol{v}_{n}\right)  \tag{59}\\
\Pi_{i j} & =p \delta_{i j}-\delta_{i, x} h_{j} \rightarrow\left(\begin{array}{cc}
p-B u^{\prime} & 0 \\
0 & p
\end{array}\right) \text { since } \boldsymbol{h}=B u^{\prime} \boldsymbol{e}_{x}  \tag{60}\\
\boldsymbol{j}^{\varepsilon} & =(\varepsilon+p) \boldsymbol{v}_{n}+\mu \boldsymbol{j}_{s} / m . \tag{61}
\end{align*}
$$

The fact that the explicit form of the currents in equations (54)-56) are completely fixed by the condition of a vanishing entropy production is quite important from a conceptual point of view: it shows that - apart from dissipative coefficients like viscosity or heat conductivity which lead to irreversible damping of the modes - hydrodynamics shares the same level of universality than equilibrium thermodynamics itself. For the specific case of the superfluid smectic, the only information needed is the equation of state together with the elastic constant $B$ and the superfluid mass-density tensor $\underline{\rho_{s}}$, which determine the conjugate fields for the spatial respectively the superfluid order in the thermodynamic relation (52).

[^18]The linearized hydrodynamic equations of motion are obtained by substituting the constitutive relations in the dynamic equations and expanding the thermodynamic forces to leading order in the hydrodynamic variables around equilibrium. For motion along the direction of the smectic order, the resulting equations only involve the $x x$ - component of the superfluid mass density tensor, which we denote by $\rho_{s}=\rho-\rho_{n}$. In this configuration, the transverse momentum degree of freedom decouples and gives rise to a diffusion mode which - at the level of reversible hydrodynamics - appears at $\omega=0$. For the remaining five degrees of freedom, the characteristic equation reads

$$
\left(\begin{array}{ccccc}
-\omega / q & 1 & 0 & 0 & 0  \tag{62}\\
K / \rho & -\omega / q & 0 & 0 & -B \\
0 & \tilde{s} T \frac{\rho_{s}}{\rho_{n}} & -\omega / q & -\rho \tilde{s} T \frac{\rho_{s}}{\rho_{n}} & 0 \\
K / \rho^{2} & 0 & -\tilde{s} / \rho c_{V} & -\omega / q & 0 \\
0 & 1 / \rho_{n} & 0 & -\rho_{s} / \rho_{n} & -\omega / q
\end{array}\right)\left(\begin{array}{l}
\delta \rho \\
g_{L} \\
\delta q \\
v_{s} \\
u^{\prime}
\end{array}\right)=0 .
$$

Here, $\omega$ is the frequency of the mode and $q$ the associated longitudinal momentum. Moreover, we introduce a heat current density variable $\delta q=\delta \varepsilon+\frac{\varepsilon+p}{\rho} \delta \rho$ with $\tilde{s}=s / \rho$ the entropy per particle and mass, while $c_{V}=\left.T \partial_{T} \tilde{s}\right|_{\rho}$ is the associated specific heat. Apart from the transverse momentum diffusion mode mentioned above, Eq. (62) contains another diffusive zero mode with eigenvector $\left(\delta \rho, g_{L}, \delta q, v_{s}, u^{\prime}\right)=\left(-B \rho / K, 0,-B c_{V} / \tilde{s}, 0,1\right)$. In the absence of superfluidity, this mode splits into two separate diffusion modes, one that describes heat diffusion, and one permeation mode that involves an interchange between the layer phase and the particle density. In the superfluid smectic phase, only the above combination remains, while an orthogonal complement will couple to the superfluid velocity and give rise to a propagating sound mode. The condition of a vanishing determinant of Eq. (62) which determines the propagating longitudinal hydrodynamic modes reads

$$
\begin{equation*}
\omega^{4}-\omega^{2} q^{2}\left[\frac{K}{\rho}+\frac{B}{\rho_{n}}+\bar{c}^{2}(T)\right]+q^{4}\left[\frac{B K}{\rho^{2}} \frac{\rho_{s}}{\rho_{n}}+\bar{c}^{2}(T) \frac{K-B}{\rho}\right]=0 \text { where } \bar{c}^{2}(T)=\frac{\rho_{s}}{\rho_{n}} \cdot \frac{\tilde{s}^{2} T}{c_{V}} . \tag{63}
\end{equation*}
$$

In the absence of superfluidity, the parameter $\bar{c}^{2}(T)$ - which has dimensions of a velocity squared - vanishes. Eq. (63) then reproduces the longitudinal compression mode of a smectic-A liquid crystal with velocity $c_{l}^{2}=(K+B) / \rho$ mentioned above. For a vanishing value of the layer compression modulus $B=0$, in turn, the equation describes first and second sound in a homogeneous superfluid as discussed in textbooks, e.g. the one by Pitaevskii and Stringari [143]. In particular, at low temperatures, the parameter $\bar{c}^{2}(T) \rightarrow c_{s}^{2} / d$ approaches a finite constant which is determined by the sound velocity $c_{s}$ of the superfluid at zero temperature and the dimension $d$. Due to $K / \rho \rightarrow c_{s}^{2}$, Eq. 63) then gives rise to two sound modes with velocities $c_{1}(T=0)=c_{s}$ and $c_{2}(T=0)=c_{s} / \sqrt{d}$. The question which of the two is the Goldstone mode associated with the broken gauge symmetry turns out to be subtle. The obvious answer seems to be it is second sound because this is the new mode which arises from a finite value of $\rho_{s}$ compared to the normal fluid, where only compression waves exist. In the context of ultracold gases, however, it is standard to identify the Goldstone mode associated with the broken gauge symmetry with the Bogoliubov mode $\omega_{q}=c_{s} q$, whose velocity $c_{1} \rightarrow c_{s}$ is quite generally determined by the superfluid density $n_{s}$ and the compressibility $\tilde{\kappa}=\partial n / \partial \mu$ by $m c_{s}^{2}=n_{s} / \tilde{\kappa}$. Indeed, this is the only mode which is present in the limit of zero temperature since second sound, as a counter-oscillation between the superfluid and the normal component at constant pressure, no longer exists because there is no remaining normal component. In a homogeneous superfluid, second sound thus exists as a hydrodynamic, collision dominated mode only in a finite range of temperatures below $T_{c}$. There is an avoided crossing between first and second sound at a characteristic temperature $T^{*}$ first discussed by Lee and Yang [144] ${ }^{19}$. For much lower temperatures only first sound remains which is, however, of a quite different nature than the standard compression mode in the normal fluid. As a proper Goldstone mode, it describes an elementary excitation associated with exact low-lying eigenstates of the many-body system, as exemplified in the Bogoliubov Hamiltonian (14), rather than a hydrodynamic, collision-dominated mode. In the supersolid phase, it turns out that also the second mode with velocity $c_{2}$ which arises from the solution of Eq. 63] remains well-defined at zero temperature. It is precisely the additional Goldstone mode associated with the broken symmetry variable $\nabla u$.

In the superfluid smectic phase, where the normal fluid fraction $f_{n} \geq 2 \delta^{2}$ is bounded below by the square of the density modulation $\delta \neq 0$ according to Eq. (38), the ratio $\rho_{s} / \rho_{n}$ remains finite as the temperature approaches zero. The thermodynamic factor $\tilde{s}^{2} T / c_{V}$, in turn, vanishes. In contrast to the homogeneous superfluid, the parameter $\bar{c}^{2}(T)$ in Eq. 63 ) therefore disappears at low temperatures. As a result, one obtains two undamped propagating modes $\omega= \pm c_{1,2} q$ with velocities

$$
\begin{equation*}
c_{1,2}^{2}=\frac{K}{2 \rho}+\frac{B}{2 \rho_{n}} \pm \frac{1}{2}\left[\left(\frac{K}{\rho}+\frac{B}{\rho_{n}}\right)^{2}-4 f_{s} \frac{K B}{\rho \rho_{n}}\right]^{1 / 2} . \tag{64}
\end{equation*}
$$

[^19]For an understanding of the physics underlying these two modes and in particular the connection to the classic picture of supersolids in terms of wave-like propagation of defects developed by Andreev and Lifshitz [103], it is instructive to rederive their velocities and, in particular, the associated eigenvectors with a slightly different set of variables introduced by Yoo and Dorsey [145]. They decompose small fluctuations of the mass density

$$
\begin{equation*}
\delta \rho=-\rho u^{\prime}+\delta \rho_{\Delta} \tag{65}
\end{equation*}
$$

into a contribution $-\rho u^{\prime}$ associated with deformations of the periodic structure and an additional defect density $\rho_{\Delta}$. This separates the density variation of a defect free crystal, for which a change in density is tied to the divergence of the deformation field, from the additional density change associated with the motion of defects. The defect density obeys a continuity equation $\partial_{t} \delta \rho_{\Delta}=-\partial_{x} \rho_{s}\left(v_{s}-v_{n}\right)$ whose conserved current $g_{\Delta}=\rho_{s}\left(v_{s}-v_{n}\right)$ is just the Galilean-invariant superfluid mass-current density. The second time derivative of the defect density is coupled to the strain field variable $u^{\prime}$ according to

$$
\begin{equation*}
\partial_{t}^{2} \delta \rho_{\Delta}=\rho_{s} \partial_{x}^{2}(\mu / m)+\rho_{s} \partial_{t}^{2} u^{\prime} . \tag{66}
\end{equation*}
$$

In a situation where the lattice is almost rigid, the contribution that involves the layer phase variable $u^{\prime}$ may be neglected. Using $K=\rho^{2} \partial(\mu / m) /\left.\partial \rho\right|_{s, u^{\prime}}$ for the bulk modulus, the defect density thus obeys a wave equation with a velocity given by $c_{4}^{2}=f_{s}(K / \rho)$. This is analogous to fourth sound of superfluid ${ }^{4} \mathrm{He}$ in narrow capillaries, where the normal fluid component is pinned by the walls. It describes the oscillation of the superfluid with no motion of the lattice. This limit is perfectly realized in the superfluid phase of Bosons in an optical lattice. As was mentioned in the discussion of the Bose-Hubbard model above, the superfluid fraction $f_{s}$ in a deep optical lattice is exponentially small. The sound velocity in the superfluid phase is therefore tiny compared to that in a homogeneous BEC. An explicit result can be derived in the limit of small repulsive interactions, where $f_{s} \rightarrow m / m_{B}$ is determined by the ratio between the bare and the band mass $m_{B}$ while $K / \rho \rightarrow \mu^{\mathrm{MF}} / m=U\langle\hat{n}\rangle / m$. As a result, one finds $c_{4}^{2} \rightarrow U\langle\hat{n}\rangle / m_{B}$, which directly connects the smallness of the sound velocity in an optical lattice to the large band mass. Note that there is no analog of first sound in an optical lattice, which is fixed externally. The variable $u^{\prime}$ therefore disappears and the density fluctuations in Eq. (65) only arise from defects, i.e. the particles added on top of the lattice.

An explicit result for the eigenmodes of the superfluid smectic phase requires to simultaneously solve Eq. $\sqrt{66}$ for the defect density and a corresponding one for the layer phase, which reads

$$
\begin{equation*}
\rho_{n} \partial_{t}^{2} u^{\prime}=\partial_{x}^{2}\left[-p+B u^{\prime}+\rho_{s}(\mu / m)\right] \tag{67}
\end{equation*}
$$

The solution of the coupled equations and (67) does of course reproduce the result above. The associated dimensionless eigenvectors are

$$
\begin{equation*}
\binom{\delta \rho_{\Delta} / \rho}{u^{\prime}}_{1}=\binom{c_{2}^{2} /(K / \rho)}{1} \quad \text { and } \quad\binom{\delta \rho_{\Delta} / \rho}{u^{\prime}}_{2}=\binom{c_{1}^{2} /(K / \rho)}{1} \tag{68}
\end{equation*}
$$

for the first and second sound modes with speed $c_{1}$ and $c_{2}$, respectively. Specifically, for an almost rigid lattice with $B / \rho_{n} \gg K / \rho$, the velocities reduce to $c_{1}^{2}=B / \rho_{n}+f_{n} K / \rho$ and $c_{2}^{2}=f_{s} K / \rho$ with $c_{1} \gg c_{2}$. In this limit, second - or better fourth - sound is essentially a defect density mode with no involvement of the lattice. By contrast, the eigenvector $\left(\delta \rho_{\Delta} / \rho, u^{\prime}\right)_{1}=\left(f_{s}, 1\right)$ for first sound in this limit involves the defect density with weight $f_{s}$. In supersolids with a rather small superfluid fraction $f_{s} \ll 1$, this mode predominantly involves the strain field, i.e., it describes the motion of the lattice. A rather different situation arises in the opposite limit of a small normal fraction $f_{n} \ll 1$ on top of a dominant homogeneous superfluid. Formally, in the limit $f_{n} \simeq\left(n_{1} / \bar{n}\right)^{2} \rightarrow 0$ of a vanishing density modulation, the contribution $B / \rho_{n}$ in Eq. (64) appears to diverge. This is not the case, however, since the elastic constant $B$ approaches zero as well. The way it does has been discussed in the context of the nematic-to-smectic-A transition of normal liquid crystals [97]. Within a mean-field approximation, the layer compression modulus $B \sim\left|n_{1}\right|^{2}$ vanishes like the square of the order parameter $n_{1}$. The ratio $B / \rho_{n}$ then turns out to be finite in the limit $n_{1} \rightarrow 0$ where the smectic order disappears. In particular, in the limit $K / \rho \gg B / \rho_{n}$ of a weak density modulation, the velocities (64) approach $c_{1}^{2}=(K+B) / \rho$ and $c_{2}^{2}=B / \rho_{n}$. The velocity of the compression mode is thus unchanged compared to that in the normal phase. In terms of the variables $\delta \rho_{\Delta} / \rho$ and $u^{\prime}$, the eigenvector associated with first sound is dominated by the layer phase variable with a negligible contribution from the defect density. Due to $u^{\prime} \simeq-\delta \rho / \rho$, the periodic structure of the smectic therefore adiabatically follows the density fluctuations $\delta \rho$ in this mode, which describes oscillations of the lattice. The second sound mode, by contrast, whose velocity $B / \rho_{n}$ is determined by the ratio of the layer compression modulus $B$ and the normal fluid density, involves both an oscillation in the longitudinal strain field as well as the defect density with essentially equal magnitude. In physical terms, it describes a wave-like propagation of particles in addition to that associated with variations in the smectic lattice structure, replacing the diffusive permeation mode of a normal smectic phase.


FIG. 12: Axial excitation spectrum of a dipolar gas of ${ }^{162} \mathrm{Dy}$ in a cigar-shaped trap as a function of the ratio $\varepsilon_{\mathrm{dd}}=\ell_{d} / a_{s}$ between the dipolar and the short range scattering length. In the homogeneous BEC (blue) the frequency agrees well with the result $\omega_{B}=\sqrt{5 / 2} \omega_{x}$ for the axial breathing mode of a BEC with short-range interactions. In the supersolid, the mode splits into two separate excitations. The frequency of the lower one disappears at the transition to a phase with independent droplets (red). The dotted lines have been obtained from a numerical solution of a time-dependent Gross-Pitaevskii equation including the dipolar interaction plus LHY-corrections. The Figure is taken from Ref. [146].

For a description of the low-energy excitations in the supersolid phase of dipolar gases in a cigar-shaped trap, where the uni-directional nature of the density wave is a consequence of the specific confining potential, the homogeneous model above is of course not directly applicable. Nevertheless, the analytical results for this idealized model provide some insight into the excitations of the supersolid which is independent of both the detailed trapping geometry and the description of the supersolid phase at a microscopic level. In particular, it should be emphasized that Eq. 64] for the sound velocities in a supersolid in no way depends on the assumption of translation invariance along the direction perpendicular to the density wave. In fact, as discussed by Yoo and Dorsey [145], it also applies for the longitudinal modes of supersolids in two or three dimensions. Experimentally, a characteristic signature of the supersolid state in a trap compared to the standard BEC is the emergence of an additional collective mode at low energies [146-148]. Specifically, as observed by Tanzi et al. [146] and shown in Fig. 12, the axial breathing mode of a trapped BEC with frequency $\omega_{B}=\sqrt{5 / 2} \omega_{x}$ shifts towards higher frequencies beyond the transition to a supersolid with a finite density modulation. In addition, a new mode appears whose frequency goes down as the density contrast increases. This observation can be understood on a qualitative level within the hydrodynamic description of a homogeneous system by noting that the lowest value $q_{\min } \simeq 1 / l_{x}$ of the longitudinal wave vector in the trap is set by the inverse of the axial confinement length $l_{x}$. In a trap, therefore, the splitting of the Bogoliubov mode of a homogeneous BEC into two independent propagating modes in the supersolid phase found in Eq. 64) shows up as a bifurcation into a compressional mode at $\omega_{1} \simeq c_{1} / l_{x}$ which - at least for a weak density modulation - now involves an oscillation of the lattice structure and a new second mode at lower frequencies $\omega_{2} \simeq c_{2} / l_{x}$ in which both the lattice variable $u^{\prime}$ and the defect density $\delta \rho_{\Delta}$ oscillate with essentially equal magnitude. With increasing depth of the density modulation, the frequency $\omega_{1}$ is shifted upwards due to the growing contribution of the layer compression modulus $B$ in the associated sound velocity $c_{1}$. By contrast, the lower frequency $\omega_{2}$ decreases upon entering more deeply into the supersolid phase since the velocity $c_{2}$ goes down with the superfluid fraction $f_{s}$. In particular, near the transition of the supersolid to a crystal of droplets, $c_{2}=\sqrt{f_{s} K / \rho}$ is expected to vanish proportional to the square root of the superfluid fraction $f_{s} \rightarrow 0$. A trap analog of the true Goldstone mode, which is properly defined only in a homogeneous system, has also been seen in experiments by Guo et al. [147] in a small array of three droplets. In the supersolid regime, the dipole mode associated with the axial motion of the whole cloud remains fixed at the trap-frequency. In addition, there is an out-of-phase mode whose frequency is much smaller than that of the trap and in which the oscillation of the periodic background is compensated by a corresponding one in the superfluid to maintain a fixed center-of-mass. A theoretical analysis of the full mode spectrum in a trap has been given by Hertkorn et al. [149] based on a numerical solution of the time-dependent Bogoliubov - de Gennes equations.

As a final remark, we stress that the possibility of a supersolid phase in dipolar gases relies on the fact that they form an incommensurate mass density wave. The inverse of the volume $v_{\text {cell }}$ of a unit cell thus differs from the full density $n$ by a finite defect contribution $n_{\Delta}=n-v_{\text {cell }}^{-1}$ [97]. It is only the defects which may be associated with a non-vanishing superfluid density. As shown above, for a finite $\rho_{s}$, the coupling between the lattice deformation and the defect density fluctuations in Eq. (65) gives rise to separate Goldstone modes associated with the simultaneous breaking of gauge and translation invariance. By contrast, as emphasized by Anderson [150], (see section 4.C) the ground state of a genuine solid like ${ }^{4} \mathrm{He}$ above the critical pressure or even a non-periodic rigid structure like glass has $n_{\Delta} \equiv 0$ since it is characterized by a finite energy gap for free motion of a single atom relative to the background. This is the defining property of a Mott-insulator, for which superfluidity is excluded, consistent with the original argument against the existence of supersolids by Penrose and Onsager.

## III. PROBING ULTRACOLD GASES AT SHORT DISTANCES

Pair distribution function at short distance In a classic paper by Kagan et al. [151] it was pointed out that losses due to twobody or three-body collisions will be suppressed quite strongly at the transition to the superfluid phase, as was indeed observed after BEC had been realized in ultracold gases [152]. The effect arises from the reduction of the local values $g^{(2)}(0)$ and $g^{(3)}(0)$ of the two- or three-body correlation function from $g^{(l)}(0)=l!$ in the non-condensed phase to $g_{\mathrm{BEC}}^{(l)}(0)=1$ in an ideal BEC. Quite generally, losses due to $l$-body decay give information about the probability that $l$ Bosons are close together. The question that will be discussed in the following is how these short distance correlations evolve in the presence of interactions. Now, at the level of a Gross-Pitaevskii approximation, the pair distribution function - and in fact all higher order correlations - is equal to $g^{(2)}(x) \equiv 1$ at arbitrary distances. Formally, this may be seen by using the exact connection

$$
\begin{equation*}
n\left[g^{(2)}(\boldsymbol{x})-1\right]=\int_{q} \exp (i \boldsymbol{q} \boldsymbol{x})[S(\boldsymbol{q})-1] \tag{69}
\end{equation*}
$$

between the pair distribution function and the static structure factor and recalling that Eq. 18 gives $S^{\mathrm{GP}}(q) \equiv 1$ at the Gross-Pitaevskii level, where $\lambda_{q}^{\mathrm{GP}}=0$. Including the coherent coupling between particles in the condensate and in the depletion discussed in Lecture I, the behavior of $g^{(2)}(\boldsymbol{x})$ at large distances $r>\xi$ is changed into the dependence given in Eq. (13). As pointed out above, this is an exact result valid for arbitrary interaction strength. For short distances, the asymptotic dependence $S_{\text {Bog }}(q \xi \gg 1)=1-1 /\left(q^{2} \dot{\xi}^{2}\right)$ obtained by approximating $\Delta_{q}=n_{0} V(q)$ by a constant gives rise to a behavior $g_{\text {Bog }}^{(2)}(r<\xi)=1-2 a / r$ which becomes negative at distances smaller than twice the scattering length. Apparently, the pair distribution function at short distances depends on the specific form of the interaction at large wave vectors and thus the local value $g^{(2)}(0)$ is ill-defined in general. An exact result $g_{\text {LHY }}^{(2)}(r<\xi)=(1-a / r)^{2}$ for the pair distribution function has been obtained by Lee, Huang and Yang [153] for the special case of a gas of hard spheres in the dilute limit $\left(n \sigma^{3}\right)^{1 / 2} \ll 1$. It coincides with the square of the scattering solution $1-a / r$ of the two-body Schrödinger equation at zero energy and is consistent with the result of the Bogoliubov approximation to linear order in the scattering length $a \doteq \sigma$, properly staying positive for arbitrary separations beyond the hard sphere diameter $\sigma$. In fact, the expression $g_{\mathrm{HS}}^{(2)}\left(r \rightarrow \sigma^{+}\right)=(1-\sigma / r)^{2}+\ldots$ is the low density limit of an exact result which will be derived for hard sphere interactions in Eqs. (156) and (157) below.

Somewhat surprisingly, a deeper understanding of two-particle correlations for ultracold gases with short range interactions came only through the independent work by Tan [154] and by Zhang and Leggett [155] in the context of strongly interacting twocomponent Fermi gases and the subsequent realization of the connection to the operator product expansion in QFT by Braaten and Platter [156]. In particular, it turns out that for quantum many-body systems with short range interactions, there is a well defined scaling limit where the interaction range is taken to zero at a fixed value of the scattering length. In this limit, the pair distribution function $g^{(2)}(0)$ at vanishing separation is ill-defined and must be replaced by the dimensionful Tan contact density $C_{2}$ defined in Eq. 72 below. For a derivation of the resulting short distance behavior of the pair distribution function for arbitrary interaction strength and temperature, we adapt an argument used by Zhang and Leggett [155] in the case of two-component Fermi gases and consider the spectral representation

$$
\begin{equation*}
\left\langle\hat{\psi}^{\dagger}\left(\boldsymbol{x}_{1}\right) \hat{\psi}^{\dagger}\left(\boldsymbol{x}_{2}\right) \hat{\psi}\left(\boldsymbol{x}_{2}\right) \hat{\psi}\left(\boldsymbol{x}_{1}\right)\right\rangle=\sum_{n} \lambda_{n}^{(2)}\left|\Phi_{n}(\boldsymbol{x}, \boldsymbol{R})\right|^{2} \tag{70}
\end{equation*}
$$

of the diagonal elements of the exact two-particle density matrix $\hat{\rho}_{2}$. Its eigenvalues $\lambda_{n}^{(2)}$ can be interpreted as the number of pairs of Bosons which are found in a two-particle wave function $\Phi_{n}(\boldsymbol{x}, \boldsymbol{R})$ expressed in terms of center-of-mass and relative coordinates $\boldsymbol{R}=\left(\boldsymbol{x}_{1}+\boldsymbol{x}_{2}\right) / 2$ and $\boldsymbol{x}=\boldsymbol{x}_{1}-\boldsymbol{x}_{2}$. For separations $\boldsymbol{x}$ which are smaller than the distance to all other atoms, the dependence of the exact eigenfunctions $\Phi_{n}(\boldsymbol{x}, \boldsymbol{R})$ on $r=|\boldsymbol{x}|$ is expected to be determined by two-body physics, as observed above for $g_{\text {LHY }}^{(2)}(r \ll \xi)$. This suggests that the $\Phi_{n}$ at short distances may be factorized according to

$$
\begin{equation*}
\Phi_{n}(\boldsymbol{x}, \boldsymbol{R}) \underset{|\boldsymbol{x}| \rightarrow 0}{\longrightarrow} c_{n}(\boldsymbol{R}) \psi_{0}(r) /(4 \pi), \tag{71}
\end{equation*}
$$

where the prefactor in the zero energy solution $\psi_{0}(r)=1 / r-1 / a$ of the two-particle problem and the $1 / 4 \pi$ are choosen for later convenience. The many-body nature of the problem is hidden in the complex numbers $c_{n}(\boldsymbol{R})$ which depend both on the microscopic parameters and also on temperature, since Eq. 70 holds for an arbitrary equilibrium state. The weighted sum of their squares defines the so called two-body contact density

$$
\begin{equation*}
C_{2}(\boldsymbol{R})=\sum_{n} \lambda_{n}^{(2)}\left|c_{n}(\boldsymbol{R})\right|^{2} . \tag{72}
\end{equation*}
$$

In a translation invariant situation, $C_{2}$ is independent of the center-of-mass coordinate since $\Phi_{n}(\boldsymbol{x}, \boldsymbol{R}) \sim \exp i \boldsymbol{Q} \boldsymbol{R}$ reduces to plane waves with respect to $\boldsymbol{R}$. In this case, Eqs (70) and (71) lead to a singular behavior

$$
\begin{equation*}
n^{2} g^{(2)}(r \rightarrow 0)=\frac{C_{2}}{16 \pi^{2}}\left(\frac{1}{r^{2}}-\frac{2}{a r}+\ldots\right) \underset{n a^{3} \ll 1}{\longrightarrow} n^{2}\left[\left(\frac{a}{r}\right)^{2}-2\left(\frac{a}{r}\right)+\ldots\right] \tag{73}
\end{equation*}
$$

of the pair distribution function at short distance and zero temperature. This quite general result reduces to the one obtained by LHY if the contact density is replaced by its weak coupling value $C_{2} \rightarrow(4 \pi n a)^{2}$ but it now applies for arbitrary interaction strength and also below $r=a$. Note, however, that the term of order $r^{0}$ from $\left|\psi_{0}(r)\right|^{2}$ in the expansion $\sqrt{73 \mid}$ must be omitted.


A surprising consequence of the singular behavior 73 at short distances follows from the connection $\sqrt[69]{ }$ between $g^{(2)}(x)$ and the static structure factor, which approaches its limiting value one at large momenta by a slow $\mathcal{C}_{2} / q$-decay according to

$$
\begin{equation*}
S(q \xi \gg 1)=1+\frac{C_{2}}{8 n}\left(\frac{1}{q}-\frac{4}{\pi a q^{2}}+\ldots\right) . \tag{75}
\end{equation*}
$$

This is quite different from the result of Bogoliubov theory at zero temperature, where $S_{\text {Bog }}(q \xi \gg 1)=1-1 /\left(q^{2} \xi^{2}\right)+\ldots$ approaches its trivial limit one from below. The Bogoliubov approximation which - as emphasized in Lecture I is exact in the limit $q \xi \ll 1$ - therefore misses the leading positive contribution $C_{2} / q$ to the static structure factor at large momentum. In fact, in the context of two-component Fermi gases with large scattering lengths, it is precisely the slow $\mathcal{C}_{2} / q$-decay that has been used to measure the contact density [157, 158]. Note, however, that independent of the value of $C_{2}$, the negative contribution $-1 /\left(q^{2} \xi^{2}\right)$ found within a Bogoliubov approach dominates the leading term in the conventionally accessible regime $q a \ll 1$.

Tan relations and the operator product expansion Beyond the short distance behavior of the pair distribution function, the contact density also appears in a number of thermodynamic quantities and in experimentally accessible response functions of many-body systems with short range interactions. A comprehensive review of this subject has been given by Braaten [159] and in two detailed papers by Werner and Castin, using a derivation based on many-body wave functions. They cover both Bose gases and mixtures [160] or two-component Fermi gases [161] and also the case of two dimensions, where many of the results hold without much change provided that the zero energy two-body wave function is replaced by its $d=2$ counterpart $\psi_{0}(r)=\ln \left(r / a_{2}\right)$, with $a_{2}$ the associated scattering length. The relevance of $C_{2}$ for thermodynamic properties becomes evident from Tan's adiabatic theorem [154], which relates the derivative

$$
\begin{equation*}
\left.\frac{\partial U}{\partial(1 / a)}\right|_{S}=-\frac{\hbar^{2}}{8 \pi m} \cdot \int_{\boldsymbol{R}} C_{2}(\boldsymbol{R})=-\frac{\hbar^{2}}{8 \pi m} \cdot C_{2} \tag{76}
\end{equation*}
$$

of the internal energy $U$ with respect to the inverse scattering length at constant entropy $S$ to the integrated contact $C_{2}$. The theorem follows from the Hellman-Feynman relation

$$
\begin{equation*}
d\langle\hat{H}\rangle=\frac{1}{2} \int_{x_{1}} \int_{x_{2}} d V\left(\boldsymbol{x}_{1}-\boldsymbol{x}_{2}\right)\left\langle\hat{\psi}^{\dagger}\left(\boldsymbol{x}_{1}\right) \hat{\psi}^{\dagger}\left(\boldsymbol{x}_{2}\right) \hat{\psi}\left(\boldsymbol{x}_{2}\right) \hat{\psi}\left(\boldsymbol{x}_{1}\right)\right\rangle=\int_{R} C_{2}(\boldsymbol{R}) \int_{x} d V(\boldsymbol{x}) \frac{\left|\psi_{0}(r)\right|^{2}}{32 \pi^{2}}=-\frac{\hbar^{2}}{8 \pi m} C_{2} \cdot d(1 / a) \tag{77}
\end{equation*}
$$

by using a result derived by Zhang and Leggett [155] which relates a change $d V(\boldsymbol{x})$ of a short range interaction to the one in the associated inverse scattering length by an integration of the product $d V(\boldsymbol{x})$ and $\left|\psi_{0}(r)\right|^{2}$. From Tan's adiabatic theorem, the result $C_{2} \rightarrow(4 \pi n a)^{2}$ quoted above for weakly interacting BEC's at zero temperature follows from the leading mean-field term $u(n)=g n / 2+\ldots$ of the energy per particle. A simple physical interpretation of the concept of a contact is obtained by considering the average number of pairs of Bosons with a separation $\left|x_{1}-x_{2}\right|<b$ which is smaller than the average interparticle spacing. Using the short distance behavior (73), this - of course extensive - number is given by

$$
\begin{equation*}
N_{\text {pairs }}(r<b)=\frac{1}{2!} \int_{R} \int_{|x|<b} n^{2}(\boldsymbol{R}) g^{(2)}(\boldsymbol{R}, \boldsymbol{x})=\frac{C_{2}}{8 \pi} \cdot b+O\left(b^{2}\right) \tag{78}
\end{equation*}
$$

For small separation $b$, the average number of close pairs of Bosons thus scales linearly with their maximum distance instead of the naive scaling with the associated volume. This anomalous behavior is a result of the $1 / r^{2}$-dependence of the probability density $\left|\psi_{0}(r)\right|^{2}$ at short distances, which cancels the factor $4 \pi r^{2}$ from the volume element.

It was realized by Braaten and Platter [156] that an effective and rather general approach to derive exact relations for the many-body problem with zero range interactions is provided by the operator product expansion (OPE), due independently to Wilson [162], Kadanoff [163] and Polyakov [164]. As a simple example, consider two particles whose separation $|\boldsymbol{x}|$ is smaller than the average distance to the remaining $N-2$ particles. They thus feel only their direct interaction. Extending the expansion (71) of the eigenfunctions of the two-particle density operator $\hat{\rho}_{2}$ to the level of operators, the product

$$
\begin{equation*}
\lim _{r \rightarrow 0} \hat{\psi}(\boldsymbol{R}-\boldsymbol{x} / 2) \hat{\psi}(\boldsymbol{R}+\boldsymbol{x} / 2)=\frac{\psi_{0}(r)}{4 \pi} \hat{\phi}(\boldsymbol{R})+\ldots \tag{79}
\end{equation*}
$$

of two annihilation operators at short distances factorizes into the two-body wavefunction $\psi_{0}(r)=1 / r-1 / a$ in vacuum and an operator $\hat{\phi}(\boldsymbol{R})$ which describes the dressing of the bare two-particle state by the surrounding many-body system. The physical interpretation of the operator $\hat{\phi}(\boldsymbol{R})$ is elucidated by considering the product of two density operators at small distance. Introducing the so called contact operator $\hat{O}_{c}(\boldsymbol{R})=\hat{\phi}^{\dagger}(\boldsymbol{R}) \hat{\phi}(\boldsymbol{R}), 79$ gives rise to a short distance expansion of the form

$$
\begin{equation*}
\hat{n}(\boldsymbol{R}+\boldsymbol{x} / 2) \hat{n}(\boldsymbol{R}-\boldsymbol{x} / 2)=\hat{n}(\boldsymbol{R}) \delta(\boldsymbol{x})+\hat{\psi}^{\dagger}(\boldsymbol{R}+\boldsymbol{x} / 2) \hat{\psi}^{\dagger}(\boldsymbol{R}-\boldsymbol{x} / 2) \hat{\psi}(\boldsymbol{R}-\boldsymbol{x} / 2) \hat{\psi}(\boldsymbol{R}+\boldsymbol{x} / 2) \underset{|\boldsymbol{x}| \rightarrow 0}{\longrightarrow} \delta(\boldsymbol{x}) \hat{n}(\boldsymbol{R})+\frac{\psi_{0}^{2}(r)}{16 \pi^{2}} \hat{\boldsymbol{O}}_{c}(\boldsymbol{R})+\ldots \tag{80}
\end{equation*}
$$

which is written here for a general inhomogeneous situation. Now, the second term is just the associated pair distribution function and the comparison with Eq. (73) shows that the contact density is simply $C_{2}(\boldsymbol{R})=\left\langle\hat{O}_{c}(\boldsymbol{R})\right\rangle$. Quite generally, the OPE provides an expansion of a product $\hat{O}_{a} \hat{O}_{b}$ of operators at nearby points in space in terms of local operators $\hat{O}_{\ell}$ in the form

$$
\begin{equation*}
\hat{O}_{a}\left(\boldsymbol{R}-\frac{\boldsymbol{x}}{2}\right) \hat{\boldsymbol{O}}_{b}\left(\boldsymbol{R}+\frac{\boldsymbol{x}}{2}\right)=\sum_{\ell} W_{\ell}^{(a, b)}(\boldsymbol{x}) \hat{\boldsymbol{O}}_{\ell}(\boldsymbol{R}) . \tag{81}
\end{equation*}
$$

As an identity at the level of operators, Eq. (81) is valid for expectation values between any state. The state-independent coefficients $W_{\ell}^{(a, b)}(\boldsymbol{x})$ are ordinary c-numbers and are called the Wilson coefficients. They depend both on $\ell$ and the specific operators $\hat{O}_{a}$ and $\hat{O}_{b}$ which appear on the left-hand side of Eq. 81. What distinguishes an operator relation of the type 81) from more familiar operator identities like the Baker-Campbell-Hausdorff relation is that the latter holds completely generally, while the OPE is valid only in eigenstates of a specific Hamiltonian. An important concept in an OPE is that of the scaling dimension $\Delta_{\ell}$ of an operator $\hat{\boldsymbol{O}}_{\ell}$ which determines the dependence of the Wilson coefficients at small separation $\boldsymbol{x}$ via

$$
\begin{equation*}
W_{\ell}^{(a, b)}(x) \sim|x|^{\Delta_{\ell}-\Delta_{a}-\Delta_{b}} . \tag{82}
\end{equation*}
$$

As an example, in the OPE in Eq. 80, the Wilson coefficient $W_{n}^{(n, n)}(\boldsymbol{x})=\delta(\boldsymbol{x})$ associated with the density operator scales like $|\boldsymbol{x}|^{-3}$ since $\hat{n}(\boldsymbol{R})$ has scaling dimension $\Delta_{n}=3$. The contact operator $\hat{\boldsymbol{O}}_{c}(\boldsymbol{R})$, in turn, has scaling dimension $\Delta_{c}=4$ which follows from the weaker divergence $W_{c}^{(n, n)}(\boldsymbol{x})=\left(\psi_{0}(r) / 4 \pi\right)^{2} \sim 1 /|x|^{2}$ of the associated Wilson coefficient as $|\boldsymbol{x}| \rightarrow 0$. Obviously, the operators $\hat{O}_{\ell}$ with the lowest scaling dimension dominate the behavior of an operator product at small separation. In particular, Wilson coefficients which are non-analytic in $x$ give rise to power law tails of the associated correlator $O_{a} O_{b}$ at large momentum. The most important example for this connection is provided by the OPE for the one-particle density matrix

$$
\begin{equation*}
\hat{\psi}^{\dagger}(\boldsymbol{R}-\boldsymbol{x} / 2) \hat{\psi}(\boldsymbol{R}+\boldsymbol{x} / 2)=\hat{n}(\boldsymbol{R})+(i / \hbar) \boldsymbol{x} \cdot \hat{\boldsymbol{g}}(\boldsymbol{R})-\frac{|\boldsymbol{x}|}{8 \pi} \hat{O}_{c}(\boldsymbol{R})+\ldots \tag{83}
\end{equation*}
$$

At first sight, it seems surprising that the operator which appears in the short distance expansion (80) of the product of two densities also appears in the corresponding one of two field operators. The underlying reason for this may be understood by considering the expression

$$
\begin{equation*}
n(\boldsymbol{q})=\frac{N}{V} \int_{x_{a}} \int_{x_{b}} e^{-i \boldsymbol{q}\left(\boldsymbol{x}_{a}-\boldsymbol{x}_{b}\right)} \int d 2 \ldots d N \psi^{*}\left(\boldsymbol{x}_{a}, \boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}\right) \psi\left(\boldsymbol{x}_{b}, \boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}\right) . \tag{84}
\end{equation*}
$$

for the dimensionless momentum distribution $n(\boldsymbol{q})$ of a pure state. Its asymptotic behavior for large momentum is determined by the integration regions in which both $\boldsymbol{x}_{a}$ and $\boldsymbol{x}_{b}$ approach one of the remaining particle coordinates $\boldsymbol{x}_{2} \ldots \boldsymbol{x}_{N}$ simultaneously. Factoring out the associated bare two-body wave function $\psi_{0}(\boldsymbol{x})$, this gives rise to a momentum distribution which behaves like

$$
\begin{equation*}
n(\boldsymbol{q}) \rightarrow C_{2}\left|\int_{x} \frac{\psi_{0}(\boldsymbol{x})}{4 \pi} e^{-i \boldsymbol{q} \cdot \boldsymbol{x}}\right|^{2}+\ldots=C_{2} / q^{4}+\ldots \tag{85}
\end{equation*}
$$

at large momentum, as first derived by Tan [165]. The same result is obtained from the OPE expression (83) by noting that the formal Fourier transform of $-|x|$ is $8 \pi / q^{4}$. Extending this expansion to higher order is a rather complex problem which will not be discussed here. As shown by Braaten et al. [166], the subleading correction to the result (85] is of the form

$$
\begin{equation*}
n(q)=\frac{C_{2}}{q^{4}}+\frac{C_{3}}{q^{5}} F(q)+\cdots, \tag{86}
\end{equation*}
$$

where $F(q)=A \sin \left[2 s_{0} \ln \left(q / \kappa_{*}\right)+2 \phi\right]$ is a log-periodic function that depends on the value of the three-body parameter $\kappa_{*}$, while $s_{0}=1.00624, \phi=-0.669064$, and $A=89.26260$ are universal constants. The appearance of a log-periodic function is a consequence of the Efimov effect: three-body bound states appear at a geometric sequence $a_{-}^{(n)}(3) \sim-\exp \left(\pi n / s_{0}\right)$ of increasingly negative scattering lengths, approaching the threshold $1 / a=0$ for the existence of a two-body bound state at which an infinite number of trimers is present. The overall magnitude of the subleading contribution $\sim C_{3} / q^{5}$ is determined by the three-body contact density $C_{3}$ which measures the likelihood for three particles to be close together. Within a strict zero range interaction, $C_{3}$ actually vanishes. Indeed, as realized by Werner [167], a finite range is needed to properly account for the threebody correlations associated with the Efimov effect. In the function $F(q)$ defined above, this is hidden in the parameter $\kappa_{*}$. The connection between $C_{3}$ and the finite interaction range is elucidated by the thermodynamic definition [166]

$$
\begin{equation*}
C_{3}=\int_{R} C_{3}(\boldsymbol{R})=-\left.\frac{m \kappa_{*}}{2 \hbar^{2}} \frac{\partial U}{\partial \kappa_{*}}\right|_{S, 1 / a} \tag{87}
\end{equation*}
$$

of the integrated three-body contact in terms of the derivative of the internal energy $U$ with respect to $\kappa_{*}$, in close analogy to Eq. (76) for the two-body contact $C_{2}$. A finite value of this derivative, which needs to be taken at both fixed entropy and scattering length, thus implies that the energy is sensitive to the short distance behavior of the interaction. Importantly, however, this only enters through a single parameter $\kappa_{*}$ which determines the characteristic momentum scale of the three-body bound states. The associated binding energies $E^{(n)}(3)=-\hbar^{2}\left(\kappa^{(n)}\right)^{2} / m$ right at unitarity follow from the condition $\sin \left[s_{0} \ln \left(\kappa^{(n)} / \kappa_{*}\right)\right]=0$. As discussed in Lecture I, the finite range of the interaction in ultracold gases of the order $\ell_{\mathrm{vdw}}$ provides a lower cutoff for this infinite sequence at $\kappa^{(0)}=\kappa_{*} \simeq 0.2 / \ell_{\mathrm{vdW}}$ and, correspondingly, a minimum (in magnitude) value $a_{-}^{(0)}(3)=a_{-}(3) \simeq-9 \ell_{\mathrm{vdW}}$ at which the trimers detach from the continuum [65].

Quantitative results for the contact densities $C_{2,3}$ are so far available only in the regime of weak interactions $\left(n a^{3}\right)^{1 / 2} \ll 1$ or in moderately degenerate gases with fugacities $z=\exp (\beta \mu) \lesssim 1$. In the latter case, the leading order results [168, 169]

$$
\begin{equation*}
C_{2}=\frac{16 \pi^{2}}{\lambda_{T}^{4}} \frac{\partial b_{2}}{\partial\left(\lambda_{T} / a\right)} z^{2} \underset{\lambda_{T} \ll|a|}{ } 32 \pi\left(n \lambda_{T}\right)^{2}\left(1-\frac{\lambda_{T}}{\sqrt{2}|a|}+. .\right) \text { and } C_{3}=\frac{\pi}{\lambda_{T}^{5}} \frac{\partial b_{3}}{\partial \ln \left(\lambda_{T} K_{*}\right)} z^{3} \xrightarrow[\lambda_{T} \ll|a|]{ } 3 \sqrt{3} s_{0}\left(n \lambda_{T}\right)^{4} / n \sim n^{3} / T^{2} \tag{88}
\end{equation*}
$$

exhibit a power law decay with temperature. The contact densities are expected to reach a maximum value near the BEC transition temperature and then decrease towards their finite values $C_{2} \rightarrow(4 \pi n a)^{2}$ and $C_{3} \rightarrow 2.9(n a)^{4} / n[168]$ in the ground state. Surpisingly, no quantitative results are available which cover the full temperature range, not even for $C_{2}$ and the limit of weak interactions. Essentially exact results are available in one dimension, where zero range interactions again lead to an asymptotic power law decay $n(q) \rightarrow C_{2} / q^{4}$ of the momentum distribution [170]. The result can be derived by an operator product expansion and thus holds for arbitrary states of the associated Lieb-Liniger gas [171]. Since delta function interactions give rise to wave functions which are non-singular in one dimension, exhibiting only a jump in their slope, the pair distribution function at vanishing distance $g^{(2)}(0)$ is finite. Together with the dimensionless Lieb-Liniger parameter $\gamma=2 /\left(n_{1}\left|a_{1}\right|\right)$, it determines the two-body contact density simply by $C_{2}^{\text {1d }}=n_{1}^{4} \gamma^{2} \cdot g^{(2)}(0)$. Here, $n_{1}$ is the number of particles per length and $a_{1}$ the scattering length in one dimension, which is negative for repulsive interactions ${ }^{20}$. For weak interactions $\gamma \ll 1$, where $g^{(2)}(0) \rightarrow 1-2 \sqrt{\gamma} / \pi$ and the ground state of the Lieb-Liniger gas is essentially a true BEC , the contact density at $T=0$ is given by $C_{2}^{1 \mathrm{~d}} \rightarrow\left(2 n_{1} /\left|a_{1}\right|\right)^{2}$. In the Tonks-Girardeau limit $\gamma \gg 1$ in turn, where the gas is fermionized, one obtains $C_{2}^{\text {dd }} \rightarrow 4 \pi^{2} n_{1}^{4} / 3$ which is the 1 d - analog of the result $C_{2} \rightarrow$ const $\cdot n^{4 / 3}$ expected for the unitary Bose gas in three dimensions. At finite temperature, the contact density has been determined numerically by Yao et al. [172]. In the regime where there is no Fermionization, $C_{2}^{1 d}(T)$ exhibits a broad maximum near the degeneracy temperature where $n_{1} \lambda_{T} \simeq 1$. A similar behavior is expected in three dimensions. In this case, however, there is a sharp BEC transition which results in a weak singularity of the contact density. It is determined by the one in the entropy density $s(T)$ using Gibbs-Duhem and the resulting Maxwell relation [173, 174]

$$
\begin{equation*}
d p\left(\mu, T, 1 / a, \kappa_{*}\right)=n d \mu+s d T+\frac{\hbar^{2}}{8 \pi m} C_{2} d(1 / a)+\frac{2 \hbar^{2}}{m \kappa_{*}} C_{3} d \kappa_{*} \rightarrow \frac{\hbar^{2}}{8 \pi m} \frac{\partial C_{2}(T)}{\partial T}=\frac{\partial s(T)}{\partial(1 / a)} . \tag{89}
\end{equation*}
$$

[^20]

FIG. 13: The Figure on the left shows the measured two-body contact of a moderately degenerate Bose gas of ${ }^{39} \mathrm{~K}$ near a Feshbach resonance at $B_{0} \simeq 402.7 \mathrm{G}$ in comparison with the theoretical leading order result (88) from a virial expansion. It is adapted from Ref. [177]. On the right, the solid red and blue lines show the trap averaged momentum distribution for fugacities $z=0.5$ and $z=0.4$ respectively in comparison with the experimental results obtained obtained by Makotyn et al. [178] for a unitary gas of ${ }^{85} \mathrm{Rb}$. The dashed lines show the result up to $O\left(z^{2}\right)$, where the oscillatory behavior due to three-body correlations is absent. The Figure is adapted from Ref. [169].

Experimentally, the two-body contact has been determined in weakly interacting BEC's from a measurement of the tail $\Gamma(\omega) \sim C_{2} / \omega^{3 / 2}$ of the momentum integrated RF-spectrum at high frequencies [175]. More recently, the contact of a weakly confined two-dimensional ${ }^{87} \mathrm{Rb}$ gas has been measured by Zou et al. [176], using the tiny shift in the frequency of the clocktransition $|1\rangle \rightarrow|2\rangle$ due to the fact that the scattering lengths $a_{11}$ and $a_{22}$ differ slightly. An interferometric method which allows to determine both the two- and three-body contact densities has been developed by Fletcher et al. [177]. It relies on a Ramsey scheme in a Bose gas of ${ }^{39} \mathrm{~K}$ with two internal states $|\uparrow\rangle,|\downarrow\rangle$ where the $\uparrow \uparrow$-interaction is resonant while all other interactions are negligible. The initial configuration with all atoms in the $|\downarrow\rangle$ state is changed to an equal weight superposition $(|\uparrow\rangle+|\downarrow\rangle) / \sqrt{2}$ by a $\pi / 2-\mathrm{RF}$ pulse. Due to the $\uparrow \uparrow$ - interactions, the relative phase of the superposition state precesses at a rate [177]

$$
\begin{equation*}
\Omega=\frac{d \varphi}{d t}=\frac{\hbar}{4 \pi m n a}\left(C_{2}+5 \pi^{2} a C_{3}\right) \tag{90}
\end{equation*}
$$

which depends on the two-body contact density $C_{2}$ through a generalized mean-field shift $\hbar^{2} C_{2} /(4 \pi m n a) \rightarrow g n$. In fact, this shift appears again in Eq. 106) below in the context of Bragg scattering at large momentum. In addition, three-body interactions give rise to a term proportional to $C_{3}$, which is the only contribution that remains at the unitary point $1 / a=0$. The rate is measured by mapping the accumulated phase $\Omega T$ after time $T$ to a population imbalance by a second $\pi / 2$ - pulse. In particular, using a Feshbach resonance at $B_{0} \simeq 402.7 \mathrm{G}$ where the scattering length $a=a_{\uparrow \uparrow}$ diverges, allows to disentangle $C_{2}$ and $C_{3}$. As shown in Fig. 13, the measured two-body contact is found to be in excellent agreement with the cusp-like structure predicted by the leading order expression 88 from the virial expansion ${ }^{21}$, despite the fact that the gas is initially prepared near the transition temperature $n \lambda_{T}^{3} \simeq 2.6$ of an ideal BEC where $z \simeq 1$. Moreover, also the observed value of the three-body contact is consistent with the corresponding results in Eq. (88). They were obtained by Barth and Hofmann [169] by a virial expansion of the momentum and frequency dependent Green functions up to third order in the fugacity $z$, which allows to calculate not only thermodynamic quantities but also the full momentum distribution in moderately degenerate Bose gases even at large scattering lengths. Since the contributions $\sim z^{3}$ properly account for three-body correlations, the results also determine the subleading, log-periodic oscillations of the momentum distribution discussed in Eq. (86). A corresponding experiment has been performed by Makotyn et al. [178], using a rapid quench from the weakly interacting regime $\left(n a^{3}\right)^{1 / 2} \lesssim 3 \cdot 10^{-3}$ to the unitary point $1 / a=0$ in a gas of ${ }^{85} \mathrm{Rb}$. Their results for the momentum distribution, which reached a stationary form after an equilibration time of a few hundred micro-seconds, are shown in Fig. 13 as a function of the dimensionless variable $\kappa=q / k_{n}$. At small momenta $q \lambda_{T} \lesssim 1$, three-body losses lead to a flattening of the momentum distribution [179]. This is not seen in Fig. [13, however, since the distribution is multiplied by $q^{4}$ in order to extract the behavior at large momentum where $n(q) \cdot q^{4}$ must approach the two-body contact density $C_{2}$. Apparently, at the present stage neither the flattening off at large momentum nor the rather small oscillatory contributions from the Efimov effect could be resolved, unfortunately.

[^21]Inelastic decay rates and contact coefficients Ultracold gases are generically subject to losses which - apart from trivial single-particle losses due to collisions with some background gas - dominantly occur via either two- or three-body collisions. In an inhomogeneous situation, the loss coefficients $L_{l}$ for $l$-body decay processes are conveniently defined by $d N_{l-\text {-body }} / d t=$ $-L_{l}\left\langle n^{l-1}\right\rangle N$, factoring out the expected dependence of the loss rates on the averaged density $\langle n\rangle$. The resulting characteristic power law decay $N_{l-\text { body }}(t) \sim t^{-1 /(l-1)}$ for $l \geq 2$ allows to separate the different contributions, at least in principle. For Bosons, $l$-body decay processes are possible in s-wave collisions and the associated decay rates are proportional to the probability that $l$ particles are close together. On a formal level, losses may be described by adding a finite imaginary part $\hat{H} \rightarrow \hat{H}-i \hat{K}$ to the many-body Hamiltonian. As discussed by Braaten et al. [180], the requirement that the resulting equation of motion for the reduced time-dependent density matrix $\hat{\rho}(t)$ is linear, trace preserving and Markovian leads to a Lindblad type equation

$$
\begin{equation*}
i \hbar \frac{d}{d t} \hat{\rho}=[\hat{H}, \hat{\rho}]-i\{\hat{K}, \hat{\rho}\}+2 i \sum_{l=1} \gamma_{l} \int_{R} \hat{\phi}_{l}(\boldsymbol{R}) \hat{\rho} \hat{\phi}_{l}^{\dagger}(\boldsymbol{R}) \text { with } \hat{K}=\sum_{l=1} \gamma_{l} \int_{R} \hat{\phi}_{l}^{\dagger}(\boldsymbol{R}) \hat{\phi}_{l}(\boldsymbol{R})=\sum_{l=1} \hat{K}_{l} \tag{91}
\end{equation*}
$$

which has originally been used in this context by Sinatra and Castin [181]. It involves a sum of local Lindblad or quantum jump (respectively collapse) operators $\hat{\phi}_{l}(\boldsymbol{R})$ which annihilate $l$ atoms at low energy in the initial state and thus obey $\left[\hat{\phi}_{l}, \hat{N}\right]=$ $l \cdot \hat{\phi}_{l}$. The total particle loss $d N / d t=-2 \sum_{l} l\left\langle\hat{K}_{l}\right\rangle / \hbar$ following from Eq. 91 is additive in the different channels. It allows to define individual loss rates $\Gamma_{l}$ by $d N_{l-\text { body }} / d t=-\Gamma_{l} N$ which are determined by the initial (or instantaneous) expectation values $C_{l}=\int_{R}\left\langle\hat{\phi}_{l}^{\dagger}(\boldsymbol{R}) \hat{\phi}_{l}(\boldsymbol{R})\right\rangle$ by the simple relation $\Gamma_{l} N=2 l \gamma_{l} \cdot C_{l} / \hbar$. To see that the extensive, quasi-equilibrium coefficients $C_{l}$ can indeed be identified with the integrated contacts $C_{2,3}$ discussed above, consider first the case of two-body losses which may occur via a spin or dipolar relaxation mechanism. On a phenomenological level, they may be accounted for by a finite, negative imaginary part $\operatorname{Im} a<0$ of the scattering length. The relevant collapse operator $\hat{\phi}_{2} \equiv \hat{\phi}$ associated with the annihilation of two close atoms with center-of-mass position $\boldsymbol{R}$ is precisely the one which appears in Eq. (79). Moreover, the associated kinetic coefficient $\gamma_{2}=\hbar^{2} /(8 \pi m) \cdot \operatorname{Im}(1 / a)$ may be inferred from the Tan adiabatic theorem. Indeed, the result $\left\langle\hat{K}_{2}\right\rangle=\gamma_{2} C_{2}$ must be consistent with the imaginary contribution to Eq. 76| which appears upon adding a finite imaginary part $-i \operatorname{Im} a /|a|^{2}$ in the inverse scattering length. As a result, the two-body decay rate

$$
\begin{equation*}
\Gamma_{2} N=\frac{\hbar}{4 \pi m} C_{2} \operatorname{Im}\left(\frac{1}{a}\right) \underset{\text { hom }}{\longrightarrow} \Gamma_{2}=L_{2} n=\frac{\hbar C_{2}}{4 \pi m n} \operatorname{Im}\left(\frac{1}{a}\right) \tag{92}
\end{equation*}
$$

is determined by the corresponding contact $C_{2}$. Note that the associated loss coefficient $L_{2}$ is density independent only in the regime where $C_{2} \sim n^{2}$ scales with the square of the density. This is no longer valid for strongly interacting Bose gases with $\left(n a^{3}\right)^{1 / 2} \gtrsim 1$. The extension of the result $\left\langle 92\right.$ to three-body losses and their connection with the contact density $C_{3}$ is similar but more subtle. Following the discussion by Werner and Castin [160], the origin of a finite rate of three-body decay even in the zero range interaction limit can be understood as a result of a cancellation of two effects: the deeply bound dimer states which are populated by three-body collisions have a binding energy of order $\hbar^{2} / m \ell_{\mathrm{vdW}}^{2}$. The resulting loss coefficient is expected to scale like the product $L_{3} \sim N_{\text {triplets }} \cdot \hbar^{2} / m \ell_{\mathrm{vdW}}^{2}$ of this energy times the average number of triples of Bosons within a distance of order $\ell_{\mathrm{vdW}}$. Now, in close analogy to the anomalous linear behavior on distance for the number of close pairs discussed in Eq. 78 above, a finite value of the three-body contact implies that $N_{\text {triplets }} \simeq C_{3} \ell_{\mathrm{vdW}}^{2}$ vanishes quadratically. As a result, the three-body loss coefficient $L_{3}$ remains finite in the zero-range limit $\ell_{\mathrm{vdW}} \rightarrow 0$. For three-body losses, the analog of a finite imaginary part of the scattering length is played by the so-called inelasticity parameter $\eta$ introduced first by Braaten et al. [182]. It allows to introduce a finite width of three-particle states due to decay into deeply bound molecules by the formal substitution $\ln \kappa_{*} \rightarrow \ln \kappa_{*}+i \eta / s_{0}$. The definition (56) of the three-body contact determines the associated kinetic coefficient $\gamma_{3}=\left(2 \hbar^{2} / m\right) \operatorname{Im}\left[\ln \kappa_{*}\right]$ which needs to be replaced by $\gamma_{3}=\hbar^{2} /\left(m s_{0}\right) \sinh 2 \eta$ in a more refined description [180]. The three-body decay rate is thus [160, 180]

$$
\begin{equation*}
\Gamma_{3} N=\frac{6 \hbar}{m s_{0}} C_{3} \sinh 2 \eta \underset{\mathrm{hom}}{\longrightarrow} \Gamma_{3}=L_{3} n^{2}=\frac{6 \hbar C_{3}}{m s_{0} n} \sinh 2 \eta . \tag{93}
\end{equation*}
$$

For weakly interacting BEC's, the dependence $C_{3} \simeq n^{3} a^{4}$ mentioned above leads to a density independent three-body loss coefficient $L_{3} \simeq(\hbar / m) \sinh 2 \eta \cdot a^{4}$ which diverges strongly near a Feshbach resonance. In practice, this divergence is cutoff by the thermal wavelength $\lambda_{T}$. Indeed, the corresponding result for $C_{3}(T)$ in Eq. (88) shows that in the regime $\lambda_{T} \ll|a|$, the loss rate $\Gamma_{3}(T) \sim(\hbar / m) n^{2} \lambda_{T}^{4} \sim n^{2} / T^{2}$ exhibits a power law dependence on temperature This has been verified experimentally [183, 184] for Bosons near infinite scattering length. An important point to note is that the result $L_{3} \sim \hbar a^{4} / m$ at low temperatures only describes the dependence on a on average. In fact, as realized by Esry et.al. [185] and by Bedaque et.al. [186], the Efimov effect gives rise to a log-periodic structure in the three-body loss rate which leads to pronounced maxima at scattering lengths $a_{-}^{(n)}<0$ where the Efimov states detach from the two-particle continuum. In turn, there are minima of $L_{3}(a)$ at a set of positive $a_{*}^{(n)}$, where the $n$-th trimer state crosses the two-body bound state energy. This intricate structure can be inferred directly from the solution of the three-body problem, which determines the zero density limit of the loss coefficient from the imaginary part of the scattering hypervolume $D$ introduced in the Lecture I by the relation $L_{3}(n \rightarrow 0)=-(\hbar / m) \operatorname{Im} D[72]$.

Short distance physics in the presence of power law interactions Some of the exact results which characterize quantum manybody systems with zero range interactions can be extended to power law interactions. A case of fundamental importance is the one with pure Coulomb interactions which has been discussed by Hofmann et al. [187]. From the exact solution of the two-body problem in the hydrogen atom, it is well known that only s-states have a finite probability at the origin and that the associated wave functions locally behave like $1-|x| / a_{B}+\ldots$. In the repulsive case of two electrons, the short distance behavior for their relative motion is $\psi_{0}(r)=1+|x| /\left(2 a_{B}\right)$, which replaces the $1 / r-1 / a$ result above in the case of zero range interactions. The pair distribution function $g^{(2)}(r)=g^{(2)}(0)\left(1+r / a_{B}+\ldots\right)$ at short distances e.g. in the standard jellium model is again determined by $\left|\psi_{0}(r)\right|^{2}$, which now leads to a finite value $g^{(2)}(0)$ at the origin. The fact that relations of this type are exact at the many-body level has originally been proven by Kato [188] and is called the Kato theorem in the mathematical literature. For the attractive situation with a point like nucleus of charge $Z e>0$ at the origin, this leads e.g. to Kato's cusp condition

$$
\begin{equation*}
Z=-\left.\frac{a_{B}}{2 n(\boldsymbol{x})} \frac{d n(\boldsymbol{x})}{d r}\right|_{r \rightarrow 0} \tag{94}
\end{equation*}
$$

which constrains the local upward cusp in the electron density $n(\boldsymbol{x})$ in many-body calculations. Due to the power law tail $\psi_{0}(q) \rightarrow-4 \pi / a_{B} q^{4}$ in the Fourier transform of the two-body wave function, the momentum distribution in the many-body problem asymptotically decays like $n(q) \rightarrow n^{2} g^{(2)}(0)\left|\psi_{0}(q)\right|^{2} \sim g^{(2)}(0) /\left(a_{B}^{2} q^{8}\right)$, a relation which again holds for any state, similar to the Tan result (85] in the case of zero range interactions [187]. An important difference to this latter case, however, is that there is no analog of the Tan adiabatic theorem (76) because the interaction energy now depends on the pair distribution function $g^{(2)}(r)$ at arbitrary distances and not just its value at contact. More generally, for interactions which exhibit a power law dependence $m V(r) / \hbar^{2} \rightarrow b_{s} / r^{s}$ at short distances, the associated two-body wave function $\psi_{0}(r)$ is finite at the origin if the exponent $s$ lies in the range $0<s<2$. In the many-body problem, therefore, the pair distribution function $g^{(2)}(0)$ for coincident arguments is finite. Due to the non-analytic contribution $\psi_{0}(r)=1+b_{s} r^{2-s} /[(3-s)(2-s)]+\ldots$ in the two-body wave function at short distances, a universal correction $\sim r^{2-s}$ appears in $g^{(2)}(r \rightarrow 0)=g^{(2)}(0)\left|\psi_{0}(r)\right|^{2}$. Moreover, the Fourier transform $\psi_{0}(q) \sim b_{s} / q^{5-s}$ exhibits a power law tail which leads to a characteristic decay $n(q) \sim g^{(2)}(0) b_{s}^{2} / q^{2(5-s)}$ in the momentum distribution. Note that the amplitude of the leading order non-analytic correction in $\psi_{0}(r)$ is independent of the energy. The subscript in $\psi_{0}(r)$ which indicates that we are considering the short-distance behavior of the solution at zero energy is thus superfluous. As will be discussed below, however, it plays a crucial role in the context of the original Tan relations. In cases where the exponent $s$ is larger than two, the situation is fundamentally different for repulsive or attractive interactions. In the repulsive case, the twobody wave function vanishes exponentially near the origin. This implies $g^{(2)}(0)=0$ and an exponentially decaying momentum distribution in the many-body problem. For attractive interactions with $b_{s}<0$ in turn, the two-body wave function at short distance

$$
\begin{equation*}
\psi_{0}(r \rightarrow 0)=A r^{(s-4) / 4} \cos \left[\frac{2}{s-2}\left(\frac{\left|b_{s}\right|}{r^{s-2}}\right)^{1 / 2}+B\right] \tag{95}
\end{equation*}
$$

exhibits a power law dependence with exponent $(s-4) / 4>-1 / 2$, modulated by an oscillation which becomes infinitely fast as $r \rightarrow 0$ [189]. This is a reflection of the fact that the spectrum of bound states is now no longer bounded from below. A similar 'fall into the center' appears for scale invariant attractive interactions $m V(r) / \hbar^{2} \rightarrow-\gamma / r^{2}$ in the regime $\gamma>1 / 4$ [190]. Interactions of this type arise in the three-body problem of Bosons with short range interactions. Specifically, the Efimov effect is a consequence of a supercritical inverse square interaction in a hyperradial coordinate $R=\left(r_{12}^{2}+r_{13}^{2}+r_{23}^{2}\right)^{1 / 2}$. The full momentum distribution of the three-body problem has been determined by Castin and Werner [191]. At large momenta, it is of the form stated above in Eq. (86), with integrated two- and three-body contacts $C_{2}=53.0972 \kappa_{*}$ and $C_{3}=\kappa_{*}^{2}$ [166]. The latter result follows from Eq. (87) by noting that the wave number $\kappa_{*}$ is related to the trimer energy by $E_{\text {trimer }}=-\hbar^{2} \kappa_{*}^{2} / m$. The log-periodic dependence of the universal function $F(q)$ in the subleading contribution $\sim 1 / q^{5}$ in Eq. 86, , which is invariant under the discrete scale transformation $\left(q / \kappa_{*}\right) \rightarrow\left(q / \kappa_{*}\right) \cdot \exp \left(\pi / s_{0}\right)$, is thus a consequence of the Efimov effect.

An important example for power law interactions with an exponent $s>2$, now at large distances, is provided by dipolar gases. Compared to the rotation invariant potentials discussed so far, they involve a nontrivial angular dependence. Indeed, even for aligned dipoles, the long-range potential $-2 d^{2} P_{2}(\cos \theta) / r^{3}$ depends on the angle $\theta$ between the direction of alignment and the relative separation. As a result, the angular momentum $l$ is not conserved. With $\ell_{d}=m d^{2} / \hbar^{2}$ the dipolar length, the amplitude for angular momentum changing collisions vanishes like $\sim\left(k \ell_{d}\right)^{2}$ in the ultracold limit $k \ell_{d} \ll 1$ [192]. As a result of the long range nature of the interaction, however, the s-wave scattering amplitude diverges like $\ell_{d} \ln \left(1 / k \ell_{d}\right)$ while the phase shifts for finite $l$ start at linear order $\delta_{l}(k)=-\tilde{a}_{l} k+O\left(k^{2}\right)$ for all angular momentum channels, with effective scattering lengths $\tilde{a}_{l} \simeq \ell_{d} / l^{2}$ that decay only slowly with increasing $l[190]$. Hence, unlike the case of isotropic short-range interactions with a van der Waals tail, the interaction of three dimensional dipolar gases at low energies cannot be described in terms of a single effective parameter and no universal description of the thermodynamics and short-distance correlation functions exists. An extension of the complete set of Tan relations to dipolar gases turns out to be possible only in two dimensions, where the interaction $V_{\mathrm{dd}}(r)=d^{2} / r^{3}$ for dipoles aligned perpendicular to their motion is purely repulsive in addition to some unknown short-range potential. In this case,
two-body scattering at low energies is dominated by the s-wave contribution. Moreover, at the many-body level, the interaction decays sufficiently fast to give rise to a well defined free energy per particle in the thermodynamic limit, independent of the boundary conditions [3]. The thermodynamic relation (119) can then be extended in the form [121]

$$
\begin{equation*}
d p\left(\mu, T, a_{2}, \ell_{d}\right)=n d \mu+s d T-\frac{\hbar^{2}}{4 \pi m} C_{2} d\left(\ln a_{2}\right)-\mathcal{D} d\left(\ln \ell_{d}\right) \tag{96}
\end{equation*}
$$

In addition to the two-dimensional scattering length $a_{2}$ of the combined short-range plus dipolar potential for which $C_{2}$ is the conjugate force, it involves a dipolar analog $\mathcal{D}$ of the contact associated with the dependence of the pressure on the dipolar length.

Finally, we mention an important point that distinguishes the Tan relations from the exact relations for Coulomb interactions discussed above. The latter remain valid even in a high-density Wigner crystal or a plasma phase at arbitrary temperature [187]. $\underset{\tilde{V}}{\mathrm{By}}$ contrast, the Tan relations only apply as long as the physical interactions may be replaced by a zero-range pseudopotential $\tilde{V}(\boldsymbol{x}) \psi(\boldsymbol{x})=g \delta(\boldsymbol{x}) \partial_{r}(r \psi(\boldsymbol{x}))$ [37]. Similar to the scaling limit in statistical physics, where the lattice constant is taken to zero at a fixed value of the correlation length, the finite effective range $r_{e}$ of the true interaction is set to zero and the strength $g=4 \pi \hbar^{2} a / m$ of $\tilde{V}(x)$ is adjusted to reproduce its experimentally determined scattering length. The Tan relations, which follow as an exact consequence of the pseudopotential description, are thus restricted to low densities and temperatures, where the finite effective range and the presence of interactions beyond s-wave scattering are negligible. On a formal level, the restrictions due to the finite range arise from the deviation in the regime $r \lesssim r_{e}$ between the solution of the Schrödinger equation with the true interaction from the exact two-body wave function $\tilde{\psi}(r)=1-a / r^{22}$ at zero energy with interaction $\tilde{V}(\boldsymbol{x})$, where the asymptotic form remains valid at arbitrary small distances. In order to determine the range of densities where such deviations show up, consider the ground state energy per particle $u(n)=g n / 2[1+\ldots]$ of a dilute Bose gas with density $n$. Both the mean-field contribution and the first two leading corrections are fully determined by the scattering length $a$ alone. By contrast, as shown by Tan [70] within a fully analytical calculation, the correction of order $n a^{3}$, which physically describes three-body forces, is sensitive to both the range of the interaction and the three-body scattering hypervolume $D$ defined in Eq. 30). For ultracold atoms, which have van der Waals interactions supporting many bound states, the effective range in two-body scattering at low energies lies between $1.4 \ell_{\mathrm{vdW}}$ and $2.8 \ell_{\mathrm{vdW}}$ in the relevant range of positive scattering lengths $a>\ell_{\mathrm{vdW}}$, irrespective of the details of the interaction at short distances [193]. Moreover, as discussed in Lecture I, the van der Waals length also determines the typical magnitude $D \simeq \ell_{\mathrm{vdW}}^{4}$ of the three-body scattering hypervolume. As a result, the Tan relations only apply for densities $n \ll \ell_{\mathrm{vdW}}^{-3}$. In practice, this limitation is not of relevance because densities where $n \ell_{\mathrm{vdW}}^{3}$ becomes of order one are inaccessible due to their short lifetime. A similar conclusion holds for the consequences of the finite range for power laws like $n(q) \rightarrow C_{2} / q^{4}$ which only hold if $q r_{e} \ll 1$. With $r_{e} \simeq \ell_{\mathrm{vdW}}$, this is again not an important limitation in practice. In particular, due to the strong inequality $\ell_{\mathrm{vdW}} \ll n^{-1 / 3}$, there is a wide range $\xi^{-1} \ll q \ll 1 / \ell_{\mathrm{vdW}}$ where the $C_{2} / q^{4}$-tail in the momentum distribution is observable even in weakly interacting BEC's like ${ }^{87} \mathrm{Rb}$, where $a \simeq \ell_{\mathrm{vdW}}$ and the strength $C_{2, \mathrm{Bog}}=1 / 4 \xi^{4}$ is determined by the healing length $\xi$. By contrast, the contribution $\sim C_{2} / r^{2}$ in the short-distance behavior of the pair distribution function, which implies an effective bunching of atoms with separations $\ell_{\mathrm{vdW}} \ll r<a$, requires large scattering lengths $a \gg \ell_{\mathrm{vdW}}$ because this term dominates the subleading contribution in $g^{(2)}(r)$ of order $-2 C_{2} / a r$ only at distances $r<a / 2$, which must still be much larger than the interaction range $r_{e} \simeq \ell_{\mathrm{vdW}}$.

Bragg scattering at large momentum In the following, based on work together with J. Hofmann [194], it will be shown that the short distance expansions discussed above determine the behavior of the dynamic structure factor $S(\omega, q)$ of strongly interacting Bose gases at large momentum transfer. The approach is analogous to the one which is used to understand deep inelastic scattering in high energy physics. The dynamic structure factor can be determined from a measurement of the rate at which momentum is imparted on a cloud of atoms subject to two intersecting light fields with a difference $\boldsymbol{q}$ in wave vectors and $\omega$ in frequency by a two-photon transition via

$$
\begin{equation*}
\frac{d \boldsymbol{P}}{d t}=-\frac{\hbar \boldsymbol{q} \Omega^{2}}{2} \operatorname{Im} \chi(\omega, q), \quad\left(S(\omega, q)=\frac{1}{\pi}\left[1-e^{-\beta \hbar \omega}\right]^{-1} \operatorname{Im} \chi(\omega, q) \underset{T \rightarrow 0}{\longrightarrow} \operatorname{Im} \chi(\omega, q) \cdot \theta(\omega) / \pi\right) \tag{97}
\end{equation*}
$$

where $\Omega$ is an effective Rabi frequency and $\chi(\omega, q)$ the density response function. Within the Bogoliubov approximation, the bosonic quasiparticles with spectrum $E_{q}$ exhaust the complete set of excitations. As a result, the dynamic structure factor

$$
\begin{equation*}
S_{\mathrm{Bog}}(\omega, q)=n S_{\mathrm{Bog}}(q) \delta\left(\hbar \omega-E_{q}\right) \text { with } E_{q}=\varepsilon_{q} / S_{\mathrm{Bog}}(q)=\left[\varepsilon_{q}\left(\varepsilon_{q}+2 \Delta_{q}\right)\right]^{1 / 2} \underset{q \xi \gg 1}{ } \varepsilon_{q}+g n+\ldots \tag{98}
\end{equation*}
$$

[^22]

FIG. 14: The Figure on the left shows the lineshift measured in Bragg scattering at a fixed large momentum transfer $q$ in a gas of ${ }^{39} \mathrm{~K}$ atoms as a function of the scattering length for two different densities. The Figure on the right shows the line shift at lower densities and over a larger range of scattering lengths, reaching values $q a \simeq 7$. The data are compared with the theoretical prediction from a single mode approximation (full curves) and the result 106 from the operator product expansion (dashed curve on the right). Both Figures are adapted from Ref. [200].
exhibits a sharp single peak, whose position evolves smoothly from the linear, phonon like behavior $E_{q} \rightarrow \hbar c_{s} q$ at small momenta to single particle like behavior $E_{q} \rightarrow \varepsilon_{q}+g n$ in the regime of large wave vectors $q \xi \gg 1$. This behavior has been observed in early Bragg scattering experiments in weakly interacting BEC's [195, 196]. In particular, it has been nicely verified that for large wave vectors, the shift $\Delta\left(\hbar \omega_{q}\right) \rightarrow g n$ of the Bragg peak with respect to the bare single particle energy $\varepsilon_{q}$ is twice as large as the mean-field energy $u^{\mathrm{MF}}=g n / 2$ per particle in the condensate [197, 198]. Within the Bogoliubov approximation, the dynamic structure factor (98) exhibits only a single peak of zero width for arbitrary values of the momentum transfer $q$. The single-mode approximation $S(\omega, q) \sim Z_{q} \delta\left(\hbar \omega-\hbar \omega_{q}\right)$, where both the excitation energy $\hbar \omega_{q}^{S M}=\varepsilon_{q} / S(q)$ and the quasiparticle weight $Z_{q}^{\mathrm{SM}}=n S(q)$ are fixed completely by the two lowest sum rules

$$
\begin{equation*}
m_{0}(q)=\hbar \int_{-\infty}^{\infty} d \omega S(\omega, q)=n S(q) \text { and } m_{1}(q)=\hbar^{2} \int_{-\infty}^{\infty} d \omega \omega S(\omega, q)=n \varepsilon_{q} \tag{99}
\end{equation*}
$$

therefore turns out to be exact at arbitrary momenta. As will be shown below, this simple structure is not valid in general. It is a deep result due to Feynman [23], however, that even in strongly interacting superfluids like ${ }^{4} \mathrm{He}$, phonon excitations exhaust the $f$-sum rule in the limit of small wave vectors. The single-mode approximation is thus exact quite generally in the long-wavelength limit $q \rightarrow 0$. Feynman's arguments rely on showing that - as a result of the symmetry and positivity of the ground state wave function $\left\langle x_{1} \ldots x_{N} \mid \psi_{0}\right\rangle$ - the only excitations above the exact ground state as $q \rightarrow 0$ are of the form $\hat{\rho}_{q}^{\dagger}\left|\psi_{0}\right\rangle$. Such a simple description no longer holds at large momentum $q \xi \gg 1$ unless the parameter $q a=q \xi\left(8 \pi n a^{3}\right)^{1 / 2}$ remains much smaller than one. Indeed, the failure of Bogoliubov theory in the regime $q a=O(1)$ was observed some time ago in a Bragg scattering experiment on ${ }^{85} \mathrm{Rb}$ by Papp et al. [199]. The experiment measures the so-called line shift $\Delta \omega_{q}=\omega_{q}-\varepsilon_{q} / \hbar$ defined by the deviation of the peak position $\hbar \omega_{q}$ in the dynamic structure factor from the single-particle energy $\varepsilon_{q}$. The measurement is carried out at fixed large momentum as a function of the scattering length. For small values $q a \ll 1$, the observed line shift follows the linear dependence $\Delta\left(\hbar \omega_{q}\right)=g n \sim a$ on the scattering length predicted by the Bogoliubov theory. For increasing scattering lengths, however, the shift reaches a maximum near $q a \simeq 1$ and then starts to decrease slightly [199]. A detailed analysis of this quite unexpected phenomenon has been performed more recently by Lopes et al. [200] with a gas of ${ }^{39} \mathrm{~K}$ in a box geometry. Changing the scattering length near a Feshbach resonance at $B_{0} \simeq 402.7 \mathrm{G}$, they have measured the line shift at fixed wave vector of the Bragg pulse for scattering lengths up to $q a \simeq 7$. As shown in Fig. 14, the line shift exhibits a non-monotonic behavior and assumes negative values beyond a characteristic dimensionless product $q \bar{a} \simeq 1.3$. The data in the left figure for two different densities reach up to $q a \simeq 2.5$ and $\left(n a^{3}\right)^{1 / 2} \simeq 0.05$ (orange diamonds). They are in quite good agreement with the prediction of the single mode approximation (full curves) which is called Feynman-Tan in Ref. [200]. The data in the Figure on the right is taken at densities about a factor ten lower than those of the blue circles on the left. Due to the strongly reduced three-body decay rate, this allows to reach scattering lengths up to $q a \simeq 7$ and a dimensionless interaction parameter $\left(n a^{3}\right)^{1 / 2} \simeq 0.1$. Apparently, over this much wider range the single mode approximation fails and the data are consistent with the result in Eq. 106 below derived from a systematic expansion in inverse powers of the momentum. As will be discussed below, the considerable deviation at large values $q a \gtrsim 4$ is likely to be caused by three-particle or higher order correlations.

The physical origin of the eventually negative line shift can be understood in qualitative terms by considering the thresholds in energy above which processes where a given momentum $q$ is imparted on $n=1,2,3 \ldots$ particles are possible. Obviously, for $n=1$, the threshold is at the single particle energy $\varepsilon_{q}$ to lowest order because for $q \xi \gg 1$, the characteristic momenta $\simeq 1 / \xi$ of Bosons in the initial state are negligible (a more refined description is provided by the impulse approximation below).


For $n=2$, however, the threshold is only half as large. Indeed, as indicated in the Figure, for an initial momentum $\boldsymbol{q}=\boldsymbol{q}_{1}+\boldsymbol{q}_{2}$ which - for simplicity - is assumed to be distributed symmetrically to two atoms at rest, the necessary energy $\hbar \omega_{q}^{(2)}(\alpha)=\varepsilon_{q_{1}}+\varepsilon_{q_{2}}=\varepsilon_{q} /\left(2 \cos ^{2} \alpha / 2\right)$ depends on the angle $\alpha$ between $\boldsymbol{q}_{1}$ and $\boldsymbol{q}_{2}$. It becomes arbitrarily large as $\alpha \rightarrow \pi$, is equal to $\varepsilon_{q}$ at $\alpha=\pi / 2$ and assumes its minimum value $\hbar \omega_{\min }^{(2)}=\varepsilon_{q} / 2$ for collinear processes with $\alpha=0$. Thus, taking into account processes in which two atoms are involved in taking up the momentum, gives rise to an energetic threshold a factor two below the single particle energy $\varepsilon_{q}$. The likelihood for such processes is proportional to the probability that two particles are close together, which is quantified by the two-body contact density $C_{2}$. As discussed above, the contact density increases with the scattering length, thus making two-particle processes more likely for larger values of $q a$.

This provides a simple picture for the origin of the downturn of the line shift as a function of $q a$ and its eventual negative values shown in Fig. 14 As will be discussed below, a quantitative description of the observed behavior requires to include also processes in which the momentum is transferred to more than two particles. The associated threshold energies $\hbar \omega_{\min }^{(n)}=\varepsilon_{q} / n$ form a cascade, decreasing inversely with the number $n$ of particles involved in higher order collinear processes [201].

Line shift from an operator product expansion For a quantitative description of the relative weight of single- versus multiparticle contributions to the dynamic density response, the operator product expansion in Eq. 80p must be extended to the product of two density operators at points which are close both in space and also in time. The product is again expanded as a sum of local operators $\hat{O}_{\ell}(0,0)$. The resulting Wilson coefficients $W_{\ell}(t, r, a)$ then also involve a dependence on the time difference. In addition, as usual, they depend parametrically on the scattering length $a$. Upon Fourier transformation, the OPE thus gives rise to an expansion

$$
\begin{equation*}
\chi(\omega, q)=\sum_{\ell} \frac{m}{\hbar^{2} q^{\Delta_{\ell}-1}} J_{\ell}\left(Z, \frac{1}{q a}\right)\left\langle\hat{O}_{\ell}\right\rangle, \tag{100}
\end{equation*}
$$

of the density response function in inverse powers of the momentum, where the momentum dependence arising from the Wilson coefficients $W_{\ell}(t, r, a)$ is factored out. The remainder is written in terms of a dimensionless scaling function $J_{\ell}$ which depends on the dimensionless momentum and frequency variables $(q a)^{-1}$ and $Z=\left(\hbar \omega-\varepsilon_{q}\right) / \varepsilon_{q}$. The exponents in the prefactor depend on the scaling dimension $\Delta_{\ell}$ of the operators $\hat{O}_{\ell}$. Its leading-order terms are formed by the operators with the lowest scaling dimension, which are the density operator with $\Delta_{n}=3$ and the two-body contact operator $\hat{O}_{c}$ with dimension $\Delta_{C_{2}}=4$ as discussed above. Up to this order, the dynamic structure factor is of the form

$$
\begin{equation*}
S(\omega, q)=\frac{m n}{\hbar^{2} q^{2}} \cdot \frac{1}{\pi} \operatorname{Im} J_{n}(Z)+\frac{m C_{2}}{\hbar^{2} q^{3}} \cdot \frac{1}{\pi} \operatorname{Im} J_{c}\left(Z, \frac{1}{q a}\right)+\ldots=n \delta\left(\hbar \omega-\varepsilon_{q}\right)+\frac{m C_{2}}{\hbar^{2} q^{3}} J_{\mathrm{OPE}}\left(Z, \frac{1}{q a}\right)+\ldots \tag{101}
\end{equation*}
$$

Due to $\operatorname{Im} J_{n}(Z)=2 \pi \delta(Z)$, the leading term does not depend on the interaction strength and gives rise to a sharp single particle peak at the energy of a free particle with momentum $q$. Apparently, at this level, the OPE does not account even for the mean-field shift $g n$ which appears within the Bogoliubov approximation. To determine this shift and also the associated finite broadening for arbitrary values of $q a$, the density response function near the single-particle peak is expressed in the form

$$
\begin{equation*}
\chi(\omega, q)=-\frac{Z_{q}}{\hbar \omega-\varepsilon_{q}-\Pi(\omega, q)}+\chi^{\mathrm{inc}}(\omega, q) \tag{102}
\end{equation*}
$$

Here, $\Pi(\omega, q)$ is a self-energy while $\chi^{\text {inc }}$ denotes an incoherent part that will be discussed in more detail below. The peak position at $\omega=\omega_{q}$ is defined by the zeros of the denominator $\hbar \omega-\varepsilon_{q}-\operatorname{Re} \Pi(\omega, q)=0$. Moreover, the width $\Gamma$ of the peak is determined by the associated imaginary part $\Gamma=-\operatorname{Im} \Pi\left(\omega_{q}, q\right)$ at the peak position $\omega_{q}$. At large momentum, the many-body correction induced by $\Pi$ is subleading, and we can determine the shift $\Delta(\hbar \omega)=\operatorname{Re} \Pi\left(\varepsilon_{q}, q\right)$ of the single particle peak in the on-shell approximation. Expanding the density response to leading order in $\Pi$,

$$
\begin{equation*}
\chi(\omega, q)=-\frac{Z_{q}}{\hbar \omega-\varepsilon_{q}}-\frac{Z_{q} \Pi\left(\varepsilon_{q}, q\right)}{\left(\hbar \omega-\varepsilon_{q}\right)^{2}}+\cdots, \tag{103}
\end{equation*}
$$

we may infer the high-momentum structure of $Z_{q}$ and $\Pi$ by computing the contribution proportional to $1 / Z^{2}$ of the universal scaling functions $J_{\ell}$ in Eq. (100). At the leading non-trivial level associated with two-particle processes described by the contact operator $\hat{O}_{c}$, the result is [194, 202]

$$
\begin{equation*}
J_{c}\left(Z \rightarrow 0, \frac{1}{q a}\right)=\frac{1}{\pi Z^{2}}\left[\frac{i}{2}+\frac{1}{q a}-\frac{2}{q a} \frac{1}{1+i q a / 2}\right] \tag{104}
\end{equation*}
$$

To leading order, the pole strength $Z_{q}=n$ is thus unchanged but there is a finite, complex self energy

$$
\begin{equation*}
\Pi\left(\varepsilon_{q}, q\right)=\frac{\hbar^{2} C_{2}}{4 \pi m n a}\left[\frac{2}{1+i q a / 2}-\frac{i q a}{2}-1\right] . \tag{105}
\end{equation*}
$$

Its real and imaginary part

$$
\begin{equation*}
\Delta\left(\hbar \omega_{q}\right)=\frac{\hbar^{2} C_{2}}{4 \pi m n a}\left[\frac{2}{1+(q a / 2)^{2}}-1\right] \quad \text { and } \quad \Gamma_{q}=\frac{\hbar^{2} C_{2} q}{8 \pi m n}\left[\frac{2}{1+(q a / 2)^{2}}+1\right] . \tag{106}
\end{equation*}
$$

then determine the line shift and the associated finite peak width. For small momenta $q a \ll 1$, the width $\Gamma_{q} \rightarrow g n \cdot 3 q a / 2$ vanishes while it increases linearly with momentum $\Gamma_{q} \sim q$ in the opposite limit, a dependence that will be discussed in more detail below. Regarding the line shift, the OPE result (106) coincides with the Bogoliubov prediction $\Delta\left(\hbar \omega_{q}\right) \rightarrow g n$ in the limit $q a \ll 1$, where $C_{2} \rightarrow(4 \pi n a)^{2}$ may be replaced by the contact density of a weakly interacting gas. With increasing values of $q a$, however, the line shift approaches a maximum close to $q a \simeq 1$, then bends downwards and changes sign at $q \bar{a}=2$. In the regime $q a \gg 1$, Eq. 106 predicts a negative line shift

$$
\begin{equation*}
\lim _{q a \gg 1} \Delta\left(\hbar \omega_{q}\right)=-\frac{\hbar^{2} C_{2}(a)}{4 \pi m n a} \rightarrow 0 \tag{107}
\end{equation*}
$$

independent of the momentum tranfer $q$ which approaches zero in the unitary limit. This result is likely to be changed at higher order in the OPE. In particular, the contribution from the three-body contact is expected to lead to a finite line shift $\simeq \hbar^{2} C_{3} /(\mathrm{mn})$ in the limit $q a \gg 1$, similar to the result (90) for the Ramsey precession rate.

As shown in Fig. 14. a comparison between theory and the experimental results reveals that, up to values $q a \simeq 2-3$, they agree quite well with the assumption that the single-mode approximation remains valid even at large momenta. Indeed, using the exact asymptotic dependence $(75)$ of the static structure factor, the resulting peak position

$$
\begin{equation*}
\hbar \omega_{q}^{\mathrm{SM}}=\varepsilon_{q} / S(q) \underset{q \xi \gg 1}{\longrightarrow} \varepsilon_{q}+\frac{\hbar^{2} C_{2}}{4 \pi m n a}\left(1-\frac{\pi q a}{4}+O(1 / q)\right) / S(q) \tag{108}
\end{equation*}
$$

correctly captures the mean-field shift $g n$ obtained within Bogoliubov theory if the contact density is replaced by its weak coupling value. It also accounts for the observed backbending in the regime $q a \simeq 1$ through the factor $1-\pi q a / 4$ which arises from the slow $C_{2} / q$-decay of $S(q)$ towards one. In particular, neglecting the $1 / q$-corrections from both the numerator and denominator, Eq. (108) predicts a zero crossing of the line shift at $q \bar{a}=4 / \pi$ which is close to the observed value. By contrast, the result $q \bar{a}=2$ for the zero crossing obtained from the OPE above is appreciably larger. The apparent success of the single-mode approximation even at large momenta has a historical precursor in Feynman's famous explanation [23] for the roton minimum in the excitation spectrum of ${ }^{4} \mathrm{He}$ as a result of the peak in the static structure factor. In practice, the peak is hardly pronounced, with a maximum value $S\left(q_{0}\right) \simeq 1.4$ around $q_{0} \sigma \simeq 5$ [6]. In a dilute quantum gas, where $\sigma$ is essentially zero, such a maximum appears in the presence of dipolar interactions, whose Fourier transform $V_{\mathrm{dd}}(q)$ is negative in a finite range of wave vectors, see the discussion in Lecture II. In fact, however, a broad maximum in the static structure factor is present even for gases with short-range interactions and large scattering lengths. Using the asymptotic expansion (75), it appears at $q_{0} a=8 / \pi$ with a value

$$
\begin{equation*}
S\left(q_{0}\right)=1+\frac{\pi a}{128 n} C_{2}(a) \xrightarrow[n a^{3} \ll 1]{ } 1+\frac{\pi^{3}}{8} n a^{3} . \tag{109}
\end{equation*}
$$

Within the single mode approximation (108), the presence of a maximum in the level shift shown in Fig. 14 might thus be interpreted as a roton precursor in a dilute but strongly interacting BEC. In the following, we will show that this interpretation fails in the regime where $q a$ becomes appreciably larger than one. Indeed, the single mode approximation misses the fact that the complete spectrum for large momentum is spread over an energy range which increases proportional to $q^{2}$. Its failure at large momenta is evident already from the limiting behavior $\Delta\left(\hbar \omega_{q}^{\mathrm{SM}}\right) \rightarrow-\varepsilon_{q} C_{2}(a) /(8 n q) \sim-q \mathcal{C}_{2}(a)$ of the associated line shift, which becomes increasingly negative in the limit $q a \gg 1$ instead of approaching a momentum independent value as predicted by the OPE result (107). A proper theory of Bragg scattering at large momentum requires to combine the classic impulse approximation due to Hohenberg and Platzman [203] to describe the behavior near the single particle peak with a systematic OPE which properly accounts for multi-particle excitations. Taken together, this gives rise to a complex spectrum which cannot be captured within a single mode approximation. An important point to note is that the associated characteristic frequency $\hbar \omega_{q}^{S M}=\varepsilon_{q} / S(q)=m_{1} / m_{0}$ provides an exact result for the normalized first moment of the spectrum as the ratio of the two sum rules in Eq. (99), irrespective of how complex the spectrum may be. However, unless there is a single dominant peak, the first moment contains litte information: neither the position nor the width of peaks in a more complex structure can be extracted.

Impulse approximation and the parton model The expansion (101) shows that the leading term in the OPE does not capture any details of the spectrum near the single particle peak and, in particular, it is insensitive to the specific momentum distribution of the many-body system. As shown above, however, the corrections which are described by the contact operator allow to determine the overall shift and broadening of the peak due to two-particle processes. Now, it turns out that in the vicinity of the single particle peak at large momenta a more refined description is possible where a rather different universal scaling function emerges. Indeed, as anticipated by Miller, Pines, and Nozières [204] and then shown in detail by Hohenberg and Platzman [203], the dynamic structure factor at large wave vectors provides a direct measure of the momentum distribution. This prediction is based on the so-called impulse approximation (IA) which assumes that, at large wave vectors $q$, the response is given by a Fermi golden rule expression for exciting a single atom with small momentum $k$ to a large momentum $k+q$. Neglecting interactions between the final and initial state atoms, this yields the impulse approximation

$$
\begin{equation*}
S_{\mathrm{IA}}(\omega, q)=\int_{k} n(k) \delta\left(\hbar \omega+\varepsilon_{k}-\varepsilon_{k+q}\right), \tag{110}
\end{equation*}
$$

in which the dynamic structure factor is completely determined by the momentum distribution $n(k)$ of the strongly interacting quantum fluid. As will be indicated schematically in Eq. (116) below, this approximation is analogous to the parton model of high-energy physics introduced by Feynman [205] and Bjorken and Paschos [206]. The IA does not take into account interactions between the scattered state and the initial state. As a result, it carries information about the time-dependent density correlations only through the equal-time momentum distribution. A crucial prediction of the IA is a spectrum which is perfectly symmetric around the single particle energy and, moreover, obeys a particular form of scaling. Specifically, assuming a rotationally invariant system with a finite condensate density, the general form $n(k)=(2 \pi)^{3} n_{0} \delta(\boldsymbol{k})+\tilde{n}(k)$ of the momentum distribution gives rise to a scaling form of the dynamic structure factor

$$
\begin{equation*}
S_{\mathrm{IA}}(\omega, q)=\frac{m}{\hbar^{2} \tilde{\xi}^{2}} \frac{1}{q} J_{\mathrm{IA}}(Y)=\frac{m}{\hbar^{2} \tilde{\xi}^{2}} \frac{1}{q}\left[n_{0} \tilde{\xi}^{3} \delta(Y)+\frac{\tilde{\xi}^{2}}{4 \pi^{2}} \int_{|Y| / \tilde{\xi}}^{\infty} d k k \tilde{n}(k)\right] . \tag{111}
\end{equation*}
$$

It has a singular part arising from the condensate and a smooth background associated with particles in the depletion. Formally, this expression is of the form (100) with a scaling dimension $\Delta=2$. The associated scaling variable

$$
\begin{equation*}
Y=\frac{m \tilde{\xi}}{\hbar^{2}} \frac{\hbar \omega-\varepsilon_{q}}{q}=\frac{q \tilde{\xi}}{2} \cdot Z \tag{112}
\end{equation*}
$$

however, differs from that in (100) by a factor $q \tilde{\xi}$. It involves a length scale $\tilde{\xi}$ whose inverse is the characteristic scale over which the momentum distribution varies. For weakly interacting Bosons, a convenient choice for $\tilde{\xi}$ is the standard healing length $\xi=(8 \pi n a)^{-1 / 2}$ which determines the associated contact density by $C_{2}=1 /\left(4 \xi^{4}\right)$. In the limit of a unitary Bose gas, in turn, the characteristic wave number is determined by the inverse $1 / \tilde{\xi} \simeq n^{1 / 3}$ of the average interparticle spacing. Apparently, the IA exhibits a delta peak in the spectrum at $\hbar \omega=\varepsilon_{q}$ only in the presence of a finite condensate density $n_{0}$. This is quite different from the prediction (101) of the OPE, where the spectrum near the single-particle energy consists of a delta function with full weight $n$ plus a background proportional to $1 / Z^{2}$. Remarkably, it turns out, that the latter dependence is identical to what is obtained in the limiting regime $|Y| \gg 1$ of the range $Y=O(1)$ of validity of the IA. As a result, there is a smooth crossover from the IA to the OPE which is reached when the associated scaling variable $|Z|=2|Y| / q \tilde{\xi}$ is of order $1 / q \tilde{\xi}$, as indicated schematically in Fig. 15 . On a formal level, the fact that the IA and the OPE are smoothly connected follows by realizing that both the OPE scaling function (104) and that of the impulse approximation exhibit a quadratic divergence $\sim 1 / Z^{2} \simeq(q \tilde{\xi})^{2} / Y^{2}$ near the single particle peak. Indeed, in the limit $|Y| \gg 1$, the scaling function $J_{\mathrm{IA}}(Y)$ in Eq. (111) depends on the behavior of the momentum distribution at large momenta. As derived in Eq. 85), for zero range interactions $n(k) \rightarrow C_{2} / k^{4}$ exhibits a universal power-law decay determined by the contact density $C_{2}$. For large values $|Y| \gg 1$, the scaling function $J_{\mathrm{IA}}(Y)$ thus acquires a universal form

$$
\begin{equation*}
\lim _{|Y| \gg 1} J_{\mathrm{IA}}(Y)=\frac{\tilde{\xi}^{4} C_{2}}{8 \pi^{2} Y^{2}} \tag{113}
\end{equation*}
$$

Combining this result with that of Eq. [104], the dynamic structure factor near the single particle peak

$$
\begin{equation*}
\lim _{|Z|<1} S_{\mathrm{OPE}}(\omega, \boldsymbol{q})=\frac{m C_{2}}{\hbar^{2} q^{3}} \frac{1}{2 \pi^{2} Z^{2}} \equiv \lim _{|Y| \gg 1} S_{\mathrm{IA}}(\omega, \boldsymbol{q})=\frac{m}{\hbar^{2} \tilde{\xi}^{2}} \frac{1}{q} \frac{\tilde{\xi}^{4} C_{2}}{8 \pi^{2} Y^{2}} \tag{114}
\end{equation*}
$$

turns out to exactly coincide in both approaches, independent of the choice for the characteristic length $\tilde{\xi}$. For large momentum $q a \gg 1$, the IA and OPE are therefore complementary scaling functions that describe separate asymptotic high-momentum regimes. They match smoothly in the regime where $|Z|=2|Y| /(q \tilde{\xi})$ is of order $1 /(q \tilde{\xi})$, which is precisely beyond the width $\Gamma_{q} \simeq \hbar^{2} C_{2} /(m n) \cdot q$ of the single particle peak as determined above in Eq. 106.



FIG. 15: The diagram on the left shows the crossover between the regimes where the dynamic structure factor at large momentum is described by either the operator product expansion (OPE) or the impulse approximation (IA), whose range of applicability shrinks with increasing momentum. The OPE properly describes the spectrum away from the single-particle peak, which is strongly asymmetric. This is shown explicitely on the right for the special case of a unitary Bose gas. The OPE and IA results are represented by the dashed green or red lines, respectively, with a sharp threshold in the OPE at $\varepsilon_{q} / 2$ due to collinear two-particle processs. A uniform approximation which captures the complete spectrum including the condensate delta-peak and which is rather close to the OPE unless $\hbar \omega \approx \varepsilon_{q}$ is shown in blue (from Ref.[194]).

The underlying physics behind this crossover can be understood by noting that the impulse approximation describes the dynamic structure factor in the regime where an excitation is created by transferring the large probe wave vector $q$ to a single atom with wave vector $k$, which is drawn from an initial momentum distribution $n(k)$ that is concentrated in a momentum range $\tilde{\xi}^{-1} \ll q$. Energy conservation then implies $\hbar \omega+\varepsilon_{k}=\varepsilon_{k+q}$ to leading order, giving rise to a deviation $\hbar \omega-\varepsilon_{q}=O\left(q^{1}\right)$ of the excitation energy from the single-particle energy which scales linearly with the wave vector $q$. The scaling variable $Y \sim\left(\hbar \omega-\varepsilon_{q}\right) / q$ is therefore of order one, which defines the regime where the impulse approximation applies. As discussed above, the behavior further away from the single particle peak involves excitations in which a large momentum is transferred to two or more particles. The associated excitation energies are then of order $\hbar \omega-\varepsilon_{q}=O\left(q^{2}\right)$, corresponding to $Y=O(q \tilde{\xi})$ and thus $Z=O(1)$. This regime is properly described by the OPE, which provides a systematic treatment of multi-particle excitations through the expansion (100) in inverse powers of the momentum. Explicit results for the dynamic structure factor away from the single-particle peak are so far restricted to the leading order associated with two-particle processes, where

$$
\begin{equation*}
\left.S(\omega, q)\right|_{Z \ngtr 0}=\frac{m C_{2}}{\hbar^{2} q^{3}} J_{\mathrm{OPE}}\left(Z, \frac{1}{q a}\right)+\ldots \underset{Z \gg 1}{\longrightarrow} \frac{2 \hbar^{3 / 2}}{45 \pi^{2} m^{5 / 2}} \frac{q^{4} C_{2}}{\omega^{7 / 2}}+\ldots \tag{115}
\end{equation*}
$$

is proportional to the two-body contact density $C_{2}$. The associated scaling function $J_{\text {OPE }}$ can be determined in analytical form (for the special case of a unitary gas, see Eq. (16) of Ref. [194]). As shown in Fig. 15]on the right, the resulting full spectrum is far more complicated than what is captured by a single mode approximation, which only describes a single peak but no incoherent background $S^{\text {inc }}(\omega, q)$ arising from the corresponding contribution in Eq. (102). In contrast to the prediction of the impulse approximation, the incoherent background is apparently strongly asymmetric. In particular, in the deep inelastic limit $Z \gg 1$ far to the right of the single-particle peak, it decays with a power law $\sim \omega^{-7 / 2}$ as derived originally by Son and Thompson [201]. Physically, this decay arises from processes in which a given momentum $\boldsymbol{q}$ is transferred to two particles with momenta $\boldsymbol{q}_{1}$ and $\boldsymbol{q}-\boldsymbol{q}_{1}$ such that $\left|\boldsymbol{q}_{1}\right| \gg \boldsymbol{q}$. As discussed above for the special symmetric case $\left|\boldsymbol{q}_{1}\right|=\left|\boldsymbol{q}-\boldsymbol{q}_{1}\right|$, this requires an energy transfer $\hbar \omega$ which is far larger than the single particle energy. To the left of the single particle peak, there is a sharp onset at $\hbar \omega_{\min }^{(2)}=\varepsilon_{q} / 2$ which is due to collinear processes in which the initial momentum is distributed to two atoms at rest. In the special case of a unitary gas which is shown in Fig. 15, the dynamic structure factor just above threshold exhibits a square root divergence since the two-body scattering amplitude diverges near zero energy. For finite scattering length, this divergence disappears and is replaced by the standard Wigner threshold law.

The presence of an appreciable weight in the spectrum at energies which are a factor two below the single particle energy provides a simple explanation for the observation above that the formal line shift $\Delta\left(\hbar \omega_{q}^{\mathrm{SM}}\right) \rightarrow-q \mathcal{C}_{2}(a)$ in the single mode approximation diverges linearly with the momentum. Indeed, if the spectrum is modelled in terms of two sharp peaks at $\varepsilon_{q}$ and $\varepsilon_{q} / 2$ with respective weights $Z_{q}^{(1)}$ and $Z_{q}^{(2)}$, the sum rule $m_{0}=n S(q) \simeq n+C_{2} / 8 q$ for the static structure factor together with the f-sum rule $m_{1}=n \varepsilon_{q}$ uniquely determine both $Z_{q}^{(1)}=n-C_{2} / 8 q$ and $Z_{q}^{(2)}=C_{2} / 4 q$ to leading order in the inverse momentum. That the normalized first moment of the spectrum is below the single particle energy by an amount of order $q C_{2}$ is thus a simple consequence of the presence of spectral weight of order $C_{2} / q$ at energies of about $\varepsilon_{q} / 2$ below the single particle peak. The argument also shows that the single mode energy $\hbar \omega_{q}^{\mathrm{SM}}$, which provides an upper bound to the minimum energy associated with
an excited state of the form $\hat{\rho}_{q}^{\dagger}\left|\psi_{0}\right\rangle$ [143], may be very far above the true onset of the spectrum. The bound is therefore useful only the limit $q \rightarrow 0$, where $m_{1}(q) / m_{0}(q) \rightarrow \hbar c_{s} q$ and the dominant peak in the dynamic structure factor approaches zero energy.

Experimentally, the nontrivial form of the spectrum shown in Fig. 15 on the right, has not been resolved and it is only the shift of the single-particle peak and its width which is accessible so far. Moreover, even within this limited information, the regime where the line shift bends upwards again according to Eq. 107) has apparently not been reached in the results shown in Fig. 14. Theoretically, the line shift is expected to approach a finite value of order $\simeq \hbar^{2} C_{3} /(m n)$ in the unitary Bose gas due to processes where the large momentum is distributed to three particles. Unfortunately, a quantitative prediction in this limit requires to extend the OPE result (106) to include three-particle and maybe even higher order processes. Since the associated threshold energies $\varepsilon_{q} / n$ are below the two-particle onset at $\varepsilon_{q} / 2$, this will lead to a more negative line shift for large scattering lengths compared with the two-particle result (106). This provides a qualitative explanation for the observation that the shifts shown in Fig. 14 on the right are more negative than the OPE result in the regime $q a \gtrsim 4$. Higher order effects will also change the prediction $q \bar{a}=2$ for the characteristic dimensionless scattering length $q \bar{a}$ where the line shift crosses zero to lower and density dependent values, again qualitatively consistent with the observed behavior. On a formal level, the problem of accounting for processes which involve more than two particles is similar to the one which appears in the extension of the parton model in high-energy physics by incorporating interactions between the constituents in a systematic expansion in powers of the coupling constant $\alpha_{s}$ of the strong interactions. Indeed, as mentioned above, the impulse approximation of Hohenberg and Platzman is historically a precursor of the parton model for the scattering of electrons on protons at energies $E>m_{p} c^{2}$, where the proton behaves as a composite object. Specifically, introducing a distribution function $f(\xi)$ for the constituents to carry a fraction $\xi \in(0,1)$ of the proton's four momentum, a process at large momentum transfer $Q$ associated with a loss $v=E-E^{\prime}$ of the electron energy in the lab frame gives rise to a cross section which involves [207]

$$
\begin{equation*}
f(\xi) \delta\left(v-\frac{\hbar^{2} Q^{2}}{2 m_{p} \xi}\right)=\frac{2 m_{p} x^{2}}{\hbar^{2} Q^{2}} f(\xi) \delta(x-\xi) \text { with } x=\frac{\hbar^{2} Q^{2}}{2 m_{p} v} \in(0,1) \tag{116}
\end{equation*}
$$

in analogy to what appears in the impulse approximation 110 for scattering from a non-relativistic quantum fluid with an unknown momentum distribution $n(k)$. In high-energy physics, the expression (116) leads to a cross section which - at a fixed value of the Bjorken scaling variable $x$ - turns out to be independent of $Q$. This is analogous to the description of the dynamic structure factor in the vicinity of the single particle peak in terms of a scaling function (111) which only depends on the variable $Y \sim\left(\hbar \omega-\varepsilon_{q}\right) / q$. As discussed above, this scaling is replaced by a different one in the regime $|Y| \gg 1$, where multi-particle processes become relevant ${ }^{23}$. The OPE results so far properly account for this at the two-particle level. Their extension to include the effects of the three-body and higher order contacts remains an open problem, however.

[^23]
## IV. SCALE AND CONFORMAL INVARIANCE IN ULTRACOLD GASES

Hidden symmetries in the 4D hydrogen atom, in electrodynamics and the theory of elastic continua The subject of scale and conformal invariance has been a central theme in statistical physics and QFT for quite some time, for an introduction see e.g. the book by Cardy [208] or the Lecture notes by Nakayama [209] and by Rychkov [210]. Its relevance in the context of ultracold atoms has been realized by Pitaevskii and Rosch [211] for Bose gases in two dimensions and independently by Werner and Castin [212] and by Son and coworkers [107, 213] in the context of two-component Fermi gases at infinite scattering length. As an introduction, we start with the elementary example of the hydrogen atom. In the standard case of three dimensions, a solution of the Schrödinger equation requires to determine the eigenfunctions and eigenvalues of the operator $\tilde{H}\left[r^{-1}, 3\right]=-\nabla^{2}-2 /\left(a_{B} r\right)$, where $a_{B}$ is the Bohr radius. There is a continuous spectrum of scattering states at positive energies and a discrete spectrum $-\mathrm{Ry} / n^{2}$ of bound states with $\mathrm{Ry}=\hbar^{2} /\left(2 m a_{B}^{2}\right)$ as the characteristic energy scale ( $m$ is the reduced mass). Now, something quite surprising happens in four dimensions, where Gauss' law $\nabla \boldsymbol{E}=\rho(\boldsymbol{x}) \rightarrow e_{4} \delta(\boldsymbol{x})$ leads to an inverse square attractive interaction $V(\boldsymbol{x})=-e_{4}^{2} /\left(2 \pi^{2} r^{2}\right)$ between two opposite point charges $\pm e_{4}$. Introducing a dimensionless equivalent $\tilde{a}_{B}=2 \pi^{2} \hbar^{2} /\left(m e_{4}^{2}\right)$ of the Bohr radius, the Schrödinger operator $\tilde{H}\left[r^{-2}, 4\right]=-\nabla^{2}-2 /\left(\tilde{a}_{B} r^{2}\right)$ is now scale invariant: under a rescaling $\boldsymbol{x} \rightarrow b \boldsymbol{x}$ of lengths by a continuous parameter $b$, the operator changes just by a factor $1 / b^{2}$. For every solution $\psi_{E}(\boldsymbol{x})$ of the stationary Schrödinger equation with energy $E$, there is thus a solution $\psi_{E}(\boldsymbol{x} / b)$ with energy $E / b^{2}$. Since $b$ is continuous, this implies that the hydrogen atom in four dimensions and - more generally - the operator $\tilde{H}\left[r^{-2}, d\right]$ in any dimension $d$ does not have a discrete spectrum! In fact, it turns out that this is not the full story: the result only holds if the strength $\kappa=2 / \tilde{a}_{B}$ is below a critical value $\kappa<\kappa_{c}=(d-2)^{2} / 4$ beyond which the continuous scale symmetry is replaced by a discrete one [214, 215]. In the case of the hydrogen atom in four dimensions, there are thus no bound states at all if $\tilde{a}_{B}>2$, i.e. when the reduced mass is below a limit $m<\pi^{2} \hbar^{2} / e_{4}^{2}=m_{c}$ for a given value of the charge. In the case $m>m_{c}$, an anomaly appears and there is an infinite number of bound states (classically, this corresponds to a fall into the center). In particular, the energy is then bounded below only if one introduces a finite lower cutoff $r \geq r_{p}$ i.e. an effective proton radius. Fortunately, this problem does not show up in three dimensions, however a Hamiltonian of the type $\tilde{H}\left[r^{-2}, 3\right]$ with an angular dependent strength $\kappa \sim p \cos \theta$ describes the interaction between a charge and a dipole. The resulting spectrum turns out to be scale invariant without any bound states only for sufficiently small values of the dipole moment. Specifically, for an electron interacting with a polar molecule, the appearance of bound states requires the electric dipole moment to be larger than about two Debye [216, 217].

From the point of view in QFT, single-particle quantum mechanics corresponds to a zero-dimensional situation. In this case, scale invariance is present if the classical action $S=\int d t L$ remains unchanged under a rescaling $\boldsymbol{x} \rightarrow b \boldsymbol{x}$ and $t \rightarrow b^{2} t$ of lengths and time. An extension of this to the larger symmetry group associated with conformal invariance, where $b \rightarrow b(x)$ acquires a spatial dependence, is possible in field theory, where the action $S=\int \mathcal{L}$ involves an integration over spatial variables $x$ and - in the case of electrodynamics or genuine QFT-problems - also time. In the context of statistical physics, scale invariance appears right at the critical point of a continuous phase transition, where the coefficient $m$ in the quadratic contribution to the effective potential $V_{\text {eff }}[\phi] \sim m \phi^{2}$ in the order parameter $\phi$ vanishes. In analogy with high-energy physics, this is called a massless theory.


Generically, such points exhibit not only scale but also conformal invariance. The associated symmetry is an extension of the simple examples of scale invariance discussed above, where the rescaling factor $b(x)$ acquires an arbitrary spatial dependence and, moreover, also local rotations $R(x)$ are incorporated. Schematically, this is indicated in the Figure on the left, taken from Ref. [209]. On a formal level, conformal transformations may be defined by the fact that they change the trivial metric tensor of an Euclidean geometry (or Minkowski, where $\delta_{\mu \nu} \rightarrow \eta_{\mu \nu}$ ) to a new metric of the form $g_{\mu \nu}(x)=b^{2}(x) \delta_{\mu \nu}$ by a so-called Weyl transformation. Expressing the associated change of coordinates $x^{\mu} \rightarrow x^{\mu}+\varepsilon \xi^{\mu}(x)$ in infinitesimal form, there are four possible types of transformations (in the special case of two dimensions, there is in fact an infinite number of them associated with an arbitrary analytic function $\xi(z)$ of the complex variable $z=x+i y)$. They may be expressed in the form [218]

$$
\begin{equation*}
\delta g_{\mu \nu}(x)=\varepsilon\left(\partial_{\mu} \xi_{v}+\partial_{\nu} \xi_{\mu}\right)=\delta b^{2}(x) \delta_{\mu \nu} \rightarrow \xi^{\mu}(x)=a^{\mu}+b_{\nu}^{\mu} x^{\nu}+c x^{\mu}+d_{v}\left(\delta^{\mu v} x^{2}-2 x^{\mu} x^{v}\right) . \tag{117}
\end{equation*}
$$

The first two terms describe translations by a vector $a^{\mu}$ and rotations, which are generated by an antisymmetric tensor $b^{\mu \nu}$. In both cases, the metric does not change at all, that is $\delta b^{2}(x) \equiv 0$. The contribution $c x^{\mu}$ corresponds to scale transformations with a constant rescaling factor $\delta b^{2}=2 \varepsilon c$. The genuine conformal transformations, for which $\delta b^{2}(x)=-4 \varepsilon d_{\mu} x^{\mu}$ is spatially varying, are associated with the last term, which involves $d$ additional parameters $d_{v}$. Quite generally, by Noether's theorem, invariance of the action $S$ under continuous transformations generated by some vector $\xi^{\mu}(x)$ gives rise to conservation laws $\partial_{\mu} J^{\mu}=0$. The associated currents $J^{\mu}$ may be expressed in terms of the different so-called Killing vectors $\xi_{v}$ and the stress-
energy tensor $\theta^{\mu \nu}$ in the form $J^{\mu}=\xi_{v} \theta^{\mu \nu}$. Each of the four contributions in Eq. (117) thus leads to separate conservation laws. In particular, invariance under translations by a constant vector $a^{\mu}$ implies conservation of momentum, which is equivalent to a vanishing divergence $\partial_{\mu} \theta^{\mu \nu}=0$ of the stress-energy tensor. Similarly, the contribution $b^{\mu}{ }_{\nu} x^{\nu}$ associated with rotations leads to angular momentum conservation. In order to determine the conservation laws associated with invariance under conformal transformations, it is convenient to use the fundamental definition of the stress-energy tensor $\theta^{\mu \nu}$ of a field theory through the change of the action $S$ under a small change $\delta g_{\mu \nu}$ of the metric [218]

$$
\begin{equation*}
\delta S=\frac{1}{2} \int_{x} \theta^{\mu \nu}(x) \delta g_{\mu \nu}(x)+\ldots \xrightarrow[\text { conformal }]{ } \frac{1}{2} \int_{x} \theta^{\mu}{ }_{\mu}(x) \delta b^{2}(x)=0 \rightarrow \theta^{\mu}{ }_{\mu}(x) \equiv 0 . \tag{118}
\end{equation*}
$$

Invariance of the action under a Weyl transformation with a spatially varying rescaling factor $b(x)$ thus requires the stress-energy tensor to be traceless $\theta^{\mu}{ }_{\mu}(x) \equiv 0$. The argument also shows the difference between conformal and scale invariance. In the latter case, we only need that the action remains unchanged under a rescaling of the coordinates by a constant $\delta b^{2}=2 \varepsilon c$. A sufficient condition for $\delta S=0$ in this case is that the trace $\theta^{\mu}{ }_{\mu}(x)=\partial_{\mu} V^{\mu}$ of the stress-energy tensor can be written as the divergence of a local, so-called virial current $V^{\mu}(x)$. Two elementary examples which elucidate the difference between scale and conformal invariance are provided by the electromagnetic field in vacuum and the theory of elastic continua. In electrodynamics, the familiar vector potential $A^{\mu}(x)$ is a massless field without any fine tuning of the parameters to some critical point. The associated action

$$
\begin{equation*}
S_{\mathrm{em}}=\frac{1}{2} \int d^{4} x\left(\boldsymbol{E}^{2}-\boldsymbol{B}^{2}\right)=-\frac{1}{4} \int d^{4} x\left(\partial^{\mu} A^{v}-\partial^{\nu} A^{\mu}\right)\left(\partial_{\mu} A_{v}-\partial_{v} A_{\mu}\right) \underset{\mathrm{GR}}{\longrightarrow}-\frac{1}{4} \int d^{4} x \sqrt{-g} g^{\mu v} g^{\sigma \rho} F_{\mu \sigma} F_{v \rho} \tag{119}
\end{equation*}
$$

is invariant under conformal transformations in $3+1$ dimensions. Indeed, since $g=\operatorname{det} g_{\mu \nu}$ is multiplied by a factor $b^{8}(x)$, the combination $\sqrt{-g} g^{\mu \nu} g^{\sigma \rho} \rightarrow b^{4}(x) b^{-2}(x) b^{-2}(x) \sqrt{-g} g^{\mu \nu} g^{\sigma \rho}$ remains unchanged under Weyl transformations ${ }^{24}$. Physically, scale invariance in electrodynamics is a consequence of the fact that photons are massless. The possibility to extend this to conformal invariance was realized by Weyl in 1918. An important observation which will reappear in a different form in non-relativistic many-body physics is that conformal transformations can be expressed as the combination of an inversion $x^{\mu} \rightarrow x^{\mu} / x^{2}$, a translation by a vector $d^{\mu}$ and a second inversion. Indeed, these three consecutive transformations can be written in infinitesimal form by noting that

$$
\begin{equation*}
x^{\mu} \underset{\operatorname{Inv}}{\longrightarrow} \frac{x^{\mu}}{x^{2}} \underset{\operatorname{Trans}}{ } \frac{x^{\mu}}{x^{2}}+d^{\mu} \underset{\operatorname{Inv}}{\longrightarrow} \frac{\frac{x^{\mu}}{x^{2}}+d^{\mu}}{\left(\frac{x^{\mu}}{x^{2}}+d^{\mu}\right)\left(\frac{x_{\mu}}{x^{2}}+d_{\mu}\right)}=\frac{x^{\mu}+d^{\mu} x^{2}}{\left(1+2 d^{\mu} x_{\mu}+d^{\mu} d_{\mu} x^{2}\right)}=x^{\mu}+d_{v}\left(\delta^{\mu v} x^{2}-2 x^{\mu} x^{v}\right)+\ldots . \tag{120}
\end{equation*}
$$

The genuine conformal transformations associated with the last term in Eq. 117p thus arise from two inversions with a small translation in between. Concerning the conserved currents $J^{\mu}=\xi_{v} \theta^{\mu \nu}$, the contribution $\xi_{v}=c x_{v}$ associated with pure scale transformations leads to the condition $\theta^{\mu}{ }_{\mu}(x) \equiv 0$ of a traceless stress-energy tensor mentioned above. For the specific case of electrodynamics, this conservation law expresses the relation $\varepsilon_{\mathrm{em}}+T_{i i}^{\mathrm{M}}=0$ between the energy density $\varepsilon_{\mathrm{em}}=\left(\boldsymbol{E}^{2}+\boldsymbol{B}^{2}\right) / 2$ and the trace of the Maxwell stress tensor $T_{i j}^{\mathrm{M}}$, which appears in the law of momentum conservation $\partial_{t} S_{i}-c^{2} \partial_{j} T_{i j}^{\mathrm{M}}=0$ ( $\boldsymbol{S}=c \boldsymbol{E} \wedge \boldsymbol{B}$ is the Poynting vector). Note that the identity holds for arbitrary space and time dependent configurations of the electromagnetic field. In a thermal equilibrium state, it implies the well known relation $\varepsilon_{\mathrm{em}}(T)=3 p_{\mathrm{rad}}(T)$ between the energy density and the radiation pressure. The presence of conformal invariance gives rise to four additional conservation laws which are connected with the contribution proportional to the four-vector $d^{\mu}$ in Eq. 117), where $\delta^{\mu \nu} \rightarrow \eta^{\mu \nu}$ is replaced by the Minkowski metric. Unfortunately, the subject of conformal currents in electrodynamics and the physical interpetation of their conservation is hardly discussed in the standard literature.

A simple criterion that allows to distinguish conformally invariant theories for vector fields $\phi_{\mu}(x)$ from those which are only scale invariant is that the tensor structure in the associated two-point functions $\left\langle\phi_{\mu}(x) \phi_{\nu}(y)\right\rangle \sim I_{\mu \nu}(x-y) /(x-y)^{2 \Delta}$ is completely fixed by the dimensionless factor $I_{\mu \nu}(x)=\delta_{\mu \nu}-2 \hat{x}_{\mu} \hat{x}_{\nu}$ which appears in the Jacobian $\partial x^{\prime \mu} / \partial x^{\nu}=I_{\mu \nu}(x) / x^{2}$ of the inversion $x^{\prime \mu}=x^{\mu} / x^{2}$ [210]. This criterion turns out to be violated in the theory of elastic continua, whose deformation field $u^{\mu}(x)$ is again a massless vector field. In the notation of Landau/Lifschitz Volume VII, the free energy density associated with finite strain in an elastic continuum gives rise to an effective Lagrange density $\mathcal{L}$ and Euclidean action

$$
\begin{equation*}
S_{\mathrm{el}}=\int d^{d} x \mathcal{L} \quad \text { with } \quad \mathcal{L}=\frac{\lambda}{2}\left(u^{\nu}{ }_{\nu}\right)^{2}+\mu u_{\mu \nu} u^{\mu \nu}=\frac{K}{2}\left(u^{\nu}{ }_{\nu}\right)^{2}+\mu \tilde{u}_{\mu \nu} \tilde{u}^{\mu \nu} . \tag{121}
\end{equation*}
$$

[^24]Here, $u_{\mu \nu}=\left(\partial_{\mu} u_{\nu}+\partial_{\nu} u_{\mu}\right) / 2$ is the symmetric and dimensionless strain tensor, with $\tilde{u}_{\mu \nu}$ its traceless part. The coefficients $\lambda$ and $\mu$ are the Lamé coefficients, where $\mu$ is the shear and $K=\lambda+2 \mu / d=1 / \kappa_{T}$ the compression modulus. Both must be positive for thermodynamic stability, while $\lambda$ may take negative values. For the issue of scale and conformal invariance, only the ratio $\mu / K$ matters. We may thus take $\mu>0$ as the characteristic scale for the free energy density (note that by its very definition, any elastic medium has a finite shear modulus) and trade the compression modulus $K$ by the dimensionless Poisson ratio

$$
\begin{equation*}
\sigma=-\frac{u_{x x}}{u_{z z}}=\frac{\lambda}{(d-1) \lambda+2 \mu}=\frac{d K-2 \mu}{d(d-1) K+2 \mu} \in\left(-1, \frac{1}{d-1}\right) . \tag{122}
\end{equation*}
$$

It determines the relative transverse contraction $u_{x x}<0$ (provided that $\sigma>0$ !) induced by a uniform, homogeneous deformation $u_{z z}>0$ along the $z$-direction (see Landau/Lifshitz VII, Section I.5). In particular, for infinite bulk compressibility $\kappa_{T}$, which corresponds to $K=0$, the Poisson ratio is $\sigma(K=0)=-1$, while the opposite limit of an incompressible medium is described by $\sigma(K=\infty)=1 /(d-1)$. In contrast to electrodynamics, it turns out that generically, the theory is only scale but not conformally invariant. On a qualitative level, this may be understood by noting that in an elastic medium the sound velocities for longitudinal and transverse modes differ because compression and shear involve two independent elastic constants. In order to see in explicit form that scale invariance does not extend to conformal invariance here, we consider the correlation function $\left\langle u_{\mu}(x) u_{\nu}(0)\right\rangle_{\mathrm{eq}}$ in thermal equilibrium in the classical limit. Since the action $S_{\text {el }} \sim \sum_{q} u^{\mu}(\boldsymbol{q}) D_{\mu v}(\boldsymbol{q}) u^{\nu}(-\boldsymbol{q})$ is a quadratic form in momentum space, with $D_{\mu \nu}(\boldsymbol{q})=(\lambda+\mu) q_{\mu} q_{\nu}+\mu q^{2} \delta_{\mu \nu}$ the dynamical matrix of the elastic continuum, this correlation function follows from the equipartition theorem. In real space, the fact that $D(\boldsymbol{q}) \sim q^{2}$ leads to a two-point function in $d=3$

$$
\begin{equation*}
\left\langle u_{\mu}(\boldsymbol{q}) u_{v}(-\boldsymbol{q})\right\rangle_{\mathrm{eq}}=k_{B} T(\underline{D})_{\mu \nu}^{-1}(\boldsymbol{q}) \quad \rightarrow \quad\left\langle u_{\mu}(\boldsymbol{x}) u_{v}(0)\right\rangle_{\mathrm{eq}}=\frac{k_{B} T}{\mu \cdot 4 \pi|\boldsymbol{x}|}\left[\frac{3-4 \sigma}{4(1-\sigma)} \delta_{\mu v}+\frac{1}{4(1-\sigma)} \hat{x}_{\mu} \hat{x}_{v}\right] \tag{123}
\end{equation*}
$$

which decays with a simple power law $\sim 1 / r$ and thus is clearly scale invariant. However in the physically allowed range $-1<\sigma<1 / 2$ of the Poisson ratio, the anisotropic tensor structure is apparently not consistent with the generic behavior $\left\langle u_{\mu}(x) u_{\mu}(0)\right\rangle \sim\left(\delta_{\mu \nu}-2 \hat{x}_{\mu} \hat{x}_{\nu}\right) / x^{2 \Delta}$ of a two-point function in position space for conformally invariant vector fields. Such a form only appears for $\sigma=7 / 8$ which is unphysical, however, since the stability conditions $\mu, K>0$ restrict the Poisson ratio in three dimensions to $\sigma<1 / 2$. A rather special situation appears in two dimensions, where the longitudinal and transverse sound velocites coincide for a medium with infinite compressibility $K=0$. In this limit, Eq. (121) gives rise to a conformal field theory, as was noted by Riva and Cardy [219]. Indeed, this also holds in the physically more relevant limit $K \rightarrow \infty$ of an incompressible medium, where the longitudinal sound velocity is infinite and only a transverse sound mode remains.

Scale and conformal invariance in non-relativistic many-body physics The conformal transformations in statistical physics and in relativistic theories like electrodynamics differ substantially from those in a non-relativistic context, where space and time appear in an asymmetric manner. This is obvious already from the one-particle Schrödinger equation $i \hbar \partial_{t} \psi=-\left(\hbar^{2} / 2 m\right) \nabla^{2} \psi$ which is invariant under homogeneous scale transformations $\boldsymbol{x} \rightarrow b \boldsymbol{x}$ if time is rescaled according to $t \rightarrow b^{2} t$. A systematic study of the invariance group of non-relativistic single-particle quantum mechanics and of field theories has been given by Niederer [220] and by Hagen [221]. They have shown that for free particles the Galilei group can always be extended by two additional generators which are associated with homogeneous scale transformations and a so-called expansion, properly defined in Eq. 126 below. Out of the four conformal transformations of a relativistic field theory in $3+1$ dimensions, there is thus only a single one which remains a symmetry in the non-relativistic limit. For a discussion of scale and conformal invariance in non-relativistic many-body physics and, in particular, its consequences for time-dependent problems, we follow an approach due to Son and Wingate [107]. They study how a complex scalar field $\psi(x)$ and a general metric $g_{i j}(x)$ with determinant $g=\operatorname{det} g_{i j}$ changes under infinitesimal transformations $x^{i} \rightarrow x^{i}+\varepsilon \xi^{i}(x)$ and $t \rightarrow t+\varepsilon \xi^{0}(x)$ of the spatial coordinates $x^{i}$ and also of time $t$ (here, similar to the case of electrodynamics above, $x$ stands for both space and time coordinates). As the example of transforming to a rotating frame shows, such transformations may give rise to gauge fields which may have both spatial and temporal components $A^{i}$ and $A^{0}$. In the absence of any interactions, the most general form of the action is [107]

$$
\begin{equation*}
S_{0}=\int d t \int d^{d} x \sqrt{g}\left[i \hbar \psi^{*}\left(\partial_{t} \psi+i A_{0} \psi\right)-\frac{\hbar^{2}}{2 m} g^{i j}\left(\partial_{i} \psi^{*}-i A_{i} \psi^{*}\right)\left(\partial_{j} \psi+i A_{j} \psi\right)\right] . \tag{124}
\end{equation*}
$$

Now, for an arbitrary choice of the functions $\xi^{i}(x)$ and $\xi^{0}(x)$, it is always possible to find a new set of fields $\psi^{\prime}, g_{i j}^{\prime}, A_{0}^{\prime}, A_{i}^{\prime}$ such that $S_{0}$ remains invariant. This just expresses the freedom to choose any set of coordinates and time that appear convenient without changing the physics. The underlying symmetries follow from the requirement that the associated transformations leave the original metric $g_{i j}=\delta_{i j}$ unchanged and, moreover, do not introduce finite gauge fields. In analogy to the Weyl transformation above, this condition determines a restricted set of possible transformations by

$$
\begin{equation*}
\delta g_{i j}(x)=\varepsilon\left(\partial_{i} \xi_{j}+\partial_{j} \xi_{i}-\partial_{t} \xi^{0} \delta_{i j}\right) \equiv 0 \rightarrow \xi^{i}(x)=a^{i}+b^{i}{ }_{j} x^{j}+c x^{i}+v^{i} t-d_{0} t x^{i} \text { and } \xi^{0}(x)=a^{0}+2 c t-d_{0} t^{2} . \tag{125}
\end{equation*}
$$

The first four terms in $\xi^{i}(x)$ describe translations and rotations in space, a homogeneous scale transformation and Galilei transformations, respectively. Similarly, the first two terms in $\xi^{0}(x)$ are associated with translations in time and the homogeneous scale transformation, whose effect on the metric is now absorbed by a rescaling of time rather than a finite $\delta b^{2}=2 \varepsilon c$ as in Eq. (117). The last term with a single scalar parameter $d_{0}$ is the special conformal transformation, whose finite version is

$$
\begin{equation*}
(t, \boldsymbol{x}) \rightarrow\left(\frac{t}{1+d_{0} t}, \frac{\boldsymbol{x}}{1+d_{0} t}\right) \tag{126}
\end{equation*}
$$

It is formally equivalent to a Galilei transformation with a spatially varying velocity field $\boldsymbol{v}(\boldsymbol{x})=-d_{0} \boldsymbol{x}$. Physically, $\boldsymbol{x}^{\prime}=\boldsymbol{x} /\left(1+d_{0} t\right)$ is the coordinate for a comoving observer in a fluid which expands according to a Hubble flow $\boldsymbol{v}_{H}(\boldsymbol{x})=d_{0} \boldsymbol{x}{ }^{25}$. The special conformal transformation (126) is thus sometimes called an expansion. Similar to the relativistic case discussed in Eq. (120), this transformation can be expressed as the combination of two inversions with an intermediate translation of the time coordinate. Indeed, as observed by Niederer [220], defining a non-relativistic analog of the inversion by $\Sigma(t, \boldsymbol{x})=(-1 / t, \boldsymbol{x} / t)$, the sequence

$$
\begin{equation*}
(t, \boldsymbol{x}) \underset{\Sigma}{\vec{\Sigma}}(-1 / t, \boldsymbol{x} / t) \underset{\text { Trans }}{\longrightarrow}\left(-1 / t-d_{0}, \boldsymbol{x} / t\right) \underset{\Sigma^{-1}}{\longrightarrow}\left(\frac{t}{1+d_{0} t}, \frac{\boldsymbol{x}}{1+d_{0} t}\right) \tag{127}
\end{equation*}
$$

gives rise to the expansion with a parameter $d_{0}$ which formally arises from the intermediate translation of the inverse time. Note that in contrast to the euclidean or relativistic inversion in 120 , the square $\Sigma^{2}(t, x)=(t,-x)$ corresponds to a parity transformation rather than the identity, which is reached only with $\Sigma^{4}=I d$. The appearance of a time dependence in the shift vector $\xi^{i}(x)$ for both Galilei transformations and the expansion implies that the corresponding symmetries are realized with a projective representation. Indeed, as shown by Son and Wingate [107], the condition of a vanishing change $\hbar \delta A_{i}=-\partial_{i} \alpha+m \partial_{t} \xi_{i}=0$ in the spatial components of the gauge field fixes the phase factor $\psi \rightarrow \psi \exp \{i \alpha / \hbar\}$ required to leave $S_{0}$ invariant. For Galilei transformations, this gives the standard transformation law with $\alpha=m \boldsymbol{v} \cdot \boldsymbol{x}$ up to linear order in the velocity while $\alpha=-(m / 2) d_{0} x^{2}$ in the case of the expansion. For homogeneous scale transformations, the phase $\alpha$ vanishes and the action is invariant under $\boldsymbol{x} \rightarrow b \boldsymbol{x}$ and $t \rightarrow b^{2} t$ provided the fields are rescaled in their canonical form $\psi \rightarrow \psi b^{-d / 2}$.

For non-interacting particles, the Galilei group can thus always be extended by two further elements, namely homogeneous scale transformations with a constant rescaling factor $b$ and the special conformal transformation (126). Now, the crucial question is whether these symmetries can still survive in the presence of interactions, where the Lagrange density is no longer quadratic as in Eq. (124). As indicated by the hydrogen atom in 4D above, a straightforward example is provided by inverse square interactions $\sim 1 /|x|^{2}$ which - at the many-body level - give rise to a proper thermodynamic limit only in one dimension. This case has been studied in considerable detail in the context of the integrable Calagero-Sutherland-Moser model or the related Haldane-Shastry spin chain but unfortunately no physical realization of these models is known so far. By contrast, it turns out that ultracold gases provide a number of concrete examples for scale invariant many-body systems: in one dimension, it is the Tonks-Girardeau gas of impenetrable Bosons, which is related to non-interacting Fermions (see e.g. Ref. [49] for a detailed discussion in the context of ultracold gases). As noted in Ref. [171], scale invariance in one dimension is also present in two-component Fermi gases with a zero-range interaction $g_{1} \delta(x)$ in the limits $g_{1} \rightarrow \pm \infty$ of both infinite repulsion or attraction. An example in two dimensions which was suggested by Pitaevskii and Rosch [211] is provided by a zero range interaction $\bar{g}_{2} \delta^{(2)}(\boldsymbol{x})$ for an arbitrary value of the coupling strength $\bar{g}_{2}$. This realization turns out to be a subtle one due to the fact that a delta function in 2D needs to be properly regularized to give rise to a finite scattering amplitude. The regularization leads to a so-called anomaly that will be discussed in more detail below. In three dimensions, the standard example of a scale invariant many-body problem is the unitary two-component Fermi gas, which has no anomaly. The additional symmetry at infinite scattering length was noted independently by Werner and Castin [212] and by Son and Wingate [107]. Finally, a number of further scale and conformally invariant non-relativistic many-body problems might be realized with ultracold gases in mixed dimensions, as was pointed out by Nishida and Tan [223].

In order to show that the symmetries of a non-interacting many-body system extend to the case of zero range interactions at infinite scattering length in $d=3$ or at infinite strength $g_{1}$ in $d=1$, it is convenient to go back to first quantization. As noted by Nishida and Son [213] and by Castin and Werner [224], the interactions in this limit are fully accounted for by keeping the non-interacting Hamiltonian but restricting the set of possible many-body wave functions to a subclass $\mathcal{D}(H)$ of those in the standard Hilbert space which is defined by

$$
\begin{equation*}
\psi \in \mathcal{D}(H) \text { if } \psi\left(x_{1} \ldots x_{N}\right) \rightarrow \text { const } \cdot\left|x_{i}-x_{j}\right|^{2-d}+O\left(\left|x_{i}-x_{j}\right|^{4-d}\right) \text { for }\left|x_{i}-x_{j}\right| \rightarrow 0 \quad(d \neq 2) . \tag{128}
\end{equation*}
$$

The crucial observation is now that for such a generalized unitary system, the many-body version (128) of the Bethe-Peierls boundary condition is scale invariant. Indeed, for an arbitrary many-body wave function $\psi \in \mathcal{D}(H)$, the rescaled function

[^25]| unitary quantum gases | Bose | repulsive Fermi | attractive Fermi |
| :--- | :---: | :---: | :---: |
| one dimension | Tonks-Girardeau gas, $\mu=\varepsilon_{F}^{\text {eff }}$ | $\uparrow \downarrow$ - Fermionization, $\mu=4 \varepsilon_{F}$ | TG gas of dimers, $\mu_{\text {eff }}=\varepsilon_{F} / 4$ |
| three dimensions | stable only for $n \lambda_{T}^{3} \lesssim 1$ | unstable repulsive branch of FBR | unitary Fermi gas, $\mu \simeq 0.37 \varepsilon_{F}$ |

$\psi_{b}\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)=\psi\left(\boldsymbol{x}_{1} / b \ldots \boldsymbol{x}_{N} / b\right) / b^{d N / 2}$ is again in $\mathcal{D}(H)$ for all $b>0$. The preceeding argument is applicable both for Bosons and Fermions where, in the latter case, two coordinates can of course be close only for different spin components. In particular, for two-component Fermi gases in one dimension, the many-body version (128) of the Bethe-Peierls boundary condition gives rise to two physically different limits depending on whether infinite interaction strength is approached from either the repulsive or the attractive side. This differs fundamentally from the situation in 3D, where the two-body bound state energy vanishes at infinite scattering length and - at least in equilibrium - the limits $a \rightarrow \pm \infty$ lead to the same ground state. To understand the different behavior in one dimension, we consider atoms which are tightly confined to individual 'quantum wires' by a strong optical lattice. A strictly 1D description is then applicable provided that only the lowest eigenstate of the quantized motion in the transverse direction is occupied. For a harmonic confinement with radial frequency $\omega_{\perp} / 2 \pi$ and associated oscillator length $\ell_{\perp}=\sqrt{\hbar / m \omega_{\perp}}$, this requires $\varepsilon_{F} \ll \hbar \omega_{\perp}$ or - equivalently $-n_{1} \ell_{\perp} \ll 1$, where $n_{1} \equiv N / L$ is the density at total particle number $N=N_{\uparrow}+N_{\downarrow}$. The low energy scattering properties in such a geometry have been discussed by Olshanii [225]. They can be described by an effective delta function interaction $g_{1} \delta(x)$ whose strength and scattering amplitude $f(k)$ are given by

$$
\begin{equation*}
g_{1}(a)=-\frac{2 \hbar^{2}}{m a_{1}(a)}=\frac{2 \hbar \omega_{\perp} a}{1-A a / \ell_{\perp}} \quad \leftrightarrow \quad f(k)=\frac{-1}{1+i k a_{1}(a)} \quad \text { with } \quad a_{1}(a)=-\frac{\ell_{\perp}^{2}}{a}+A \ell_{\perp} . \tag{129}
\end{equation*}
$$

Here, $A=-\zeta(1 / 2) / \sqrt{2} \simeq 1.0326$ is a numerical constant. As expected, an attractive 3D scattering length $a<0$ gives rise to a negative interaction strength $g_{1}<0$ which is associated with a positive value of the effective 1D scattering length $a_{1}$. For positive scattering lengths $a>0$, in turn, there is a confinement induced resonance (CIR) at $\ell_{\perp} / a=A \simeq 1.0326$ where $a_{1}$ approaches zero while $g_{1}$ jumps from $-\infty$ to $+\infty$, similar to a standard Feshbach resonance in 3D. For $g_{1}>0$, the short-range potential $g_{1} \delta(x)$ no longer has a bound state. It is still present, however, with finite radial trapping. Indeed, as shown by Bergeman et al. [226], the exact solution of the two-body scattering problem in a tightly confined geometry always exhibits precisely one bound state, whatever the 3D scattering length $a$. Remarkably, both the binding energy $\tilde{\varepsilon}_{b}=0.606 \hbar \omega_{\perp}$ and $g_{1}$ remain finite at a Feshbach resonance $a= \pm \infty$, a prediction that has been verified experimentally by Moritz et al. [227]. Near the CIR at $g_{1}= \pm \infty$, the exact bound-state energy $\tilde{\epsilon}_{b}=2 \hbar \omega_{\perp}$ is much larger than the Fermi energy and the dimers in this regime are thus essentially unbreakable Bosons. As shown by Fuchs et al. [228] and by Tokatly [229], the attractive Fermi gas in the limit $a_{1} \rightarrow 0^{+}$defines a scale invariant Tonks-Girardeau gas of dimers which form the strong coupling limit of the so-called Luther-Emery liquid in one dimension. Its effective chemical potential $\mu_{\mathrm{eff}}=\varepsilon_{F} / 4$ is a factor four smaller than that of the non-interacting two-component Fermi gas at the same density because the formation of tightly bound pairs effectively reduces the Fermi momentum by a factor two. The opposite behavior is found in the repulsive limit $a_{1} \rightarrow 0^{-}$of the integrable Gaudin - Yang model [230, 231] ${ }^{26}$, where even Fermions with opposite spin cannot be at the same point in space. The ground state thus behaves like a non-interacting single-component gas with an effectively doubled Fermi wave vector $k_{F}=\pi n_{1}$ and chemical potential $\mu=4 \varepsilon_{F}$. In fact, this state turns out to be degenerate with a fully ferromagnetic one, as discussed in some detail in Ref. [171]. In three dimensions, the two-body bound state at infinite scattering length lies at zero energy rather than being separated by $2 \hbar \omega_{\perp}$ from the states in the continuum in the case of the confinement induced resonance. Both limits $a \rightarrow \pm \infty$ of attraction or repulsion thus give rise to the stable unitary Fermi gas, which is a superfluid of repulsively interacting pairs [232]. In analogy to the limit $g_{1} \rightarrow \infty$ in one dimension, one may however also consider following the repulsive branch of the Feshbach resonance and formally eliminate the two-body bound state at negative energy $-\hbar^{2} / m a^{2}$ of two Fermions with opposite spin, which exists for $a>0$. This limit defines a scale invariant repulsive Fermi gas in 3D. The associated many-body problem has been studied intensively following experiments by Jo et al. [233] which indicated ferromagnetic order of the Stoner type above a critical value $\left.k_{F} a\right|_{c} \simeq 1.9$. While the observations turned out to be consistent with formation of pairs rather than a ferromagnetic instability [234], the theoretical problem of determining the ratio $\mu / \varepsilon_{F}$ of the repulsive Fermi gas in the limit $k_{F} a \gg 1$ with two-body bound states projected out is still open. In particular, saturated ferromagnetism on the repulsive branch of the Feshbach resonance is ruled out even in the absence of a decay to the lower branch if $\mu / \varepsilon_{F}<2^{2 / 3}=1.5874$ [171]. For Bosons in $d=3$, the zero range interaction model with infinite scattering length gives rise to a Hamiltonian which is unbounded below. Indeed, as discussed in Lecture I, the unitary Bose gas exhibits $N$-body bound states for arbitrary $N \geq 3$ and it can in practice be realized in a gaseous effective equilibrium configuration only in the non-degenerate regime $n \lambda_{T}^{3} \lesssim 1$. For a well defined ground state, however, one needs a finite range repulsion as exemplified by superfluid ${ }^{4} \mathrm{He}$, which is a nearly unitary Bose liquid whose equilibrium density $\bar{n} \sigma^{3}=0.364$ at vanishing pressure is set by the length scale $\sigma$ of the repulsion at short distances.

[^26]In order to determine the non-relativistic analog of the vanishing $\theta^{\mu}{ }_{\mu}(x) \equiv 0$ of the trace of the stress-energy tensor for scale invariant relativistic systems derived in Eq. 118, we consider the law of momentum conservation $\partial_{t} \hat{\boldsymbol{g}}_{i}+\partial_{j} \hat{\Pi}_{i j}=0$ within a many-body formulation. For rotation invariant short range interactions with a potential $V(r)$, a microscopic expression for the associated stress tensor operator $\hat{\Pi}_{i j}$ has been derived by Martin and Schwinger [235]. Its trace

$$
\begin{equation*}
\hat{\Pi}_{i i}(\boldsymbol{R})=\frac{\hbar^{2}}{m}\left[\left(\nabla \hat{\psi}^{\dagger}\right) \cdot(\nabla \hat{\psi})-\frac{d}{4} \nabla^{2}\left(\hat{\psi}^{\dagger} \hat{\psi}\right)\right]-\frac{1}{2} \int_{x} \hat{\psi}^{\dagger}(\boldsymbol{R}+\boldsymbol{x} / 2) \hat{\psi}^{\dagger}(\boldsymbol{R}-\boldsymbol{x} / 2) r \partial_{r} V(r) \hat{\psi}(\boldsymbol{R}-\boldsymbol{x} / 2) \hat{\psi}(\boldsymbol{R}+\boldsymbol{x} / 2) \rightarrow 2 \hat{\varepsilon}(\boldsymbol{R}) \tag{130}
\end{equation*}
$$

turns out to coincide with the operator $\hat{\varepsilon}(\boldsymbol{R})$ of the local energy density up to a factor of two for the particular case of scale invariant interactions, where $r \partial_{r} V(r) \equiv-2 V(r)$. It is important to note that Eq. (130) is an operator identity which is valid for arbitrary, even time dependent situations, similar to the relation $\varepsilon_{\mathrm{em}}+T_{i i}^{\mathrm{M}}=0$ between the energy density and the trace of the Maxwell stress tensor in electrodynamics. The fact that the energy density in 130) appears with a factor two is a consequence of the rescaling of $\hat{H}$ with $1 / b^{2}$ instead of $1 / b$ in the relativistic case. In thermal equilibrium, $\left\langle\hat{\Pi}_{i i}\right\rangle=d \cdot p$ is proportional to the pressure. Scale invariance thus implies that the equation of state of a non-relativistic many-body system in $d$ dimensions obeys $p=2 \varepsilon / d$, formally identical to that in the non-interacting case. For the example of a unitary Fermi gas in 3D, this relation, which holds at arbitrary temperature and independent of the precise state, has been derived first by Ho [236] based on simple dimensional analysis at infinite scattering length. More generally, it is a special case of the exact relation

$$
\begin{equation*}
p=\frac{2}{3} \varepsilon+\frac{\hbar^{2}}{24 \pi m a} C_{2}+\frac{2 \hbar^{2}}{3 m} C_{3} \quad(\text { Bosons, 3D) } \tag{131}
\end{equation*}
$$

which generalizes the Tan pressure relation [237] for two-component Fermi gases with zero-range interactions. Evidently, for Bosons, a finite value of the three-body contact density $C_{3}$, which - according to Eq. (87) - accounts for the dependence of the thermodynamic functions on the three-body parameter $\kappa_{*}$, gives rise to a violation of scale invariance even at infinite or at vanishing scattering length. An example for the latter case is the 3D Bose fluid at $a=0$ and finite three-body scattering hypervolume $D$ discussed in Lecture I, whose equation of state (31) leads to $p=2 \varepsilon$ rather than $p=2 \varepsilon / 3$.

The argument above for scale invariance requires interactions which are homogeneous of degree -2 , like an inverse square potential or a delta function $\bar{g}_{2} \delta^{(2)}(\boldsymbol{x})$ in two dimensions. A formal proof that the operator identity $2 \hat{\varepsilon}=\hat{\Pi}_{i i}$ also holds for zero range interactions in one or in three dimensions at fine-tuned values $a_{1}=0$ or $a= \pm \infty$ of the respective scattering lengths is nontrivial. In one dimension, its validity relies on the fact that the interaction contribution $\hat{\Pi}_{\text {int }}(\boldsymbol{R}) \rightarrow \hat{\varepsilon}_{\text {int }}(\boldsymbol{R})=-\left(\hbar^{2} / 4 m\right) a_{1} \hat{\mathcal{C}}_{2}^{1 \mathrm{~d}}(\boldsymbol{R}) \rightarrow 0$ to the stress tensor vanishes in the limit $a_{1} \rightarrow 0$ because the corresponding contact density operator is finite there. The energy density thus eventually arises only from the kinetic contribution in Eq. (130), which always enters with a factor two. The fact that the energy density in a Tonks-Girardeau gas is of purely kinetic origin has been verified experimentally through the observation by Kinoshita et al. [238] that the axial expansion energy saturates with increasing strength of the transverse confinement. A different situation arises in 3D, where only the sum of the kinetic and the interaction energy is well defined in the zero range limit. For a proof of the operator identity $2 \hat{\varepsilon}=\hat{\Pi}_{i i}$ at infinite scattering length in this case see [213].

A more detailed understanding of scale and conformal invariance in non-relativistic many-body problems and the underlying conservation laws is provided by considering the algebra of the associated operators. In the Heisenberg representation, the generator $\hat{D}(t)$ of scale transformations with a constant rescaling factor $b=e^{-\lambda}$ is defined by the transformation law [213]

$$
\begin{equation*}
\exp \{-i \lambda \hat{D}(t)\} \hat{\psi}(t, \boldsymbol{x}) \exp \{i \lambda \hat{D}(t)\}=e^{\lambda d / 2} \hat{\psi}\left(e^{2 \lambda} t, e^{\lambda} \boldsymbol{x}\right) \quad \text { and } \quad \exp \{-i \lambda \hat{D}\} \hat{H} \exp \{i \lambda \hat{D}\} \underset{\text { scale inv. }}{ } e^{2 \lambda} \hat{H} \tag{132}
\end{equation*}
$$

of the time dependent field operator or the Hamiltonian. It gives rise to a symmetry provided the latter is reproduced up to a factor $1 / b^{2}$. Scale invariant many-body Hamiltonians are thus characterized by the commutation relation $i[\hat{H}, \hat{D}] \equiv 2 \hat{H}$. Note the contrast to standard symmetries like rotation invariance, where the angular momentum operator as the generator of rotations commutes with $\hat{H}$. Here, it is only the action $S$ which is invariant, not the Hamiltonian itself. Nevertheless, as a continuous symmetry, scale invariance gives rise to a local conservation law of the form $\partial_{t} \hat{\rho}_{D}+\operatorname{div} \hat{\boldsymbol{j}}_{D}=0$. The associated dilatation density and dilatation current density operators $\hat{\rho}_{D}$ and $\hat{j}_{D}$ have been determined by Hagen [221]. In an infinitesimal transformation $\lambda \rightarrow 0$, the generator $\hat{D}(t)$ changes coordinates by $\boldsymbol{x} \rightarrow(1-\lambda) \boldsymbol{x}$ and time by $t \rightarrow(1-2 \lambda) t$. The change of time implies that $\hat{D}(t)=\hat{D}-2 t \hat{H} / \hbar$ carries an explicit time dependence. In local form, the operator $\hat{D}(t)=\int \hat{\rho}_{D}$ can be expressed as the spatial integral of a dilatation density $\hat{\rho}_{D}=(\boldsymbol{x} \cdot \hat{\boldsymbol{g}}-2 t \hat{\boldsymbol{\varepsilon}}) / \hbar$ which involves the momentum density operator $\hat{\boldsymbol{g}}$ and that of the energy density $\hat{\varepsilon}$. The expression for $\hat{\rho}_{D}$ fixes the dilatation current density operator $\hat{\boldsymbol{j}}_{D}=\left(\boldsymbol{x} \underline{\Pi}-2 t \hat{\boldsymbol{S}}_{\varepsilon}\right) / \hbar$ in the conservation law $\partial_{t} \hat{\rho}_{D}+\operatorname{div} \hat{\boldsymbol{j}}_{D}=0$ up to a divergence-less contribution. Here, as noted above, the stress tensor operator $\hat{\Pi}$ is defined by the local form $\partial_{t} \hat{\boldsymbol{g}}_{i}+\partial_{j} \hat{\Pi}_{i j}=0$ of momentum conservation while the energy current density $\hat{\boldsymbol{S}}_{\varepsilon}$ enters in the corresponding conservation law $\partial_{t} \hat{\varepsilon}+\operatorname{div} \hat{\boldsymbol{S}}_{\varepsilon}=0$. It is now straightforward to see that the conservation law associated with dilatations is obeyed in addition to these two generally valid relations if and only if the operator identity $2 \hat{\varepsilon}=\hat{\Pi}_{i i}$ for scale invariant many-body systems is valid.

In a similar manner, the generator $\hat{C}(t)$ of the special conformal transformation ind defined by the transformation law

$$
\begin{equation*}
\exp \left\{-i d_{0} \hat{C}(t) / \hbar\right\} \hat{\psi}(t, \boldsymbol{x}) \exp \left\{i d_{0} \hat{C}(t) / \hbar\right\}=\left(\frac{1}{1+d_{0} t}\right)^{d / 2} \exp \left(i \frac{d_{0} m \boldsymbol{x}^{2}}{2 \hbar\left(1+d_{0} t\right)}\right) \hat{\psi}\left(\frac{t}{1+d_{0} t}, \frac{\boldsymbol{x}}{1+d_{0} t}\right) \tag{133}
\end{equation*}
$$

It can be written in the form $\hat{C}(t)=\hat{C}-\hbar t \hat{D}+t^{2} \hat{H}$, where $\hat{C}=(1 / 2) \int x^{2} \hat{\rho}$ is the second moment of the mass density $\hat{\rho}$. The resulting commutation relations are $i[\hat{D}, \hat{C}]=2 \hat{C}$ and $i[\hat{H}, \hat{C}]=\hbar^{2} \hat{D}$. Again, there is a local conservation law $\partial_{t} \hat{\rho}_{C}+\operatorname{div} \hat{j}_{C}=0$ with a conformal density $\hat{\rho}_{C}=(1 / 2) \boldsymbol{x}^{2} \hat{\rho}-t \boldsymbol{x} \cdot \hat{\boldsymbol{g}}+t^{2} \hat{\varepsilon}$ and an associated current $\hat{\boldsymbol{j}}_{C}=(1 / 2) \boldsymbol{x}^{2} \hat{\boldsymbol{g}}-t \boldsymbol{x} \underline{\underline{\Pi}}+t^{2} \hat{\boldsymbol{S}}_{\varepsilon}$. The validity of this conservation law is a direct consequence of mass, energy and momentum conservation, together with the condition $2 \hat{\varepsilon}=\hat{\Pi}_{i i}$ for scale invariance. For complex scalar fields, non-relativistic conformal invariance therefore necessarily follows from the combination of Galilei and scale invariance and no further condition is required [221]. The unusual situation for vector fields like the elastic continuum discussed above which are only scale - but not conformally invariant therefore does not appear.

Before turning to the implications of scale and conformal invariance for dynamical properties in the following section, we briefly discuss their relevance for measurements of the equation of state of ultracold gases. In particular, it turns out that scale invariance allows to infer the complete set of thermodynamic functions from the density profile $n(x)$ in a harmonic trap with potential $V(\boldsymbol{x})$. This relies on the local density approximation, where a change $d V$ translates to a corrresponding change $d \mu=-d V$ in the local chemical potential. The thermodynamic relation $d p=n d \mu$ therefore determines the pressure of the gas from an integration $p(\mu)=\int_{-\infty}^{\mu} d \mu^{\prime} n\left(\mu^{\prime}\right)=\int_{V}^{\infty} d V^{\prime} n\left(V^{\prime}\right)$ of the density profile from a given value $V=\mu$ of the external potential to the edge of the trap, where the density vanishes. In turn, the compressibility $n^{2} \kappa(\mu)=-d n / d V$ requires to differentiate $n(V)$ once. Now, as noted by Van Houcke et al. [239] in the context of the unitary Fermi gas in 3D, in the presence of scale invariance the two observables pressure $p$ and compressibility $\kappa$ are sufficient to fully determine all thermodynamic functions. Indeed, in a scale invariant system, both the normalized pressure $\tilde{p}=p / p^{(0)}$ and the compressibility $\tilde{\kappa}=\kappa / \kappa^{(0)} 27$ only depend on a single parameter which may either be taken as the dimensionless phase space density $\mathcal{D}=n \lambda_{T}^{d}$ or - in the case of Fermi gases - the equivalent ratio $\theta=T / T_{F}$. In particular, this parameter may be eliminated from $\tilde{\kappa}$ and $\tilde{p}$, thus giving rise to a universal function $\tilde{\kappa}(\tilde{p})$ which uniquely connects compressibility and pressure. This crucial step and the associated Eq. 134 below, which allows a precise determination of temperature in an ultracold gas from the measured relation $\kappa(p)$, is in fact a special case of a concept discussed in $\S 160$ of the textbook 'Vorlesungen über Thermodynamik' by Max Planck in 1897, where he shows how to determine the absolute temperature $T(\tau)$ as a function of some experimentally accessible parameter $\tau$ from a measurement on a system with an unknown equation of state.

On a formal level, the single parameter nature of the thermodynamic functions may be derived as a consequence of universality associated with the quantum phase transition out of the vacuum state at $\mu<0$ to a finite density gas at $\mu>0$. This point of view has been discussed by Nikolic and Sachdev [241] for the Fermi gas near unitarity and by Rancon and Dupuis [242] for dilute Bose gases in two and three dimensions. Specifically, it relies on the existence of a fixed point with infinite correlation length at vanishing chemical potential, similar to the points on the line $\mu=0, g>0$ in Fig. 7 ]on the left. The fixed points are associated with only a few relevant perturbations. For Bose gases in 2 D , it turns out that the only relevant one is $\mu$ itself since the flow of the coupling constant $\bar{g}_{2}$ can be neglected. The assumption that the interactions in 2D enter only as a fixed external parameter is very well obeyed in the accessible regime of temperatures, as will be discussed in detail in the section on anomalies below. In the unitary Fermi gas in 3D, the fixed point at $\mu=0$ and infinite scattering length has three relevant perturbations which are $\mu$, a finite deviation $(1 / a) \neq 0$ from unitarity and a possible difference $2 h=\mu_{\uparrow}-\mu_{\downarrow}$ of the chemical potentials of both spincomponents [241]. Both for the balanced unitary Fermi gas and for Bose gases in 2D, the thermodynamic properties are therefore characterized by a universal function $\mathcal{D}=f_{n}(\beta \mu)$ of the single dimensionless variable $\beta \mu$. This parameter may be eliminated in favor of a direct relation between compressibility and pressure by using the fact that the internal energy $U=d p V / 2$ in a scale invariant system is simply proportional to the pressure. Following Planck's concept, the general thermodynamic relation

$$
\begin{equation*}
T\left(\frac{\partial p}{\partial T}\right)_{V}=p+\left(\frac{\partial U}{\partial V}\right)_{T} \xrightarrow[\mathrm{~s}-\mathrm{inv} .]{ } \quad T \frac{d \tilde{p}}{d T}=\frac{d+2}{2}\left(\tilde{p}-\frac{1}{\tilde{\kappa}}\right) \rightarrow \ln \left(\frac{T\left(\tilde{p}_{1}\right)}{T\left(\tilde{p}_{0}\right)}\right)=\frac{2}{d+2} \int_{\tilde{p}_{0}}^{\tilde{p}_{1}} \frac{d \tilde{p}}{\tilde{p}-1 / \tilde{\kappa}(\tilde{p})} \tag{134}
\end{equation*}
$$

then determines the temperature $T(\tilde{p})$ with respect to some reference value $T\left(\tilde{p}_{0}\right)$ from the measured function $\tilde{\kappa}(\tilde{p})$. In practice, $T\left(\tilde{p}_{0}\right)$ is fixed by the equation of state near the edge of the cloud, where $p(T)$ is known from the virial expansion. Note that the temperature of the trapped gas is constant while $\tilde{p}$ increases monotonically away from the trap center. What is obtained by inverting the relation $T(\tilde{p})$ from Eq. 134 is the normalized pressure $\tilde{p}$ as a function of temperature in a homogeneous system.

[^27]

FIG. 16: Compressibility as function of pressure for two examples of a scale invariant quantum fluid. The Figure on the left shows the unitary Fermi gas in 3D where the transition to the superfluid phase near $\tilde{p}_{c} \simeq 0.5$ is accompanied by a singular behavior of the compressibility. The data from Ku et al. [109] agree quite well with the theoretical prediction from Haussmann et al. [243] without any adjustable parameter. The corresponding results for a Bose gas in 2D obtained by Desbuquois et al. [240] are shown on the right, where the red curve displays $X_{1}=\tilde{\kappa} / 2$ as a function of $X_{-1}=\tilde{p}$. Here, the transition to the superfluid phase at $\tilde{p}_{c} \simeq 0.2$ leaves no detectable signature in the equation of state.

Experimentally, scale invariance in the equation of state has been verified both for the unitary Fermigas in 3D by Ku et al. [109] and for Bose gases in two dimensions by the groups at ENS [240] and in Chicago [244]. The measured universal function $\tilde{\kappa}(\tilde{p})$ for the unitary Fermi gas is shown in Fig. 16 . It covers the relevant range from the non-degenerate regime $\tilde{p}=4$ to the superfluid transition at $\tilde{p}_{c} \simeq 0.5$ and down to the limiting value $\tilde{p}_{\text {min }}=\xi_{s}$ set by the finite ground state pressure of the gas. Due to the attractive interactions, this pressure is below that of a non-interacting Fermi gas with the same density $n$ by a universal factor $\xi_{s} \simeq 0.37$ which is called the Bertsch parameter. It may be extracted from the limit $\tilde{\kappa}\left(\tilde{p}_{\text {min }}\right)=1 / \xi_{s} \simeq 2.70$ of the dimensionless compressibility at the minimum value of the pressure, since $\tilde{\kappa} \rightarrow 1 / \tilde{p}$ as $T \rightarrow 0$. This relation is a consequence of Eq. (134) by noting that the thermal expansion coefficient $\beta_{V}=(\partial p / \partial T)_{V} \rightarrow 2 c_{V}(T) / d-$ which, for scale invariant systems, is directly proportional to the specific heat $c_{V}(T)$ per volume - must vanish. In the case of the Bose gas in 2 D , scale invariance is well obeyed in the accessible regime of temperatures, however the function $\tilde{\kappa}(\tilde{p})$ depends on the dimensionless strength $\tilde{g}_{2}=\sqrt{8 \pi} a / \ell_{z} \simeq 0.1$ of the interaction. This is apparent in the limit of zero temperature, where $\tilde{p} \rightarrow \tilde{g}_{2} / 2$ and $\tilde{\kappa} \rightarrow 2 / \tilde{g}_{2}$ are near the theoretical values obtained from a Thomas-Fermi description of a perfect condensate, which is indicated by the black dot in Fig. 16 The superfluid transition of the homogeneous Bose fluid in 2D is of the Berezinskii-Kosterlitz-Thouless (BKT) type. It is characterized by a universal jump of the superfluid density from zero to a finite value $n_{s}\left(T_{c}\right)=4 / \lambda_{T_{c}}^{2}$ right at $T_{c}$ [245-247]. However, this leaves no sharp feature in the thermodynamics, consistent with the observed smooth dependence of the compressibility on pressure in the harmonically trapped gas.

Dynamics in scale invariant ultracold gases In the following, we will discuss some of the consequences of scale and conformal invariance for dynamical properties. The most striking among those is the fact that, in a scale invariant system, any solution of the Schrödinger equation in free space can be transformed into one in the presence of an isotropic harmonic trap with an arbitrary time-dependent frequency $\omega(t)$. This transformation has originally been discovered for free particles by Niederer [248] and has been extended to the many-body problem with scale invariant interactions by Pitaevskii and Rosch [211] and by Castin [249]. Following Ref. [250], it may be derived from the invariance of the action (124] under a transformation $x^{i} \rightarrow \lambda(t) x^{i}$ and $t \rightarrow f^{-1}(t)$ with $\dot{f}(t)=1 / \lambda^{2}(t)$ of coordinates and time with a completely general rescaling factor $\lambda(t)$. In addition, a finite gauge transformation $\psi \rightarrow \psi \exp \{i \alpha / \hbar\}$ with $\alpha=\left(m x^{2} / 2\right) \dot{\lambda} / \lambda$ is performed. With $A^{0}=A^{i}=0$ in free space, these transformations leave the metric $g_{i j}=\delta_{i j}$ invariant, however they give rise to a non-vanishing time component $\hbar A^{0}=m \omega^{2}(t) \boldsymbol{x}^{2} / 2$ of the gauge field which represents a harmonic trapping potential with frequency $\omega^{2}(t)=-\ddot{\lambda} / \lambda$. The full transformation of the field operator and the resulting one for the many-body wave function $\psi(\boldsymbol{X}, t)$ in a time-dependent harmonic trap (where $\boldsymbol{X}=\left\{\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right\}$ denotes the complete set of coordinates) has the form

$$
\begin{equation*}
\psi^{\prime}(\boldsymbol{x}, t)=\frac{1}{\lambda^{d / 2}(t)} \exp \left[\frac{i m \dot{\lambda}(t)}{2 \hbar \lambda(t)} \boldsymbol{x}^{2}\right] \psi(\boldsymbol{x} / \lambda(t), f(t)) \quad \rightarrow \quad \psi(\boldsymbol{X}, t)=\frac{e^{i \theta(t)}}{\lambda^{d N / 2}(t)} \exp \left[\frac{i m \dot{\lambda}(t)}{2 \hbar \lambda(t)} \boldsymbol{X}^{2}\right] \psi(\boldsymbol{X} / \lambda(t), 0) . \tag{135}
\end{equation*}
$$

Here, in the second form of the relation, it is assumed that the harmonic trap is already present in the initial state at time $t=0$ with a finite frequency $\omega_{\text {in }} \neq 0$. The prime on the wave function is therefore omitted, however the relation $\omega^{2}(t)=-\ddot{\lambda} / \lambda$ which connects the rescaling factor $\lambda(t)$ from a free-space situation to the frequency of the trapping potential is now changed to a nonlinear equation

$$
\begin{equation*}
\ddot{\lambda}(t)=\frac{\omega_{\text {in }}^{2}}{\lambda^{3}(t)}-\omega^{2}(t) \lambda(t) \quad \text { with } \lambda(0)=1 \text { and } \dot{\lambda}(0)=0 \tag{136}
\end{equation*}
$$

In addition, there is a phase $\theta(t)$ which arises from a shift of the time coordinate $f(t)$ in the first equation to zero. For an initial stationary state in the harmonic trap with energy $E$, its time derivative is determined by $\hbar \dot{\theta}=-E / \lambda^{2}(t)$. An example where Eq. (136) can be solved analytically is provided by a sudden change of the trap frequency from $\omega_{\text {in }}$ at $t \leq 0$ to a different constant value $\omega_{\text {fin }}$ for $t>0$. In this case, the density distribution

$$
\begin{equation*}
n(x, t)=\frac{1}{\lambda^{d}(t)} n\left(\frac{x}{\lambda(t)}, t=0\right) \quad \text { with } \quad \lambda(t)=\sqrt{\cos ^{2}\left(\omega_{\text {fin }} t\right)+\left(\frac{\omega_{\text {in }}}{\omega_{\text {fin }}}\right)^{2} \sin ^{2}\left(\omega_{\text {fin }} t\right)} \tag{137}
\end{equation*}
$$

evolves from its initial form by a simple rescaling $\boldsymbol{x} \rightarrow \boldsymbol{x} / \lambda(t)$. The frequency spectrum of the factor $\lambda(t)$ only contains integer multiples $\omega_{n}=n \cdot 2 \omega_{\text {fin }}$ of twice the trap frequency which reflects the existence of an infinite ladder of excited states for scale invariant many-body systems in an isotropic and time-independent trap that will be derived below. As a straightforward consequence of Eq. 137), the expectation value $E_{\mathrm{pot}}(t)=\omega_{\mathrm{fin}}^{2}\langle\hat{C}\rangle(t)=E_{\mathrm{pot}}(0) \cdot \lambda^{2}(t)$ of the potential energy, which is determined by the generator $C$ of the special conformal transformation (126), oscillates purely sinusoidally with period $T / 2$, where $T=2 \pi / \omega_{\text {fin }}$ is the natural period of the harmonic trap. Experimentally, this has been observed at ENS in the time evolution of the transverse mean-square radius $R_{x}^{2}(t)+R_{y}^{2}(t)$ of a gas in a cigar-shaped trap, which oscillates with frequency $2 \omega_{\perp}$ after exciting a transverse breathing mode [251]. In a much cleaner, truly 2 D setup, a related experiment has been performed recently by Saint-Jalm et al. [252], where the dynamics of essentially arbitrary initial configurations in an isotropic trap could be studied, e.g. the square shaped density shown in Fig. 17 below. The resulting energy $E_{\mathrm{pot}}(t)$ is nicely periodic despite the fact that the full density $n(\boldsymbol{x}, t)$ does not exhibit a simple time dependence. In the special case of $\omega_{\text {fin }}=0$, where the trapping potential is suddenly removed completely, Eq. 137) gives rise to a ballistic expansion with a rescaling factor $\lambda(t)=\left(1+\omega_{\text {in }}^{2} t^{2}\right)^{1 / 2}$ and a mean-square radius $\left\langle\boldsymbol{x}^{2}\right\rangle(t)=\lambda^{2}(t)\left\langle\boldsymbol{x}^{2}\right\rangle(0)$ which grows quadratically in time. A rich structure in the dynamics, involving parametric resonance and the stabilization of anti-trapped configurations, appear for scale invariant systems in a harmonic trap in the presence of external periodic driving, which has been discussed by Moroz [253].

The existence of an infinite sequence of excited eigenstates of scale invariant many-body systems in an isotropic trap was predicted by Pitaevskii and Rosch [211] for the case of Bose gases in 2D. It is based on a hidden $\mathrm{SO}(2,1)$ symmetry which is in fact present both in free space and in the presence of an isotropic trap. Indeed, as first realized by Niederer [220] and by Hagen [221], the generators $\hat{D}$ and $\hat{C}$ of dilatations and the special conformal transformation 126) together with a scale invariant Hamiltonian $\hat{H}$ form a closed $\mathrm{SO}(2,1)$ subalgebra of the full Schrödinger group. In an isotropic trap with a time independent frequency $\omega$, the symmetry gives rise to an equidistant ladder of exact eigenstates. This may be derived in a purely algebraic manner by noting that the associated Hamiltonian can be written in the form

$$
\begin{equation*}
\hat{H}_{\omega}=\hat{H}+\omega^{2} \hat{C} \quad \text { with } \quad \hat{C}=\frac{1}{2} \int_{\boldsymbol{x}} \boldsymbol{x}^{2} \hat{\rho}(\boldsymbol{x}) . \tag{138}
\end{equation*}
$$

Using the commutators $i[\hat{H}, \hat{C}]=\hbar^{2} \hat{D}$ and $i[\hat{D}, \hat{C}]=2 \hat{C}$ stated above, it is straightforward to show that the operators defined by

$$
\begin{equation*}
\hat{L}_{ \pm}=\frac{\hat{H}}{2 \hbar \omega}-\frac{\omega}{2 \hbar} \hat{C} \pm \frac{i}{2} \hat{D}=\hat{L}_{1} \pm i \hat{L}_{2} \quad \text { and } \quad \hat{L}_{3}=\hat{H}_{\omega} / 2 \hbar \omega \tag{139}
\end{equation*}
$$

behave like the generators $\hat{L}_{1,2}$ of Lorentz boosts in two directions and of rotations $\hat{L}_{3}$ in the associated plane. Indeed, they obey

$$
\begin{equation*}
\left[\hat{L}_{1}, \hat{L}_{2}\right]=-i \hat{L}_{3}, \quad\left[\hat{L}_{2}, \hat{L}_{3}\right]=i \hat{L}_{1}, \quad\left[\hat{L}_{3}, \hat{L}_{1}\right]=i \hat{L}_{2}, \tag{140}
\end{equation*}
$$

which is the algebra associated with the Lorentz group in $2+1$ dimensions or, equivalently, the group $\mathrm{SU}(1,1)$ of complex two-by-two matrices with unit determinant which leave the form $\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}$ invariant. The relation $\left[\hat{H}_{\omega}, \hat{L}_{ \pm}\right]= \pm 2 \hbar \omega \hat{L}_{ \pm}$allows to generate a tower of new eigenstates with excitation energy $2 n \hbar \omega$ from an arbitrary eigenstate $\left|\psi_{E}\right\rangle$ of $\hat{H}_{\omega}$ by repeating

$$
\begin{equation*}
\hat{H}_{\omega}\left(\hat{L}_{+}\left|\psi_{E}\right\rangle\right)=\left(\hat{L}_{+} \hat{H}_{\omega}+2 \hbar \omega \hat{L}_{+}\right)\left|\psi_{E}\right\rangle=(E+2 \hbar \omega)\left(\hat{L}_{+}\left|\psi_{E}\right\rangle\right) \tag{141}
\end{equation*}
$$

$n$ times. Similarly, the operator $\hat{L}_{-}$lowers the energy by $2 \hbar \omega$. Since the spectrum of the Hamiltonian is bounded below, the exact ground state with $\hat{H}_{\omega}|0\rangle=E_{0}|0\rangle$ obeys $\hat{L}_{-}|0\rangle=0$. Scale invariance thus implies the existence of an infinite number of excited states above $E_{0}$ at multiples of twice the trap frequency. For a more detailed understanding of the full spectrum, we consider the representations of the group $\mathrm{SO}(2,1)$ or, more precisely, its universal covering group $\overline{\mathrm{SU}(1,1)}$ which have been discussed by Bargmann [254]. As a non-compact Lie group, all unitary representations of the algebra (140) are infinite dimensional. The requirement that the spectrum of $\hat{L}_{3}=\hat{H}_{\omega} / 2 \hbar \omega$ is bounded below, uniquely fixes the representations. Following the notation in Ref. [254], they are denoted by $D^{+}(k)$ where the possible values $k>0$ are defined via the Casimir operator

$$
\begin{equation*}
\hat{Q}=\hat{L}_{3}^{2}-\hat{L}_{1}^{2}-\hat{L}_{2}^{2}=\left(\hat{H}_{\omega} / 2 \hbar \omega\right)^{2}-\frac{1}{2}\left\{\hat{L}_{+}, \hat{L}_{-}\right\}=k(k-1) \rightarrow \hat{Q}|0\rangle=k_{0}\left(k_{0}-1\right)|0\rangle \text { with } k_{0}=E_{0} /(2 \hbar \omega) . \tag{142}
\end{equation*}
$$



FIG. 17: Dynamics of two different initial configurations of a two-dimensional Bose gas in an isotropic trap with frequency $\omega$. In the left Figure the time evolution of a initially square shaped gas is shown. While the density distribution at times $\omega t=0, \pi, 2 \pi, 3 \pi, 4 \pi$ shows no special features, the potential energy is perfectly periodic with frequency $2 \omega$. Remarkably, the initial triangular shape shown on the right displays perfect revivals at times $t=n \cdot T / 2$ for $n=1 \ldots 4$, where $T=2 \pi / \omega$. Both Figures are taken from Ref. [252].

Quite generally, the spectrum of the operator $\hat{L}_{3}$ in the representation $D^{+}(k)$ is of the form $k+n$, with $n=0,1,2, \ldots$. For the specific case $k_{0}=E_{0} / 2 \hbar \omega$, this reproduces the equidistant ladder of breathing mode excitations with energy $n \cdot 2 \hbar \omega$ above the ground state $|0\rangle$. Superficially, the algebraic approach thus appears to reduce any scale invariant many-body problem in an isotropic trap to a set of harmonic oscillators. Not surprisingly, this is an illusion for two different reasons: first of all, the representation $k_{0}$ associated with the exact ground state only accounts for a subset of all states which share the property of vanishing angular momentum with the ground state $|0\rangle$. Indeed, in the trivial case of a single particle where $E_{0}^{(0)}(N=1)=d \hbar \omega / 2$, the complete set of eigenstates is generated by representations belonging to $k_{l}^{(0)}=(d / 4)+l / 2$ with an arbitrary value $l=0,1,2 \ldots$ of the angular momentum [255] ${ }^{28}$. As a result, there is an infinite sequence of equidistant ladders rather than a single one. Second, the many-body nature of the operators $\hat{L}_{j}$ does not allow to directly connect the eigenfunctions of a given representation of the $S U(1,1)$ algebra to explicit many-body wave functions. In particular, it is only the dynamics of the operators $\hat{L}_{j}$ which exhibit a simple periodicity.

In the following, we will discuss the symmetries which are necessary for an understanding of a quite surprising observation in the experiments by Saint-Jalm et al. [252] mentioned above. As shown in Fig. 17 ]on the right, the time evolution of the full density distribution turns out to be perfectly periodic with period $T / 2$ for certain special initial configurations like an equilateral triangle. The occurence of such breathers has been proposed to be analogous to spin-echoes on the familiar Bloch sphere for $\mathrm{SU}(2)$ by Lv et al. [256]. Specifically, they consider the generalized coherent states for the group $\operatorname{SU}(1,1)$ which have been determined by Perelemov [257]. For a given representation $D^{+}(k)$ with a quite general value $k>0$ associated with the Casimir operator (142), they are characterized by a complex number $z$ within the unit disc $|z|<1$. In terms of the eigenstates $|k, n\rangle$ of $\hat{L}_{3}$, they are of the form [257]

$$
\begin{equation*}
|k, z\rangle=\left(1-|z|^{2}\right)^{k} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2 k+n)}{\Gamma(n+1) \Gamma(2 k)}} z^{n}|k, n\rangle . \tag{143}
\end{equation*}
$$

The expectation values of the generators in these states are

$$
\begin{equation*}
\langle k, z| \hat{L}_{1}|k, z\rangle=k \cdot \frac{2 \operatorname{Re} z}{1-|z|^{2}}, \quad\langle k, z| \hat{L}_{2}|k, z\rangle=k \cdot \frac{2 \operatorname{Im} z}{1-|z|^{2}} \quad \text { and } \quad\langle k, z| \hat{L}_{3}|k, z\rangle=k \cdot \frac{1+|z|^{2}}{1-|z|^{2}} . \tag{144}
\end{equation*}
$$

They have the important property of being single-valued for an arbitrary representation $D^{+}(k)$. Now, a crucial advantage of using coherent states is that the time evolution operator $\hat{U}(t=T / 2)=\exp \left(-i 2 \pi \hat{L}_{3}\right)$ over half the natural period $T=2 \pi / \omega$ of the oscillator acts like a rotation by $2 \pi$ in the Poincaré - disc $|z|<1$. As will be shown below, the expectation values of the operators $\hat{L}_{j}$ therefore have period $T / 2$ for an arbitrary initial state despite the fact that the associated time evolution $\hat{U}(T / 2)=\exp \left(-i 2 \pi \hat{L}_{3}\right) \rightarrow \exp (-i 2 \pi k)$ in a given representation $D^{+}(k)$ gives rise to a phase factor that distinguishes different $k$ 's. This may be seen by noting that the ground state energy $E_{0}\left(N, \tilde{g}_{2}\right)$ of $N$ Bosons in the harmonic trap in units of $\hbar \omega$ varies

[^28](255]
continuously with the dimensionless coupling constant $\tilde{g}_{2}$. The associated value $k_{0}=E_{0} / 2 \hbar \omega$ is therefore a real number, not restricted to integer or half-integer values, which exhaust the complete set $k_{l}^{(0)}=(l+1) / 2$ of representations in the trivial singleparticle limit $N=1$. Indeed, it is precisely the appearance of arbitrary real values of $k$ which distinguishes the representations of the universal covering group $\overline{\mathrm{SU}(1,1)}$ from that of the Lorentz group $\mathrm{SO}(2,1)$, where $k$ only assumes integer or half-integer values [254]. In order to show that the global observables associated with the group generators exhibit period $T / 2$, we decompose a general many-body initial state $|\psi\rangle(t=0)=\sum_{\{k\}}\left|\psi_{k}\right\rangle$ in terms of the discrete set $\{k\}$ of all irreducible representations. With $c_{k}(z)=\langle k, z \mid \psi\rangle$, the standard resolution of the identity in terms of coherent states [257] shows that
\[

$$
\begin{equation*}
\left|\psi_{k}\right\rangle=\frac{2 k-1}{\pi} \int_{|z|<1} \frac{d^{2} z}{\left(1-|z|^{2}\right)^{2}} c_{k}(z)|k, z\rangle \quad \rightarrow \quad\langle\psi(0)| \hat{L}_{j}|\psi(0)\rangle=\sum_{\{k\}} \frac{2 k-1}{\pi} \int_{|z|<1} \frac{d^{2} z}{\left(1-|z|^{2}\right)^{2}}\left|c_{k}(z)\right|^{2}\langle k, z| \hat{L}_{j}|k, z\rangle . \tag{145}
\end{equation*}
$$

\]

Since the matrix elements in Eq. 144 are single-valued for arbitrary $k$, the expectation values $\left\langle\hat{L}_{j}\right\rangle$ are evidently invariant under a rotation of $z$ by $2 \pi$, which describes the time evolution of the coherent states over a time $t=T / 2$. Thus, the observables $\hat{L}_{j}$ or linear combinations thereof evolve with period $T / 2$ for arbitrary initial states despite the generically multi-valued nature of the representations $D^{+}(k)$. An example that was mentioned above is the potential energy $E_{\text {pot }}(t)=\omega^{2}\langle\hat{C}\rangle(t)$, which is represented by the generator $\hat{C}=\hbar\left(\hat{L}_{3}-\hat{L}_{1}\right) / \omega$ of the special conformal transformations (due to energy conservation, a nontrivial time evolution appears, of course, only in $\hat{L}_{1,2}$ ). For many-body observables which cannot be expressed as linear combinations of the $\mathrm{SO}(2,1)$ generators, however, no simple periodicity is expected. Indeed, if the operators $\hat{L}_{j}$ in 145 are replaced by the local density operator $\hat{n}(\boldsymbol{x})=\hat{\psi}^{\dagger}(\boldsymbol{x}) \hat{\psi}(\boldsymbol{x})$, the matrix elements $\langle k, z| \hat{n}(\boldsymbol{x})|k, z\rangle$ will in general not be invariant under $2 \pi$ - rotations of $z$. The observation shown in Fig. 17 that for the special initial condition of a triangular shape, the full density exhibits revivals at multiples of $T / 2$ is therefore not a consequence of the $\mathrm{SU}(1,1)$ algebra 140 , which only constrains the time evolution of the global observables $\hat{L}_{j}$. A clue towards a theoretical understanding of these revivals has recently been provided by Shi et al. [258]. It is based on the equivalence of a time-dependent Gross-Pitaevskii (GP) description with a classical Vlasov equation [259]

$$
\begin{equation*}
\partial_{t} f_{t}+\frac{\boldsymbol{p}}{m} \nabla_{x} f_{t}-\nabla_{x} V_{\mathrm{eff}}\left[f_{t}\right] \cdot \nabla_{p} f_{t}=0 \quad \text { where } \quad f_{t}(\boldsymbol{x}, \boldsymbol{p})=\int \frac{d^{d} y}{(2 \pi \hbar)^{d}} \exp (i \boldsymbol{p} \cdot \boldsymbol{y} / \hbar) \bar{\phi}_{t}(\boldsymbol{x}+\boldsymbol{y} / 2) \phi_{t}(\boldsymbol{x}-\boldsymbol{y} / 2) \tag{146}
\end{equation*}
$$

for the Wigner function $f_{t}(\boldsymbol{x}, \boldsymbol{p})$ associated with a coherent state which follows from the solution $\phi_{t}(\boldsymbol{x})$ of the time-dependent GP equation. More precisely, as discussed e.g. by Fröhlich et al. [260], this equivalence holds in the formal limit $\hbar \rightarrow 0$, where the difference $V_{\mathrm{mf}}(\boldsymbol{x}+\boldsymbol{y} / 2)-V_{\mathrm{mf}}(\boldsymbol{x}-\boldsymbol{y} / 2) \rightarrow \boldsymbol{y} \cdot \nabla_{x} V_{\mathrm{mf}}(\boldsymbol{x})$ of the mean-field contribution $V_{\mathrm{mf}}(\boldsymbol{x}, t)=g n_{t}(\boldsymbol{x})$ to the total effective potential $V_{\text {eff }}(\boldsymbol{x}, t)=V(\boldsymbol{x})+V_{\mathrm{mf}}(\boldsymbol{x}, t)$ is determined by the gradient of the local density $n_{t}(\boldsymbol{x})=\int_{\boldsymbol{p}} f_{t}(\boldsymbol{x}, \boldsymbol{p})=\mid \phi_{t}\left(\left.\boldsymbol{x}\right|^{2}\right.$. In this limit, any initial configuration with a positive Wigner function evolves into one where $f_{t}(\boldsymbol{x}, \boldsymbol{p})$ remains positive for all times $t>0$, a property which is not guaranteed by GP dynamics in general. Note also that Eq. 146) describes a collisionless situation characteristic for superfluid hydrodynamics. In contrast to a Boltzmann equation, therefore, both energy and entropy are conserved. The equation can formally be rewritten in hydrodynamic form with a momentum density $\boldsymbol{g}_{t}(\boldsymbol{x})=n_{t}(\boldsymbol{x})\langle\boldsymbol{p}\rangle_{t}(\boldsymbol{x})$ which obeys a local conservation law $\partial_{t} \boldsymbol{g}+\operatorname{div} \boldsymbol{\Pi}=\left(\partial_{t} \boldsymbol{g}\right)_{\text {trap }}$ [143]. The associated stress tensor $\boldsymbol{\Pi}$ is obtained from the one in Eq. 130 above for the genuine many-body problem by simply replacing field operators $\hat{\psi}(\boldsymbol{x}), \hat{\psi}^{\dagger}(\boldsymbol{x}) \rightarrow \phi(\boldsymbol{x}), \bar{\phi}(\boldsymbol{x})$ by the GP wave function or its complex conjugate, as expected for a coherent state. Its trace

$$
\begin{equation*}
\boldsymbol{\Pi}_{i i}^{\mathrm{GP}}(\boldsymbol{R})=\frac{\hbar^{2}}{m} \nabla_{\boldsymbol{R}} \phi \cdot \nabla_{\boldsymbol{R}} \bar{\phi}+d \cdot \frac{g_{d}}{2}|\phi(\boldsymbol{R})|^{4} \underset{d=2}{\longrightarrow} 2 \varepsilon^{\mathrm{GP}}(\boldsymbol{R}) \tag{147}
\end{equation*}
$$

obeys the condition of scale invariance discussed in Eq. 130) for the special case of two dimensions, where the connection $p_{\mathrm{GP}}(n, T=0) \sim n^{2}$ between the ground state pressure and density within GP coincides with the dependence $p(n, T=0) \sim n^{1+2 / d}$ which follows from the relation $p=2 \varepsilon / d$ for scale invariant many-body systems in general. Here, the second equality in (147) relies on the assumption that $g_{2}$ does not depend on density. This is valid in a weakly confined situation, as will be discussed in detail in the following section. A further subtle point that should be emphasized is that - despite formal appearance - the gradient and quartic terms in Eq. 147 ) do not provide a separation of the microscopic energy density into a kinetic and an interaction part. In fact, as discussed by Lieb, Seiringer and Yngvason [261, 262] and mentioned also in connection with Eq. (152) below, such a separation is impossible for zero range interactions. For an understanding of why an initial triangle with a flat density shows revivals after multiples of $T / 2$, a crucial insight by Shi et al. [258] is the observation that the gradient of the mean-field potential $V_{\mathrm{mf}}(\boldsymbol{x}, t)$ may be assumed to vanish not only at $t=0$ but for all times up to $t=T / 4$, where the triangle has turned upside down, see Fig. 17 . As a result, the solution of the Vlasov equation (146) is obtained from the initial distribution $f_{0}(\boldsymbol{x}, \boldsymbol{p})$ upon replacing $\boldsymbol{x}, \boldsymbol{p} \rightarrow \boldsymbol{x}(t), \boldsymbol{p}(t)$ by the single particle dynamics in the external harmonic oscillator potential. The second observation, which singles out the triangle from other initial configurations like the square shown on the left side in Fig. 17 is that, for equilateral triangles, the overlap between two triangles of any size which are displaced by an arbitrary vector is again an equilateral triangle. This property allows an explicit solution of the Vlasov equation for an initial Wigner function which is proportional to the indicator function $\Delta_{L}(\boldsymbol{x})$ of an equilateral triangle with side length $L$ and its complement $\nabla_{p_{F}}(\boldsymbol{p})$ in momentum space, whose
characteristic scale $p_{F}$ will be discussed below. Indeed, under the free time evolution $\boldsymbol{x}(t)=\boldsymbol{x} \cos \omega t-(\boldsymbol{p} / m \omega) \sin \omega t$ and $\boldsymbol{p}(t)=\boldsymbol{p} \cos \omega t+m \omega \boldsymbol{x} \sin \omega t$ in a harmonic trap, the product $\Delta_{L}(\boldsymbol{x}(t)) \nabla_{p_{F}}(\boldsymbol{p}(t))$ can be expressed as a product of two triangles in momentum space which are shifted and rescaled in size. The fact that their overlap is again a triangle therefore allows to represent the Wigner function

$$
\begin{equation*}
f_{0}(\boldsymbol{x}, \boldsymbol{p})=\frac{4 \bar{n}}{\sqrt{3} p_{F}^{2}} \Delta_{L}(\boldsymbol{x}) \nabla_{p_{F}}(\boldsymbol{p}) \rightarrow f_{t}(\boldsymbol{x}, \boldsymbol{p})=\frac{4 \bar{n}}{\sqrt{3} p_{F}^{2}} \Delta_{L}(\boldsymbol{x}(t)) \nabla_{p_{F}}(\boldsymbol{p}(t))=\frac{4 \bar{n}}{\sqrt{3} p_{F}^{2}} \nabla_{p_{F}(t)}(\boldsymbol{p}-\overline{\boldsymbol{p}}(\boldsymbol{x}, t)) \tag{148}
\end{equation*}
$$

in the form of a single indicator function in momentum space. Its side length $p_{F}(t)$ and mean position $\overline{\boldsymbol{p}}(\boldsymbol{x}, t)$ are uniquely determined up to the maximum time $t=T / 4$, where the solution describes an inverted triangle in position space with uniform particle and vanishing momentum density $\boldsymbol{g}(\boldsymbol{x}, T / 4) \equiv 0$. The exact revival of the initial distribution at $t=T / 2$ may then be inferred from the fact that the free oscillator dynamics leads to a density $n(\boldsymbol{x}, T / 4+\delta t)=n(\boldsymbol{x}, T / 4-\delta t)$ which is invariant under a reversal of time at $t=T / 4$. More precisely, as discussed by Olshanii et al. [263], connecting the two solutions at $t=T / 4$ requires to properly deal with the singularities encountered at the sharp edges of the distribution, where the Thomas-Fermi dynamics described by the Vlasov equation becomes invalid. Now, apparently, the arguments above are independent of the choice made for $p_{F}$ and they also do not rely on the scale invariance condition (147). In fact, the strength $g_{2}$ of the interactions in this approach is determined only a posteriori by connecting it to the momentum scale $p_{F}$ in the form $g_{2}=p_{F}^{2} /(24 m \bar{n})$ [258]. In particular, by choosing $p_{F}^{2} \sim \hbar^{2} \bar{n}$ to depend on density like the Fermi momentum of a non-interacting Fermi gas, the interaction contribution to the trace of the GP stress tensor 147) scales like $\bar{n}^{2}$, as expected for a scale invariant GP functional in 2D. For the validity of Eq. 148, however, an arbitrary choice for $p_{F}$ is possible. Moreover, in the Thomas-Fermi limit, the dynamics of the 2D Bose gas follows by balancing the inward force due to the harmonic trap with the interaction contribution to the divergence div $\Pi^{\text {int }}$ of the stress tensor while the kinetic contribution is irrelevant. In the solution by Shi et al. [258], div $\Pi{ }^{\mathrm{int}}$ is replaced by the gradient of an effective Fermi pressure, a procedure which gives rise to a scale invariant equation of state in any dimension. A deeper understanding of the experimental observations, which are perfectly reproduced by a numerical solution not only at the Gross-Pitaevskii level but also for the classical N-body problem with scale invariant interactions [252], is therefore still an open challenge. It is possible that the revivals seen in the time evolution of the density are connected with the decomposition of special conformal transformations in terms of two inversions and a translation discussed in Eq. 127.

Broken scale invariance and anomalies in two dimensions Beyond the rather special case of one dimension, the two standard examples for scale invariant many-body systems are the unitary Fermi gas in 3D and both Bose or two-component Fermi gases in 2D. Apparently, the latter case appears more rich, in particular since zero range interactions $V(\boldsymbol{x})=\bar{g}_{2} \delta(\boldsymbol{x})$ in 2D seem to be scale invariant without the necessity for any fine tuning of the interaction strength $\bar{g}_{2}$. Moreover, in contrast to the situation in 3D, it is not difficult to realize an isotropic harmonic trap. As discussed above, this allows to directly observe the consequences of scale and conformal invariance in the dynamics which is formally related to the fact that the generator $\hat{C}$ of the special conformal transformation (126) appears as the trapping part in the many-body Hamiltonian (138). A careful analysis of scale invariance in two dimensions shows, however, that it is strictly valid only in the trivial limit of vanishing interactions. For an understanding of this caveat and its fortunately often negligible consequences, it is convenient to consider an elementary criterion for the existence of scale invariance $i[\hat{H}, \hat{D}]=2 \hat{H}$ in a many-body system due to Holstein [264]. The argument relies on the observation that scale invariance survives in the presence of interactions provided that the associated phase shifts become independent of momentum and thus do not involve any intrinsic length scale. For low energy scattering of ultracold atoms in 3D, where only s-wave scattering survives, the standard expansion $k \cot \delta_{0}(k) \rightarrow-1 / a+r_{e} k^{2} / 2+\ldots$ shows that this condition is obeyed at infinite scattering length $1 / a=0$. Indeed, since the effective range correction is negligible in the ultracold limit, no intrinsic length scale remains in the phase shift $\delta_{0}(k)=\pi / 2-r_{e} k / 2+\ldots$ Similarly, in 1 D , the phase shift from $\cot \delta(k)=k a_{1}+\ldots$ which appears in the backscattering amplitude $f(k)=-(1+i \cot \delta(k))^{-1}$ of Eq. 129, approaches $\delta(k \rightarrow 0)=\pi / 2$ at low energies. A rather peculiar behavior is found in two dimensions where the scattering of an incoming plane wave $\exp (i \boldsymbol{k} \cdot \boldsymbol{x})$ gives rise to an outgoing cylindrical wave. Asymptotically, this can be expressed in the form ${ }^{29}$

$$
\begin{equation*}
\psi_{k}(x) \rightarrow e^{i \boldsymbol{k} \cdot \boldsymbol{x}}-\sqrt{\frac{i}{8 \pi}} f(k, \theta) \frac{e^{i k r}}{\sqrt{k r}} \quad \text { where } \quad f(k, \theta) \rightarrow f_{0}(k)=\frac{4}{-\cot \delta_{0}(k)+i} \underset{k \rightarrow 0}{\longrightarrow} \frac{4 \pi}{2 \ln \left(1 / k a_{2}\right)+i \pi} \tag{149}
\end{equation*}
$$

At low energy, the associated dimensionless scattering amplitude $f(k, \theta)$ becomes independent of the scattering angle $\theta$ and exhibits a logarithmic dependence on momentum, which defines the 2D scattering length $a_{2}$. For any finite value $a_{2} \neq 0$, the resulting s-wave phase shift $\delta_{0}(k) \rightarrow-\pi /\left[2 \ln \left(1 / k a_{2}\right)\right]$ therefore never looses its dependence on $k$ and scale invariance is violated. On a formal level, this arises from the fact that a delta function in 2D does not give rise to a finite low energy scattering

[^29]amplitude unless its strength $\bar{g}_{2} \rightarrow \bar{g}_{2}(\Lambda)$ is made cutoff dependent, see Eq. (151) below. Fortunately, however, the violation of scale invariance is unobservable in the weak-confinement limit, where both the effective range $\simeq \ell_{\mathrm{vdW}}$ and the 3D scattering length $a$ are considerably smaller than the transverse confinement length $\ell_{z}$. The two-particle scattering problem is therefore of a 3D nature. Its truncation to the lowest transverse eigenstate allows to express the 2D scattering length $a_{2}$ in terms of its 3D counterpart $a$ in the form [49, 265]
\[

$$
\begin{equation*}
a_{2}(a)=\ell_{z} \sqrt{\frac{\pi}{B}} \exp \left(-\sqrt{\frac{\pi}{2}} \frac{\ell_{z}}{a}\right) \quad \text { with } \quad B=0.905 \ldots \tag{150}
\end{equation*}
$$

\]

Note that $a_{2}$ is always positive, which implies that there is a two-body bound state for arbitrary values of the 3D scattering length. Formally, the bound state arises from the pole of $f_{0}(k)$ at $k=i / a_{2}^{\mathrm{b}}$, however the associated scattering length $a_{2}^{\mathrm{b}}$ coincides with $a_{2}$ from Eq. (150] only for small negative 3D scattering lengths $-\ell_{z} / a \gg 1$ [49]. In the generic opposite case where $0<a \ll \ell_{z}$, the 2D scattering length is exponentially small. The exponential factor in 150 can then be separated off in $-\cot \delta_{0}(k)=(2 / \pi) \ln \left(1 / k a_{2}\right) \rightarrow 4 / \tilde{g}_{2} \gg 1$, leaving a negligible momentum dependent correction $\sim \ln \left(1 / k \ell_{z}\right)$. As a result, the s-wave scattering amplitude $f_{0}(k) \rightarrow \tilde{g}_{2}=\sqrt{8 \pi} a / \ell_{z}$ is finite and real, equivalent to a constant phase shift $\delta_{0}(k) \rightarrow-\tilde{g}_{2} / 4$ at low energies. In the standard range $0<a \ll \ell_{z}$ of scattering lengths, the fact that the 2D delta function is homogeneous with degree -2 therefore indeed leads to an effectively scale invariant many-body Hamiltonian. In particular, as discussed above, there is an undamped breathing mode at $2 \omega$ and an equation of state $n \lambda_{T}^{2}=\left.f_{n}(\beta \mu)\right|_{\tilde{g}_{2}}$ which obeys a single parameter scaling. This simple description breaks down when the momentum dependence of the phase shift, which drives $\delta_{0}(k)=-\pi /\left[2 \ln \left(1 / k a_{2}\right)\right]$ to zero eventually, can not be neglected. Now, at a given temperature $T$, the characteristic momenta $k \simeq 1 / \lambda_{T}$ scale inversely with the associated thermal length $\lambda_{T}$. The appparent scale invariance is thus violated when $\ln \left(\lambda_{T} / a_{2}\right)$ can no longer be replaced by a temperature independent constant. Using the result 150 for the 2 D scattering length, this leads to $\ln \left(\lambda_{T} / \ell_{z}\right) \simeq \ell_{z} / a$ which, in the weak-confinement limit $\ell_{z} \gg a$, requires exponentially small temperatures that are far beyond reach.

A quantitative discussion of this problem has been given by Rancon and Dupuis [242], both in two and the rather different case of three dimensions. Quite generally, the density equation of state of dilute gases with zero range interactions can be written in the form $n \lambda_{T}^{d}=f_{n}\left(\beta \mu, u_{d}(T)\right)$. The associated universal scaling function $f_{n}(x, y)$ depends on temperature both via $x=\beta \mu$ and also via $y=u_{d}(T)$, which describes the effects of interactions. The associated parameter $u_{d}(T)=u_{d}\left(\kappa=1 / \lambda_{T}\right)$ is determined by the renormalized value of the bare dimensionless strength $u_{d}(\Lambda)=\left(2 m S_{d} / \hbar^{2}\right) \Lambda^{d-2} \bar{g}_{d}(\Lambda)$ of the zero range interactions at a momentum scale $\kappa \ll \Lambda$ which is set by the inverse of the thermal length $\lambda_{T}\left(2 S_{2}=1 / \pi\right.$ and $2 S_{3}=1 / \pi^{2}$ in $d=2$ or $d=3$ are numerical factors). Now, as derived by Fisher and Hohenberg [267], the momentum dependence of the coupling constant $u_{2}(\kappa)$ in two dimensions obeys the flow equation $d u_{2} / d l=-u_{2}^{2} / 2$. By integrating this between a UV cutoff $\Lambda$ at $l=0$ down to a momentum scale $\kappa=\Lambda e^{-l}$, one obtains an inverse logarithmic dependence

$$
\begin{equation*}
u_{2}(\kappa)=\frac{u_{2}(\Lambda)}{1+\frac{u_{2}(\Lambda)}{2} \ln (\Lambda / \kappa)} \rightarrow \frac{2}{\ln \left(1 /\left(\kappa a_{2}\right)\right)} \quad \text { for } \quad \bar{g}_{2}(\Lambda)=-\frac{2 \pi \hbar^{2}}{m \ln \left(\Lambda a_{2}\right)} \tag{151}
\end{equation*}
$$

of the coupling constant. It is independent of the short distance scale $\Lambda \simeq 1 / \ell_{\mathrm{vdW}}$ if the bare strength $\bar{g}_{2}(\Lambda)$ of the delta function interaction is choosen appropriately. Consistent with the qualitative discussion above, $u_{2}(T) \rightarrow \tilde{g}_{2} / \pi$ can be replaced by a temperature independent constant in the weak confinement limit unless the temperature reaches values where $\lambda_{T} \simeq \ell_{z} \exp \left(\ell_{z} / a\right)$. The equation of state is then effectively like the one for the scale invariant non-interacting system with $\beta \mu$ as a single relevant variable, while $\tilde{g}_{2}$ only appears as a fixed external parameter. To see deviations from single parameter scaling requires exponentially small temperatures. This may be viewed as a low energy counterpart of asymptotic freedom, where the strength $\alpha_{s}(\kappa)$ of the strong interactions approaches zero for momenta $\kappa$ much larger than the QCD scale $\Lambda_{\mathrm{QCD}}$ (defined by $\alpha_{s}\left(\Lambda_{\mathrm{QCD}}\right) \simeq 1$ ), following an inverse logarithmic dependence as in Eq. (151] with $a_{2} \rightarrow 1 / \Lambda_{\mathrm{QCD}}$ [207]. An exponentially small 2D scattering length $a_{2}$ thus corresponds to a large value of the QCD scale $\Lambda_{\mathrm{QCD}}$. In three dimensions, the flow equation $d u_{3} / d l=-u_{3}-u_{3}^{2} / 2$ has again $u^{*}=0$ as a fixed point but the approach is linear. Thus, $u_{3}(T) \sim a / \lambda_{T}$ leads to an equation of state $n \lambda_{T}^{3}=f_{n}\left(\beta \mu, a / \lambda_{T}\right)$ which has the form expected from simple dimensional analysis. It is reduced to a scale invariant form with a single temperature dependent variable $\beta \mu$ only at fine tuned values $a=0$ or $1 / a=0$ of the scattering length.

The experimental consequences of the violation of scale invariance in 2D beyond the weak confinement limit have been discussed first by Olshanii et al. [268] and by Hofmann [269] for Bose or two-component Fermi gases. On a formal level, the basic operator relation $i[\hat{H}, \hat{D}]=2 \hat{H}$ turns out to be replaced by $i[\hat{H}, \hat{D}]=2 \hat{H}+\partial \hat{H} / \partial \ln a_{2}$, a relation that was originally derived within a quantum hydrodynamic approach [268]. Any dependence of the Hamiltonian on the 2D scattering length therefore necessarily destroys scale invariance. The additional term is an example of an anomaly, where quantum fluctuations break a symmetry of the classical Lagrangian. The concept has its origin in QFT, the most notable example being the Adler-BellJackiw anomaly which gives rise to a finite amplitude for the decay $\pi^{0} \rightarrow \gamma+\gamma$ of a neutral pion into two $\gamma$ 's [207]. In the context of AMO physics, the standard example is the Efimov effect, where the appearance of an infinite number of trimers right at the


FIG. 18: The left Figure, taken from Ref. [270], shows the breathing mode frequency (blue dots) in a two-component Fermi gas of ${ }^{6} \mathrm{Li}$ in 2 D . The shift with respect to twice the dipole mode frequency $\omega_{\text {dip }}=\omega$ (red dots) exhibits a maximum near the Feshbach resonance at $B_{0}=832 \mathrm{G}$. The right Figure shows the RF spectrum associated with the continuum of bound-free transitions from Ref. [271]. Both the smooth onset at the threshold $\omega=E_{d}$ for breaking a molecule and the asymptotic dependence $1 /\left(\omega^{2} \ln ^{2} \omega\right)$ agree well the theory prediction [272].
threshold for the existence of a two-body bound state breaks the scale invariance associated with the $1 / R^{2}$ - effective interaction of three particles in terms of the associated hyper-radius $R$ [214, 215]. The physical meaning of the additional contribution to the commutator $i[\hat{H}, \hat{D}]$ is elucidated by noting that the 2D version of the Tan adiabatic theorem (76) connects its expectation value $\partial\langle\hat{H}\rangle / \partial \ln a_{2}=\hbar^{2} C_{2} /(4 \pi m)$ to the integrated two-body contact. Based on the microscopic expression f130p for the trace of the stress tensor, the violation of scale invariance in 2D may thus be formulated in terms of local operators in the form ${ }^{30}$

$$
\begin{equation*}
\hat{\Pi}_{i i}(\boldsymbol{R})=2 \hat{\varepsilon}(\boldsymbol{R})+\frac{\hbar^{2}}{4 \pi m} \hat{C}_{2}(\boldsymbol{R}) \rightarrow i[\hat{H}, \hat{D}]=2 \hat{H}+\frac{\hbar^{2}}{4 \pi m} \hat{C}_{2} . \tag{152}
\end{equation*}
$$

In a thermal equilibrium state, the expectation value of the trace $\hat{\Pi}_{i i}$ of the stress tensor in 2D gives twice the pressure. As a result, Eq. 152) immediately implies

$$
\begin{equation*}
p=\varepsilon+\frac{\hbar^{2}}{8 \pi m} C_{2} \quad(\text { Bosons, 2D) } \tag{153}
\end{equation*}
$$

which is the 2D version of the Tan pressure relation (131). Since there is no Efimov effect in 2D, the introduction of a three-body parameter $\kappa_{*}$ and the associated three-body contact density $C_{3}$ as in Eq. 131) is unnecessary. Scale invariance in 2D requires $C_{2}$ to vanish, which is strictly true only for non-interacting particles. For finite interactions, the breathing mode frequency $\omega_{B}=2 \omega+\delta \omega$ is thus expected to deviate from its ideal value. The shift $\delta \omega$ at $T=0$ has been determined by Hofmann [269] for two-component Fermi gases, where the scattering length can be changed over a wide range using Feshbach resonances. In terms of the dimensionless interaction parameter $\eta=\ln \left(k_{F} a_{2}\right)$, one obtains a blue-shift with a maximum value $\delta \omega / \omega \simeq 0.2$ near $\eta=0$. For weak interactions, the shift vanishes like $\delta \omega / \omega \rightarrow 1 / 4 \eta^{2}$ in the Fermi gas or BCS regime $\eta \gg 1$ and like $\delta \omega / \omega \rightarrow-1 / 4 \eta$ in the opposite BEC limit $\eta \ll-1$ [269].

Experimentally, this has been tested by Vogt et al. [273], who found that the breathing mode frequency stays close to the scale invariant value $2 \omega$ over a rather wide range of the coupling constant $\ln \left(k_{F} a_{2}\right)$. More recent measurements by Holten et al. [270] which are shown in Fig. 18 have indeed observed the predicted blue-shift. As expected, the maximum appears near the Feshbach resonance, where $\eta \simeq 1$. In quantitative terms, the effect turns out to be about an order of magnitude smaller than predicted from the zero temperature calculation. A possible explanation of this discrepancy relies on an observation due to Taylor and Randeria [274]. Using sum rules, they have shown that the breathing mode frequency stays near $2 \omega$ provided the deviation

$$
\begin{equation*}
K-2 p=\rho\left(\frac{\partial p}{\partial \rho}\right)_{s}-2 p=\frac{\hbar^{2}}{4 \pi m}\left[C_{2}+\frac{a_{2}}{2}\left(\frac{\partial C_{2}}{\partial a_{2}}\right)_{s}\right] \tag{154}
\end{equation*}
$$

of the isentropic bulk modulus $K$ from its value $K^{(0)}=2 p$ in a scale-invariant system is small. This is indeed the case because the two contributions on the right hand side of Eq. 154 largely cancel. A quite different option to test the deviations from

[^30]scale invariance in 2D gases is provided by the RF spectrum in the presence of a non-vanishing final state interaction. This has been measured again with ${ }^{6} \mathrm{Li}$ atoms by Sommer et al. [271]. At the 3D Feshbach resonance, the binding energy of a two-body bound state is $E_{d}=0.244 \hbar \omega_{z}$ [49]. For the parameters used in the experiment, this is about six times the Fermi energy. The confined, resonant Fermi gas at these low densities is therefore essentially in the BEC limit, with a dimensionless coupling constant $\ln \left(k_{F} a_{2}\right) \simeq-0.55$. In the presence of final state interactions, the dissociation spectrum exhibits a smooth onset $\sim 1 / \ln ^{2}\left[\left(\omega-E_{d}\right)\right]$ near the threshold $\omega=E_{d}$, quite different from the jump due to the 2D density of states expected from Wigner's threshold law [272]. Moreover, the violation of scale invariance leads to a decay $\sim 1 /\left(\omega^{2} \ln ^{2} \omega\right)$ of the RF spectrum for large frequencies which is faster than the $1 / \omega^{2}$ - tail obtained from a simple dimensional analysis. As shown in Fig. 18, both features are consistent with the measured data. More recently, the effects of broken scale invariance have been observed in a rather direct form in the real time dynamics of 2D Fermi gases by Murthy et al. [275]. The experiment relies on the extension of Eq. (137) to the time-dependent momentum distribution
\[

$$
\begin{equation*}
n(\boldsymbol{k}, t)=\lambda^{2}(t) \int d^{2} x f(\boldsymbol{x}, \lambda(t)(\boldsymbol{k}+m \dot{\lambda}(t) \boldsymbol{x} / \hbar), t=0) \tag{155}
\end{equation*}
$$

\]

which follows from the exact mapping (135) of the many-body wave function between free space and a time-dependent harmonic trap $(f(\boldsymbol{x}, \boldsymbol{k}, t)$ is the Wigner function). By measuring $n(\boldsymbol{k}, t)$ at the inner and outer turning points of a breathing mode cycle, where $\dot{\lambda}(t)=0$, deviations from the simple scaling (137) of the associated density show up at intermediate values of $\ln \left(k_{F} a_{2}\right)$ with maximum deviation by a factor $\simeq 0.8$ near $\ln \left(k_{F} a_{2}\right) \simeq 1.3$ [275].

Finally, we mention an analog of the anomalous contribution in Eq. (153) due to the breaking of scale invariance which was effectively discovered by Fierz [276] in the context of quantum fluids with hard-sphere interactions. Similar to the many-body version $\sqrt{128}$ of the Bethe-Peierls boundary condition for unitary gases, the effect of interactions in this case is fully accounted for by requiring that the many-body wave function $\psi\left(\boldsymbol{x}_{1} \ldots \boldsymbol{x}_{N}\right)$ vanishes when the distance $\left|\boldsymbol{x}_{\boldsymbol{i}}-\boldsymbol{x}_{\boldsymbol{j}}\right| \leq \sigma$ between any two coordinates is below the hard-sphere diameter $\sigma$. Now, at the two-body level, the associated wave function $\psi_{0}(r) \rightarrow(|x|-\sigma)+\ldots$ exhibits a discontinuity in its first derivative at contact $r=\sigma$. This leads to a power law decay $\psi_{0}(q) \rightarrow-4 \pi \sigma \sin (q \sigma) / q^{3}$ of its Fourier transform at large momentum, modulated with an oscillating factor $\sin (q \sigma)$. The consequences of this behavior for the quantum fluid at finite density $n$ become apparent by defining a hard-sphere analog $\mathcal{C}_{\mathrm{HS}}$ of the two-body contact density $\mathcal{C}_{2}$ for zero-range interactions by the relation

$$
\begin{equation*}
n^{2} g^{(2)}\left(r \rightarrow \sigma^{+}\right)=\frac{C_{\mathrm{HS}}}{16 \pi^{2}}\left(1-\frac{r}{\sigma}\right)^{2}+\ldots=n^{2} \frac{g^{\prime \prime}(\sigma)}{2}(r-\sigma)^{2}+\ldots \tag{156}
\end{equation*}
$$

Using the short-distance factorization arguments in Lecture III, both the momentum distribution and the static structure factor

$$
\begin{equation*}
\lim _{q \sigma \gg 1} n(q)=C_{\mathrm{HS}}\left(\frac{\sin (q \sigma)}{q^{3}}\right)^{2} \text { and } \lim _{q \sigma \gg 1} S(q)=1-\frac{C_{\mathrm{HS}}}{2 \pi n \sigma} \frac{\cos (q \sigma)}{q^{4}}+\ldots \tag{157}
\end{equation*}
$$

of quantum hard-sphere fluids in 3D exhibit a power law decay with an oscillating amplitude. Its strength is determined by the curvature $g^{\prime \prime}(\sigma)=2 C_{\mathrm{HS}} /(4 \pi n \sigma)^{2}$ of the pair distribution function at contact which appears as an anomalous contribution in the 3D virial theorem

$$
\begin{equation*}
p_{\mathrm{HS}}=\frac{2}{3} \varepsilon+\frac{\pi \hbar^{2} n^{2} \sigma^{3}}{3 m} g^{\prime \prime}(\sigma) \tag{158}
\end{equation*}
$$

that was derived by Fierz in 1957 [276]. Note that in the case of hard spheres, the full energy density in the first term of the virial theorem may be replaced by the kinetic energy contribution $\varepsilon_{\mathrm{kin}}=\left\langle\hat{H}_{0}\right\rangle / V$. Indeed, the quantum hard-sphere system has no interaction energy at all. Yet, it is of course very different from a non-interacting system. The additional contribution proportional to $g^{\prime \prime}(\sigma)$ in Eq. (158) accounts for that. Similar to the correction involving the two-body contact in Eq. (153), it is an effective anomaly which reflects the fact that the finite hard-sphere diameter $\sigma$ breaks scale invariance. In the low-density limit $n \sigma^{3} \ll 1$, one has $\mathcal{C}_{\mathrm{HS}} \rightarrow(4 \pi n)^{2}$ as a simple consequence of the leading order result $g^{(2)}(r) \rightarrow(1-r / \sigma)^{2}$ derived by Lee, Huang and Yang [153]. This result is equivalent to the mean-field expression $C_{2} \rightarrow(4 \pi n a)^{2}$ for the two-body contact density of a dilute BEC since the scattering length $a_{\mathrm{HS}}=\sigma$ of hard spheres is just the associated diameter $\sigma$. More generally, it turns out that the Tan result (85) for the leading power law decay of the momentum distribution is a special case of Eq. (157) which is obtained by taking the limit $\sigma \rightarrow 0$ with a fixed value of $\sigma^{2} \mathcal{C}_{\mathrm{HS}} \rightarrow \mathcal{C}_{2}$. An interesting issue arises in the classical limit, where the virial theorem for hard spheres can be expressed in the form [19]

$$
\begin{equation*}
p_{\mathrm{HS}}^{\mathrm{cl}}=\frac{2}{3} \varepsilon_{\mathrm{kin}}^{\mathrm{cl}}\left[1+\frac{2 \pi}{3} n \sigma^{3} g_{\mathrm{cl}}^{(2)}(\sigma)\right], \tag{159}
\end{equation*}
$$

with $2 \varepsilon_{\text {kin }}^{\mathrm{cl}} / 3=n k_{B} T$ by the equipartition theorem. The classical hard-sphere fluid thus has a finite value of the pair distribution function $g_{\mathrm{cl}}^{(2)}(\sigma)$ right at contact which describes the interaction contribution to the pressure. In fact, it even takes its maximum value there [19]. It is an open problem to see how the result (159) emerges as a classical limit of the Fierz relation (158).


FIG. 19: The left Figure shows the time evolution of the aspect ratio in a Fermi gas after removing the trap with an initial anisotropy $1 / 2.7 \simeq 0.37$. The lowest curve at $a=0$ exhibits purely ballistic expansion while the four upper ones are for a unitary gas at different average energies $\tilde{E} / E_{F}$ per particle. They display hydrodynamic flow, where the inversion of the aspect ratio is slowed down by a finite shear viscosity. The trap-averaged values of the shear (blue) and bulk (red) viscosities are shown in the right. Both Figures are taken from Elliott et al. [277].

Bulk viscosity and contact correlations An important application of scale and conformal invariance in ultracold gases that will be touched only briefly in this final section concerns the issue of transport coefficients. A standard example are the shear and bulk viscosity $\eta$ and $\zeta$ which appear in the stress tensor

$$
\begin{equation*}
\left\langle\hat{\Pi}_{i j}\right\rangle=p \delta_{i j}+\rho v_{i} v_{j}-\eta\left(\partial_{i} v_{j}+\partial_{j} v_{i}-\frac{2}{3} \delta_{i j} \cdot \operatorname{div} \boldsymbol{v}\right)-\zeta \delta_{i j} \cdot \operatorname{div} \boldsymbol{v} \tag{160}
\end{equation*}
$$

of a Newtonian fluid in the presence of small but non-vanishing gradients in the velocity field $\boldsymbol{v}(\boldsymbol{x})$. It was shown by Son [278], that the bulk viscosity $\zeta(T) \equiv 0$ of a conformally invariant fluid vanishes at arbitrary temperature ${ }^{31}$. Physically, this implies that there is no entropy production $\dot{s}=\zeta / T \cdot(\operatorname{div} \boldsymbol{v})^{2} \equiv 0$ in a uniform expansion like the Hubble flow, where div $\boldsymbol{v}$ is nonzero. Experimentally, the prediction of a vanishing bulk viscosity $\zeta$ has been tested for the unitary Fermi gas by Elliott et al. [277]. As shown in Fig. 19, a hydrodynamic model for the time evolution of the aspect ratio during expansion from an anisotropic trap allows to extract the averaged values of both the shear and bulk viscosity, the latter being consistent with $\zeta(T) \equiv 0$. By contrast, the shear viscosity is always finite, even in a superfluid. Remarkably, it exhibits a minimum value $\eta_{\min } \simeq 0.37 \hbar n$ in the unitary Fermi gas slightly below the superfluid transition [281] which is closely connected with the conjecture by Kovtun, Son and Starinets [282] that the ratio $\eta / s \geq \hbar /\left(4 \pi k_{B}\right)$ between the shear viscosity and the entropy density $s$ is bounded below by universal constants of nature for any fluid. For a detailed discussion of this topic see the reviews [281, 283].

In the following, we will restrict the attention to the connection between broken scale invariance and a finite bulk viscosity discussed by Dusling and Schäfer [284] and more recently by Enss [285], Hofmann [286] and by Fujii and Nishida [287]. On a microscopic level, both the shear and bulk viscosities follow from the retarded correlation functions of the stress tensor $\Pi_{i j}$ according to a Kubo formula

$$
\begin{equation*}
\chi_{i j, k l}(\boldsymbol{q}=0, \omega)=\frac{i}{\hbar} \int d t \int d^{d} x e^{i \omega t} \theta(t)\left\langle\left[\hat{\Pi}_{i j}(\boldsymbol{x}, t), \hat{\Pi}_{k l}(\mathbf{0}, 0)\right]\right\rangle \rightarrow \operatorname{Re} \eta(\omega)=\frac{\operatorname{Im} \chi_{x y, x y}(\omega)}{\omega} \text { and } \operatorname{Re} \zeta(\omega)=\frac{\operatorname{Im} \chi_{i i, i i}(\omega)}{d^{2} \omega} \tag{161}
\end{equation*}
$$

where $d=2,3$ is the dimension. An explicit evaluation of these correlators has so far only be achieved for the shear and bulk viscosity in the normal state of the unitary Fermi gas [285, 288] or within a leading order virial expansion [284-287]. A special situation arises for the bulk viscosity which vanishes identically for scale invariant systems. Formally, this follows by observing that the operator $\int_{\boldsymbol{x}} \hat{\Pi}_{i i}(\boldsymbol{x}, t) \rightarrow 2 \hat{H}$ which enters the commutator determining the frequency dependent bulk viscosity $\zeta(\omega)$ is then just the conserved Hamiltonian which commutes with its value at $t=0$. A more precise formulation must take into account the fact that for conserved or, more generally, non-ergodic variables, the Kubo relaxation function $\phi(z)=\left(\chi(z)-\chi^{\mathrm{T}}\right) / i z$ exhibits a singularity of the form $\pi\left[\chi^{\mathrm{T}}-\chi(z=0)\right] \delta(\omega)$ due to the finite difference between the thermodynamic and the static susceptibility. This has been discussed by Fujii and Nishida [287], who extend Eq. (152] to define a local pressure operator $\hat{p}(\boldsymbol{x})$ in general dimension $d$ in the form (here written for Bosons, where an additional factor 2 appears in the denominator of the term $\sim \hat{C}_{2}$ )

$$
\begin{equation*}
\hat{p}(\boldsymbol{x})=\frac{2}{d} \hat{\varepsilon}(\boldsymbol{x})+\frac{\hbar^{2}}{2 d \Omega_{d-1} m a^{d-2}} \hat{C}_{2}(\boldsymbol{x})=\hat{p}_{\mathrm{eq}}+\delta \hat{p}(\boldsymbol{x}) \quad\left(\Omega_{d-1}=2 \pi, 4 \pi \text { in } d=2,3\right) . \tag{162}
\end{equation*}
$$

[^31]Since the contribution from the energy density vanishes, the real part of the frequency dependent bulk viscosity $\operatorname{Re} \zeta(\omega)=$ $\operatorname{Re} \phi_{\zeta}(z=\omega+i 0)$ can be obtained from the relaxation function [285, 287] ${ }^{32}$

$$
\begin{equation*}
\phi_{\zeta}(z)=\left(\frac{\hbar^{2}}{2 d \Omega_{d-1} m a^{d-2}}\right)^{2} \int_{0}^{\infty} d t e^{i z t} \int d^{d} x\left(\hat{C}_{2}(\boldsymbol{x}, t) \mid \hat{C}_{2}(\mathbf{0}, 0)\right) \tag{163}
\end{equation*}
$$

of the contact densities. A finite bulk viscosity therefore only appears due to the deviation $p-2 \varepsilon / d$ of the pressure from scale invariance. In particular, for unitary gases in 3D, Eq. (163) immediately shows that $\zeta(\omega) \equiv 0$ vanishes not only at $\omega=0$ but in fact at all frequencies due to the vanishing prefactor $(1 / a)^{2}=0$. Analytical results for the bulk viscosity have been derived in leading order of a virial expansion [284-287]. Here we just quote the result in two dimensions

$$
\begin{equation*}
\zeta_{2 \mathrm{D}}(\omega=0)=\frac{\hbar n \cdot 2 \pi n \lambda_{T}^{2}}{\left[\ln ^{2}\left(k_{B} T / 2 \varepsilon_{b}\right)+\pi^{2}\right]^{2}} \tag{164}
\end{equation*}
$$

obtained by Enss [285]. It has the remarkable feature that interactions only enter through the logarithmic term in the denominator which involves the two-body bound state energy $\varepsilon_{b}=\hbar^{2} / m\left(a_{2}^{\mathrm{b}}\right)^{2}$. The dominant dependence on density and temperature, however, is completely fixed by dimensional analysis. Indeed, the units of viscosity are $\hbar n$ and it is also known from the kinetic theory of fluids that the bulk viscosity in single component gases only arises at order $n^{2}$ [19]. As a result, the leading order temperature dependence necessarily involves the factor $\lambda_{T}^{2} \sim 1 / T$. It is an open challenge to measure bulk viscosities in dilute gases, even in the regime of only moderate values of the phase space density, where concrete theoretical predictions are available.

A possible option to determine the bulk viscosity has been proposed by Fujii and Nishida [289]. They consider a situation where the scattering length $a(x)=a(x, t)$ may be changed spatially or in time. In a regime of local thermodynamic equilibrium, the relation 162 still holds, with $\hat{p}_{\mathrm{eq}} \rightarrow p(x)$ as the local pressure which enters the hydrodynamic equations of motion. A finite bulk viscosity then leads to a deviation

$$
\begin{equation*}
C_{2}(x)-C_{2, \mathrm{eq}}(x)=\left(2 m \Omega_{d-1} / \hbar^{2}\right) a^{d-2}(x) \pi_{i i}^{\mathrm{diss}}(x) \quad \rightarrow \quad\left(2 m \Omega_{d-1} / \hbar^{2}\right) d^{2} a^{d-3}(t) \zeta(t) \cdot \dot{a}(t) \tag{165}
\end{equation*}
$$

of the contact density from its instantaneous, local equilibrium value which is determined by the dissipative contributions to the trace of the stress tensor. In particular, for a slow and spatially homogeneous change of the scattering length, this can be expressed by the product of an instantaneous value $\zeta(t)$ of the bulk viscosity times the rate $\dot{a}(t)$ of change in the scattering length. Neglecting corrections of order $\dot{a}^{3}$, the associated change in the local energy and entropy density are of the form [289]

$$
\begin{equation*}
\dot{\varepsilon}(t)=\frac{\hbar^{2} C_{2, \mathrm{eq}}(t)}{2 m \Omega_{d-1} a^{d-1}(t)} \dot{a}(t)+T(t) \dot{s}(t) \quad \text { with } \quad T(t) \dot{s}(t)=d^{2} \frac{\zeta(t)}{a^{2}(t)} \dot{a}^{2}(t)=d^{2}\left(\partial_{t} a^{-1}\right)^{2} a^{2}(t) \zeta(t) . \tag{166}
\end{equation*}
$$

Note that due to $a^{2} \zeta \rightarrow$ const, a temporal modulation of the inverse scattering length around zero gives rise to a non-vanishing entropy production even in the scale invariant unitary gas. Alternatively, the viscosity and related transport coefficients like the thermal conductivity may be extracted from the damping of sound modes. This has recently been achieved for the unitary Fermi gas in a box type configuration by Patel et al. [290], which allows a precise measurement of the dispersion $\omega_{q}=c_{s} q-i D_{s} q^{2} / 2$. Since $\zeta \equiv 0$ at infinite scattering length, the associated sound diffusion constant $D_{s}=(4 / 3) D_{\eta}+\mathrm{LP} \cdot D_{T}$ only contains the kinematic viscosity $D_{\eta}=\eta /(m n)$ mentioned above and the thermal diffusion constant $D_{T}$, which enters with the Landau-Placzek ratio $\mathrm{LP}=c_{p} / c_{V}-1$ as a prefactor. The results provide support for the concept of quantum limited diffusion constants in scale invariant systems [281, 283]. Specifically, the lower bound $\left.D_{\eta}\right|_{\min } \simeq 0.37 \hbar / m$ for the kinematic viscosity is considerably smaller than the value $\left.D_{T}\right|_{\text {min }} \simeq 4.2 \hbar / m$ obtained for the thermal diffusion constant [291].

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## APPENDIX: SUPERFLUIDITY AND DISSIPATIONLESS CURRENTS

In this Appendix, the existence of persistent currents in neutral superfluids or the Meissner effect in superconductors will be derived in rather general terms within a formulation which is based on the existence of a new thermodynamic variable associated with broken gauge invariance. We start by defining neutral or charged superfluids (= superconductors) by

## A superfluid/superconductor is a system which supports dissipationless mass/electric currents in an equilibrium state.

An example is superfluid ${ }^{4} \mathrm{He}$ in a slowly rotating bucket, where part (or all at $T=0$ ) of the fluid stays at rest in the Lab-frame. The fact that this state with non-classical rotational inertia $I_{\mathrm{SF}}=I_{\mathrm{cl}} \rho_{n}(T) / \rho<I_{\mathrm{cl}}$ is reached independent of whether one starts from a fluid at rest and then slowly rotates the container or if a fully co-rotating fluid in the normal state is cooled to temperatures below $T_{c}$ shows that it is a true equilibrium state (see G.B. Hess and W.M. Fairbank, PRL 19, 216 (1967)). For superconductors, lifetimes of persistent currents in a closed ring have been estimated to be beyond $10^{15}$ years! In order to understand on a thermodynamic level what is the property which allows dissipationless particle currents, we start from an analogy and claim that

## Any system with a finite shear modulus allows dissipationless transport of angular (or transverse) momentum currents.

To show this, consider the torsion of a cylindrical rod of length $L$ and radius $R$ which is twisted by a finite angle $\theta=\int_{z} d \varphi / d z$ due a pair of external forces at $z=L$ which exert an external torque $N_{\text {ext }}=2 R F_{\text {ext }} e_{z}$ (see Figure). The rod is clamped at $z=0$ such that the local rotation angle $\varphi(z)$ with respect to the undistorted rod obeys $\varphi(z=0) \equiv 0$. Note that any system with a finite shear modulus $\mu$ will withstand the torque in a static and therefore equilibrium configuration, it does not need to be a crystalline solid. What is necessary, however, is a broken symmetry with respect to rotations about the $z$-axis: clamping the surface of the rod at $z=0$ is thus sufficient to constrain the angle difference at large lengths in the sense $\lim _{L \rightarrow \infty}\langle\exp [i(\varphi(L)-\varphi(0))]\rangle \neq 0$.


The displacement field for a finite, constant twist $\tau=d \varphi / d z$ along the rod is $\boldsymbol{u}=\tau(-y z, x z, 0)$. It gives rise to a strain tensor

$$
\underline{u}=\frac{\tau}{2}\left(\begin{array}{ccc}
0 & 0 & -y \\
0 & 0 & x \\
-y & x & 0
\end{array}\right)=\underline{u}^{t}
$$

which has $\operatorname{Tr}[\underline{u}]=\operatorname{div} \boldsymbol{u}=0$ and thus describes a pure shear deformation $\underline{u}^{t}$ with no change in volume.

The associated elastic free energy density is $f_{\mathrm{el}}[\underline{u}]=\mu \operatorname{Tr}\left[\left(\underline{u}^{t}\right)^{2}\right]=\mu \tau^{2} r^{2} / 2$. Integration over the volume of the cylinder gives the total elastic free energy

$$
F_{e l}=\int_{0}^{2 \pi} d \varphi \int_{0}^{L} d z \int_{0}^{R} r d r \frac{1}{2} \tau^{2} \mu r^{2}=\frac{1}{2} \int_{0}^{L} d z\left(\frac{d \varphi}{d z}\right)^{2} \underbrace{2 \pi \mu \int_{0}^{R} r^{3} d r}_{=C}=\frac{C}{2} \int_{z}\left(\frac{d \varphi}{d z}\right)^{2}
$$

with a torsional stiffness coefficient $C=(\pi / 2) \mu \cdot R^{4}$ which is linearly proportional to the shear modulus $\mu$. In order to determine the twist induced by the external torque, the work $N_{\text {ext }} d \theta$ done in changing the total twist angle $\theta$ by a small amount must be equal to the associated change $d F_{\text {el }}=C(\theta / L) d \theta$ in the elastic free energy. This leads to $\theta / L=N_{\text {ext }} / C$ or $\tau=N_{\text {ext }} / C$. To prove that the internal elastic forces which appear at a finite twist transmit dissipationless angular momentum currents along the $z$-direction which precisely cancel the effect of the external torque and thus give rise to a static configuration despite the non-vanishing external force, one needs the stress tensor. In the case of pure shear, it reads

$$
\underline{\sigma}=\underline{\sigma}^{t}=2 \mu \underline{u}^{t}=\mu \tau\left(\begin{array}{ccc}
0 & 0 & -y \\
0 & 0 & x \\
-y & x & 0
\end{array}\right) .
$$

On any cross section $z=$ const with normal vector $e_{z}$, the resulting internal force per area is thus $-\underline{\sigma} e_{z}=-\mu \tau r e_{\varphi}$. This is an azimuthal internal force per area which tries to untwist the rod back to its undistorted configuration. It is associated with an
internal angular momentum current density $j_{\text {int }}=r \wedge\left(-\underline{\sigma} e_{z}\right)=-\mu \tau r^{2} \boldsymbol{e}_{z}$ along the negative $z$-direction. The resulting rate of change of total angular momentum precisely cancels the external torque because

$$
\boldsymbol{N}_{\mathrm{int}}=\frac{d \boldsymbol{L}_{\mathrm{int}}}{d t}=\int \boldsymbol{j}_{\mathrm{int}} d f=-\int_{0}^{2 \pi} d \varphi \int_{0}^{R} r d r \mu \tau r^{2} \boldsymbol{e}_{z}=-\tau C \boldsymbol{e}_{z}=-\frac{d F_{\mathrm{el}}}{d \theta} \boldsymbol{e}_{z}=-\boldsymbol{N}_{\mathrm{ext}}
$$

where $d f=r d r d \varphi$ denotes an integration over the area of the cross section.
In the following, we will show that dissipationless transport of particles in superfluids can be understood in a manner which is completely analogous to the situation of a rod with a finite torsion. The twist $\tau=d \varphi / d z$ in the latter case plays the role of a non-vanishing gradient $Q_{z} \rightarrow \nabla \varphi$ of the global $U(1)$ phase which is associated with the conservation of particle number. Moreover, the torsional stiffness per area $C / A=\mu R^{2} / 2 \rightarrow \gamma$ is replaced by a phase rigidity parameter $\gamma$ which defines a nonvanishing superfluid mass (or number) density $\rho_{s}=m n_{s}$ by $\gamma=\hbar^{2} \rho_{s} / m^{2}$. This description is extended in a straightforward manner to superconductors by introducing a rigidity with respect to finite twists of the phase $\varphi_{2}$ of pairs of particles together with the requirement of electromagnetic gauge invariance, which itself is never broken. For a precise thermodynamic definition of the superfluid stiffness $\rho_{s}$ in an interacting many-body system we start from a microscopic Hamiltonian

$$
\begin{equation*}
\hat{H}=\frac{\hbar^{2}}{2 m} \int_{x} \nabla \hat{\psi}^{\dagger}(\boldsymbol{x}) \nabla \hat{\psi}(\boldsymbol{x})+\frac{1}{2} \int_{x} \int_{x^{\prime}} V\left(\boldsymbol{x}-\boldsymbol{x}^{\prime}\right) \hat{\psi}^{\dagger}(\boldsymbol{x}) \hat{\psi}^{\dagger}\left(\boldsymbol{x}^{\prime}\right) \hat{\psi}\left(\boldsymbol{x}^{\prime}\right) \hat{\psi}(\boldsymbol{x}) \tag{167}
\end{equation*}
$$

of interacting, charge neutral Bosons expressed in terms of field operators $\hat{\psi}(\boldsymbol{x})$. Because it commutes with the total particle number $[\hat{H}, \hat{N}]=0$, this Hamiltonian is invariant under a global $U(1)$ gauge transformation

$$
\begin{equation*}
\hat{\psi}(\boldsymbol{x}) \rightarrow e^{-i \varphi \hat{N}} \hat{\psi}(\boldsymbol{x}) e^{i \varphi \hat{N}}=e^{i \varphi} \hat{\psi}(\boldsymbol{x}) . \tag{168}
\end{equation*}
$$

As a result, the free energy $F=-\kappa_{B} T \ln Z$ is independent of the angle $\varphi \in(-\pi, \pi)$. This is no longer true, however, if $\varphi \rightarrow \varphi(x)$ is choosen to depend on position $x$, i.e. the global $U(1)$ is extended to a local gauge transformation. Indeed, while the interaction term in (167) is unchanged even under local gauge transformations, the kinetic energy is apparently not invariant! The exact many-body wavefunctions and corresponding energies will thus change under the local gauge transformation $\hat{\psi}(\boldsymbol{x}) \rightarrow \exp i \varphi(\boldsymbol{x}) \hat{\psi}(\boldsymbol{x})$ but they remain invariant if $\varphi(\boldsymbol{x})$ is spatially constant. The change in free energy which results from a local $U(1)$ gauge transformation can thus only depend on derivatives of the phase, i.e. on the vector $\nabla \varphi(\boldsymbol{x})$, the scalar $\nabla^{2} \varphi(\boldsymbol{x})$ or higher order derivatives. Since $\varphi(\boldsymbol{x})$ and $-\varphi(\boldsymbol{x})$ are equivalent, only the square of these variables can appear. For slowly varying $\varphi(x)$, the leading order term in the change of free energy associated with finite phase twists is thus expected to be of the form

$$
\begin{equation*}
\Delta F[\varphi(\boldsymbol{x})]=\frac{\gamma}{2} \int_{x}(\nabla \varphi(\boldsymbol{x}))^{2}+\ldots=\frac{\rho_{s}}{2} \int_{x} \boldsymbol{v}_{s}^{2}+\ldots \quad \text { with } \quad \gamma=\frac{\hbar^{2} \rho_{s}}{m^{2}} \quad \text { and } \quad \boldsymbol{v}_{s}=\frac{\hbar}{m} \nabla \varphi(\boldsymbol{x}) . \tag{169}
\end{equation*}
$$

The positive rigidity parameter $\gamma$ - which is called the 'helicity modulus' in the statistical physics literature - is an intensive variable. It is non-vanishing only in the superfluid phase and plays a role analogous to the shear modulus of a solid. States with finite rigidity spontaneously break the global gauge invariance (168) and - in three dimensions - exhibit off-diagonal long range order as defined by Eq. 11 due to $\lim _{|\boldsymbol{x}| \rightarrow \infty}\langle\exp \{i \hat{\varphi}(\boldsymbol{x})\} \exp \{-i \hat{\varphi}(0)\}\rangle \neq 0{ }^{33}$. From a thermodynamic point of view, the transition to the superfluid phase is associated with the appearance of the superfluid velocity $\boldsymbol{v}_{s}$ as a new thermodynamic variable and an associated rigidity parameter $\rho_{s}$, similar to the transverse strain tensor $\underline{u}^{t}$ and the shear modulus which distinguish the normal fluid and the solid phase, see Table Formally, the superfluid mass current density $\boldsymbol{j}_{s}^{(m)}$ is the thermodynamic variable conjugate to $\boldsymbol{v}_{s}$. It can thus be obtained from a functional derivative

$$
\boldsymbol{j}_{s}^{(m)}(\boldsymbol{x})=\frac{\delta F\left[\boldsymbol{v}_{s}\right]}{\delta \boldsymbol{v}_{s}(\boldsymbol{x})}=\rho_{s} \boldsymbol{v}_{s}(\boldsymbol{x})+\ldots \quad\left(\operatorname{cf.} \quad \underline{\sigma}^{t}(\boldsymbol{x})=\frac{\delta F_{\mathrm{el}}[\underline{u}]}{\delta \underline{u}^{t}(\boldsymbol{x})}=2 \mu \underline{u}^{t}(\boldsymbol{x})+\ldots\right)
$$

of the equilibrium free energy with respect to the superfluid velocity. States with non-vanishing particle currents $\boldsymbol{j}_{s}^{(m)}$ are therefore non-dissipative, similar to the transverse momentum currents associated with $\sigma^{t} \neq 0$ in a solid with a finite shear deformation. The thermodynamic definition of superfluidity via Eq. (169) does - of course - not answer the question what are necessary or sufficient requirements on the microscopic level to obtain a finite stiffness $\gamma$ with respect to local gauge transformations. A largely trivial observation can be made, however, which states that $\gamma \neq 0$ can never occur in a many-body

[^33]TABLE I: Thermodynamic description of solids and superfluids

|  | solid | superfluid |
| :--- | :---: | :--- |
| new th.dyn. variable | $\underline{u}^{t}$ | $\boldsymbol{v}_{s}$ |
| free energy | $F_{\mathrm{el}}\left[\underline{\underline{t}}^{t}\right]=\int_{x} f_{\mathrm{el}}\left(\underline{u}^{t}\right)$ | $F\left[\boldsymbol{v}_{s}\right]=\int_{x} f\left(\boldsymbol{v}_{s}\right)$ |
| conjugate variable | $\underline{\sigma}^{t}=\partial f_{\mathrm{el}} / \partial \underline{u}^{t}$ | $\boldsymbol{j}_{s}^{(m)}=\partial f / \partial \boldsymbol{v}_{s}$ |
| rigidity | $f_{\mathrm{el}}=\mu \cdot \operatorname{Tr}\left[\left(\underline{u}^{t}\right)^{2}\right]+\ldots$ | $f\left(\boldsymbol{v}_{s}\right)=\rho_{s} \boldsymbol{v}_{s}^{2} / 2+\ldots$ |

system which is described by classical statistical mechanics The argument is a simple variant of the one which leads to the Bohr-van-Leeuwen theorem on the absence of magnetism in classical statistical physics, where the kinetic and interaction energy parts of the total Hamiltonian commute. The partition function and free energy are therefore fully determined by the configuration integral, which only involves $H_{\text {int }}$. Since the latter is invariant under arbitrary local gauge transformations, the classical free energy cannot depend on $\varphi(\boldsymbol{x})$ at all, implying $\gamma_{\mathrm{cl}} \equiv 0$. A non-trivial issue is whether the mass $m$ which connects a finite phase twist $\nabla \varphi$ with the superfluid velocity might be a renormalized one due to interactions. Now, as long as Galilei invariance is not broken e.g. by an external random potential, $m$ must indeed be the bare mass of the Bosons since $\boldsymbol{v}_{s}$ must transform like a velocity, see M. Liu, Physical Review Letters 81, 3223 (1998).

The extension to charged superconductors in which pairs of Fermions condense is straightforward. In close analogy to (169), a spatially dependent two-particle phase $\varphi_{2}(\boldsymbol{x})$ associated with the product of two field operators is now assumed to lead to a change of the free energy in the form (in cgs units)

$$
\begin{equation*}
\Delta F\left[\varphi_{2}(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{x})\right]=\frac{\gamma}{2} \int_{x}\left(\nabla \varphi_{2}(\boldsymbol{x})+\frac{2 e}{\hbar c} \boldsymbol{A}(\boldsymbol{x})\right)^{2}+\frac{1}{8 \pi} \int_{x}(\operatorname{rot} \boldsymbol{A}(\boldsymbol{x}))^{2}=\frac{\rho_{s}}{2} \int_{x} \boldsymbol{v}_{s}^{2}+\frac{1}{8 \pi} \int_{x}(\operatorname{rot} \boldsymbol{A}(\boldsymbol{x}))^{2} \tag{170}
\end{equation*}
$$

This is invariant under static electromagnetic gauge transformations

$$
\begin{equation*}
\boldsymbol{A}(\boldsymbol{x}) \rightarrow \boldsymbol{A}^{\prime}(\boldsymbol{x})=\boldsymbol{A}(\boldsymbol{x})+\nabla \chi(x) \tag{171}
\end{equation*}
$$

which are accompanied by a phase change $\varphi(\boldsymbol{x}) \rightarrow \varphi^{\prime}(\boldsymbol{x})=\varphi(\boldsymbol{x})-e \chi(\boldsymbol{x}) /(\hbar c)$ of the field operators $\hat{\psi}_{\sigma}(\boldsymbol{x})$ for particles with charge $-e$. A change of the vector potential by $\nabla \chi(\boldsymbol{x})$ therefore changes the two-particle phase by $\varphi_{2}(\boldsymbol{x}) \rightarrow \varphi_{2}(\boldsymbol{x})-$ $2 e \chi(\boldsymbol{x}) /(\hbar c)$, leaving invariant the combination $\nabla \varphi_{2}(\boldsymbol{x})+2 e \boldsymbol{A}(\boldsymbol{x}) /(\hbar c)^{34}$. The gauge invariant superfluid velocity $\boldsymbol{v}_{s}$, as the new thermodynamic variable in superconductors, and the associated stiffness or superfluid density for pairs are now defined via

$$
\begin{equation*}
\boldsymbol{v}_{s}=\frac{\hbar}{2 m}\left(\nabla \varphi_{2}(\boldsymbol{x})+\frac{2 e}{\hbar c} \boldsymbol{A}(\boldsymbol{x})\right) \text { and } \rho_{s}=\gamma \cdot \frac{(2 m)^{2}}{\hbar^{2}} \tag{172}
\end{equation*}
$$

in complete analogy with (169). An immediate consequence of the free energy $\sqrt{170}$ ) is that the lowest energy, untwisted state of a superconductor has $\boldsymbol{v}_{s} \equiv 0$ in its bulk. The vector potential $\boldsymbol{A}(\boldsymbol{x})=-\hbar c \nabla \varphi_{2}(\boldsymbol{x}) /(2 e)$ is thus a pure gauge which immediately implies the Meissner effect $\boldsymbol{B} \equiv 0$. More precisely, using $\boldsymbol{B} \cdot \operatorname{rot} \delta \boldsymbol{A}=\operatorname{div}(\delta \boldsymbol{A} \wedge \boldsymbol{B})+\delta \boldsymbol{A} \cdot \operatorname{rot} \boldsymbol{B}$, the condition that the first order change in free energy

$$
\left.\delta F=\int_{x}\left[\frac{\hbar}{2 m} \frac{2 e}{\hbar c} \boldsymbol{j}_{s}^{(m)} \cdot \delta \boldsymbol{A}(\boldsymbol{x})+\frac{1}{4 \pi} \boldsymbol{B} \cdot \operatorname{rot} \delta \boldsymbol{A}(\boldsymbol{x})\right]+\boldsymbol{O}(\delta \boldsymbol{A})^{2}\right)
$$

with respect to variations $\delta \boldsymbol{A}(\boldsymbol{x})$ of the vector potential vanishes, gives rise to Ampère's law

$$
\operatorname{rot} \boldsymbol{B}=\frac{4 \pi}{c} \boldsymbol{j}_{s}(\boldsymbol{x}) \text { with } \boldsymbol{j}_{s}(\boldsymbol{x})=-\frac{2 e}{2 m} \boldsymbol{j}_{s}^{(m)}(\boldsymbol{x})=-e n_{s} \frac{\hbar}{2 m}\left(\nabla \varphi_{2}+\frac{2 e}{\hbar c} \boldsymbol{A}(\boldsymbol{x})\right) .
$$

Here, following conventional notation, the superfluid number density has been defined as $n_{s}=\rho_{s} / m$. Thus, $n_{s}$ is the density of single electrons which appear in the superfluid response, twice that of the 'pair-density' $\rho_{s} / 2 m$. The expression for the electric current density immediately implies the second London equation

$$
\begin{equation*}
\operatorname{rot} \boldsymbol{j}_{s}(\boldsymbol{x})=-\frac{n_{s} e^{2}}{m c} \boldsymbol{B}(\boldsymbol{x}) \tag{2}
\end{equation*}
$$

[^34]which, together with the condition of stationarity $\operatorname{div} \boldsymbol{j}_{s}(\boldsymbol{x}) \equiv 0$, uniquely fixes the superfluid current density for any given $\boldsymbol{B}(\boldsymbol{x})$. Combined with Ampère's law and rot rot $\boldsymbol{B}=-\nabla^{2} \boldsymbol{B}$, the magnetic field inside a superconductor obeys $\nabla^{2} \boldsymbol{B}=\boldsymbol{B} / \lambda_{L}^{2}$. A magnetic field parallel to the surface of a superconductor in the half space $x>0$ thus decays exponentially $\sim \exp -x / \lambda_{L}$ on the scale of the London penetration depth $\lambda_{L}$ which is defined by
\[

$$
\begin{equation*}
\frac{1}{\lambda_{L}^{2}}=\frac{4 \pi n_{s} e^{2}}{m c^{2}}=\left(\frac{\omega_{p, s}}{c}\right)^{2} \quad \text { or } \quad \frac{1}{\lambda_{L}^{2}}=16 \pi \frac{e^{2}}{\hbar c} \cdot \frac{\gamma}{\hbar c} \simeq 0.37 \frac{\gamma}{\hbar c} . \tag{174}
\end{equation*}
$$

\]

For clean systems, where $n_{s}$ is essentially the full conduction electron density, the London penetration depth is just the ratio of the velocity of light and the standard plasma frequency. It is thus a factor $c / v_{F} \simeq 10^{2}$ larger than the Thomas Fermi screening length. An important point, which should be emphasized, is evident from the second form of Eq. (174): apart from universal constants of nature, the London penetration depth is a direct measure of the phase rigidity parameter $\gamma$, unaffected by the problem of how to define an effective number density or mass in realistic superconductors: as pointed out by de Gennes [292], it is only the ratio $n_{s} / m=4 \gamma / \hbar^{2}$ which is physically meaningful in this context (note that due to $\boldsymbol{v}_{s}=\hbar\left[\nabla \varphi_{2}+2 e /(\hbar c) \boldsymbol{A}\right] /(2 m)$ there is a factor four compared to the relation between $n_{s}$ and the stiffness $\gamma$ in Eq. (23) for Bose superfluids with mass $m$ instead of $2 m$ ).

The second London equation only covers static properties like the Meissner effect. It does not, however, adress the issue of perfect conductivity. At first sight, an equation which determines the time derivative of the superfluid current density seems to require information beyond equilibrium free energies. Fortunately, this is not the case. Indeed, the crucial point is to realize that the two-particle phase $\hbar \varphi_{2}$, which becomes a relevant thermodynamic variable in the superconducting state, is canonically conjugate to the density of pairs. Its covariant time derivative ${ }^{35}$

$$
\hbar \partial_{t} \varphi_{2}-2 e \Phi=-\frac{\partial f}{\left(\rho_{s} / 2 m\right)}=-\mu_{\mathrm{pair}}
$$

defines a chemical potential of pairs of electrons. Using the free energy density 170 , one finds $\mu_{\text {pair }}=m \boldsymbol{v}_{s}^{2}$, which vanishes in the bulk of a superconductor in equilibrium. The scalar potential $\Phi_{\text {eq }}=\hbar \partial_{t} \varphi_{2} /(2 e)$ is then a pure gauge, implying $\boldsymbol{E}=$ $-\nabla \Phi-\partial_{t} \boldsymbol{A} / c=-2 m \partial_{t} \boldsymbol{v}_{s} /(2 e)=0$, as expected in a perfect conductor. More generally, in a time dependent, inhomogeneous situation, the time derivative

$$
\begin{equation*}
\partial_{t} \boldsymbol{j}_{s}(\boldsymbol{x}, t)=\frac{e n_{s}}{2 m} \nabla \mu_{\mathrm{pair}, \mathrm{el}}(\boldsymbol{x}, t) \rightarrow \frac{e^{2} n_{s}}{m} \boldsymbol{E}(\boldsymbol{x}, t) \tag{1}
\end{equation*}
$$

of the gauge invariant superfluid current density $\boldsymbol{j}_{s}=-e n_{s}\left(\nabla\left(\hbar \varphi_{2}\right)+2 e \boldsymbol{A} / c\right) /(2 m)$ can be expressed in terms of the gradient $\nabla \mu_{\text {pair,el }}=\nabla \mu_{\text {pair }}+2 e \boldsymbol{E}$ of the electrochemical potential of pairs, which reduces to $2 e \boldsymbol{E}$ if $\mu_{\text {pair }}$ is spatially constant. Eq. (175) is the first London equation and expresses the property of perfect conductivity. Both sides of the equation are even under time reversal, as is necessary for non-dissipative transport. This should be contrasted with the associated result $\boldsymbol{j}(\boldsymbol{x})=\underline{\sigma} \nabla \mu_{\mathrm{el}}(\boldsymbol{x}) / e$ for dc-currents in a normal metal, which relates the driving force $\nabla \mu_{\mathrm{el}}(\boldsymbol{x})$ with the current density itself and not its time derivative. The arguments which lead to the first London equation can also be used to derive the Josephson relation. Indeed, defining a gauge invariant phase difference $\Delta \varphi_{12}=\int_{1}^{2}\left(\nabla \varphi_{2}+2 e \boldsymbol{A} /(\hbar c)\right) \cdot d s$ between two points inside a superconductor or also across a junction between two superconductors, Eq. (175) shows that its time derivative

$$
\begin{equation*}
\hbar \frac{d}{d t} \Delta \varphi_{12}=\int_{1}^{2}(\underbrace{\nabla\left(\hbar \partial_{t} \varphi_{2}\right)+\frac{2 e}{c} \partial_{t} \boldsymbol{A}}_{-\nabla \mu_{\text {pair,cl }}}) \cdot d s=2 e V_{12} \quad \text { with } \quad 2 e V_{12}=-\int_{1}^{2} \nabla \mu_{\text {pair,el }} \cdot d s \tag{176}
\end{equation*}
$$

is determined by the gauge invariant voltage difference $V_{12}$ between these two points. This equation is a consequence of electromagnetic gauge invariance and is thus exact, allowing to define the voltage standard via the ac-Josephson effect. Note that the exactness of the Josephson relation is compleletly unaffected by the non-equilibrium dynamics of the superfluid density itself, which obeys a dissipative time-dependent Ginzburg-Landau equation. Eq. [176) also connects finite voltage differences within superconductors or chemical potential differences $\Delta \mu_{12}=h d N_{\mathrm{V}} / d t$ in neutral superfluids with the rate at which vortices cross the line connecting the points 1 and 2, see P.W. Anderson in Reviews of Modern Physics 38, 298 (1966).

[^35]
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[^1]:    ${ }^{1}$ The theorem also implies that the two-electron ground state of a spin-independent Hamiltonian is always a singlet, see problem 2 in Ref. [11] p. 689 .

[^2]:    ${ }^{2}$ Note that both $\Delta_{2} \tilde{W}_{1 / 2}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)$ and $\Delta_{1} \tilde{W}\left(\boldsymbol{x}_{1}\right)$ in Eq. $\sqrt{9}$ depend implicitely also on the coordinates $\boldsymbol{x}_{2}, \ldots \boldsymbol{x}_{N}$ but this dependence is suppressed.

[^3]:    ${ }^{3}$ See chapter 3.9 in the book by Hansen and McDonald [19]. Note that the expression 11. is applicable only in the fluid phase for $\eta<0.49$ beyond which the equilibrium state of the classical hard sphere system is a crystal with an fcc-lattice structure, reaching close packing at $\eta_{\mathrm{cp}}=\pi \sqrt{2} / 6 \simeq 0.74$.

[^4]:    ${ }^{4}$ For a strictly number conserving formulation of Bogoliubov theory see Girardeau [29] and the review by Leggett [30]. Note also that in a quantum optics context, Eq. 16] describes a two-mode squeezed state, see e.g. Walls and Milburn [31]. This analogy is discussed by Haque and Ruckenstein [32].

[^5]:    5 The periodicity in the variable $\theta$ is important e.g. for understanding the exactness of flux-quantization in superconducting rings with a thickness much larger than the London penetration depth. Using $\theta=2 \pi \phi / \phi_{0}$, this relies on the fact that the large energy associated with the Fourier coefficient $F_{l=2} \simeq \gamma A_{\perp} / L$ forces $\cos \left(2 \cdot 2 \pi \phi / \phi_{0}\right)=1$ with negligible fluctuations. The magnetic flux $\phi$ is thus pinned at an integer number times the flux quantum $\phi_{0} / 2$ in superconductivity.

[^6]:    ${ }^{6}$ This conclusion no longer holds in the presence of a magnetic field, as shown for example by the incompressible Quantum Hall state of a half filled Landau level in two dimensions described by the Laughlin wave function $\Psi_{\mathrm{L}}\left(z_{1}, \ldots z_{N}\right)=\prod_{i<j}\left(z_{i}-z_{j}\right)^{2} \cdot \exp -\sum_{i}\left|z_{i}\right|^{2} / 4$, which has a uniform density $n(z)=1 / 4 \pi$.

[^7]:    ${ }^{7}$ In the case of Fermions $\left.\Lambda_{d \mathrm{~B}}^{\mathrm{c}, \text { solid }}\right|_{F} \simeq 0.42$ is substantially larger because Fermions prefer to stay localized near a discrete set of lattice sites even for larger values of the zero point motion. The ground state of ${ }^{3} \mathrm{He}$ at zero pressure is a liquid since its de Boer parameter $\Lambda_{\mathrm{dB}} \simeq 0.45$ lies above this critical value.

[^8]:    ${ }^{8}$ For an introduction to the formalism see e.g. the book by Zee [69]. Due to Galilei invariance, derivatives only appear in the covariant form $\tilde{D}=\hbar \partial_{\tau}-\hbar^{2} \nabla^{2} / 2 m$.

[^9]:    ${ }^{9}$ For a discussion of the thermodynamics and life time of stars if a di-proton bound state would exist, see L. A. Barnes, arXiv:1512.06090 [astro-ph.SR].

[^10]:    ${ }^{10}$ We use a bar in the surface tension $\bar{\sigma}$ to distinguish it from the short distance length scale $\sigma$. Note also that the exponent $v_{u}=v_{t} / \phi_{t}=1$ for the divergence of the correlation length $1 / \kappa_{0}$ along the first-order transition line $\mu=\mu_{c}$ is a subsidiary tricritical exponent in the notation of Griffiths [80]. The relevant crossover exponent $\phi_{t}=1 / 2$ is determined by the quadratic behavior 32 of the chemical potential near the quantum tricritical point.

[^11]:    ${ }^{11}$ The corresponding phase diagram for the unitary Fermi gas has been determined by Frank et al. 95 . The critical temperature $T_{c} / \Delta_{0} \simeq 0.34$ at $h=0$ and the fields $h_{c} / \Delta_{0} \simeq 0.9$ where a finite imbalance sets in, or $h_{\mathrm{FFLO}} / \Delta_{0} \simeq 1.05$ where the FFLO-phase disappears at $T=0$, have different values. In particular, their magnitude is determined by a much larger gap $\Delta_{0} \simeq 0.46 T_{F}$. More importantly, however, the Lifshitz point no longer coincides with the tricritical point below which the transition to the homogeneous superfluid is first order. It is shifted to much lower temperatures, beyond the regime that has been explored in experiments so far, which have determined the phase diagram of Fermi superfluids with finite imbalance down to and slightly below the tricritical point 98 .

[^12]:    ${ }^{12}$ For a detailed discussion of the counting of Goldstone modes in a non-relativistic context see the review by Watanabe 102 .

[^13]:    ${ }^{13}$ Note that this identification does not work for weakly interacting BEC's in a harmonic trap, where the fact that $\mu \gg \hbar \omega$ gives rise to an inverted parabolic rather than a Gaussian density profile.

[^14]:    14 The situation is much less clear in high-temperature superconductors, where a pair-density wave seems to coexist with density order already in the anomalous normal state, see e.g. Ref. [106].

[^15]:    ${ }^{15}$ This correspondence becomes exact for the Tonks-Girardeau gas in one dimension, see e.g. Ref. [49].

[^16]:    ${ }^{16}$ A common approach uses a Gross-Pitaevskii description with additional mean-field contributions due to the dipolar interaction and the fluctuation corrections from an LHY-calculation of the 3 D homogeneous system. In the regime of interest, where $\varepsilon_{\mathrm{dd}}>1$, these corrections develop a non-vanishing imaginary part, which indicates that the configurations around which the expansion is performed are unstable, see e.g. the work by Weinberg and Wu [133] in QFT context.

[^17]:    ${ }^{17}$ The value $\varepsilon_{\mathrm{dd}}^{(c)}$ is of course not universal but it is expected to be near the value $\varepsilon_{\mathrm{dd}}^{(c)} \simeq 1.37$ observed in similar experiments discussed in Fig. 12 below.

[^18]:    ${ }^{18}$ The formal equivalence of the order in classic smectic-A liquid crystals and in superfluids was first realized by de Gennes [141]. Note that the associated angles $\varphi_{\mathrm{sm}}$ and $\varphi_{\mathrm{sf}}$ transform in an opposite manner under time reversal: $\varphi_{\mathrm{sm}}$ is a true scalar while $\varphi_{\mathrm{sf}}$ is a pseudoscalar.

[^19]:    ${ }^{19}$ For a detailed discussion of the crossover between first and second sound in the context of weakly interacting BEC's see section 6.6 of Ref. [143].

[^20]:    ${ }^{20}$ For a discussion of the exactly soluble Lieb-Liniger gas in the context of ultracold atoms, see chapter V in the review by Bloch et al. [49].

[^21]:    ${ }^{21}$ The cusp arises because the second virial coefficient $b_{2}$ in Eq. 88 is restricted to the contribution of continuum states only, i.e. upon crossing $1 / a=0$ from negative to positive values, one switches from the attractive to the repulsive branch of the Feshbach resonance, excluding the weakly bound state at $a>0$.

[^22]:    22 The replacement of this by the form $\psi_{0}(r)=1 / r-1 / a$ used above allows to apply the pseudopotential description even at infinite scattering length. Note that in the OPE of Eq. [81), it is always possible to switch a multiplicative factor between the operators $\hat{O}_{\ell}(\boldsymbol{R})$ and the associated Wilson coefficients $W_{\ell}^{(a, b)}(\boldsymbol{x})$.

[^23]:    ${ }^{23}$ Similarly, in high-energy physics, interactions between the partons lead to corrections of the form $\delta(1-x / \xi) \rightarrow \delta(1-x / \xi)-\alpha_{s} P(x / \xi) / 2 \pi+\ldots$ in Eq. 116 to linear order in the coupling constant $\alpha_{s}$. The associated function $P(z)$ then leads to logarithmic violations of Bjorken scaling, see Ref. [207].

[^24]:    ${ }^{24}$ Note that in curved spacetime, the covariant curl $D_{\mu} A_{v}-D_{v} A_{\mu}=\partial_{\mu} A_{\nu}-\partial_{v} A_{\mu}$ is identical to the ordinary one and therefore the electromagnetic field tensor $F_{\mu \nu}$ with two lower indices does not depend on the metric [218]. Moreover, conformal invariance evidently only holds in 3+1-dimensions.

[^25]:    ${ }^{25}$ In the context of ultracold gases, such flow patterns arise in the hydrodynamic expansion of a unitary Fermi gas from an isotropic trap. For a discussion of such scaling flows see the chapters by Werner and Castin and by Schäfer and Chafin in Ref. [222].

[^26]:    ${ }^{26}$ The exact solution of this model has been found independently by É. Brézin and J. Zinn-Justin but has been written up only as an internal report at Saclay.

[^27]:    ${ }^{27}$ For Fermions at unitarity, $p^{(0)}$ and $\kappa^{(0)}$ are the pressure and compressibility of the non-interacting gas. For Bose gases in 2D, a convenient choice is provided by the characteristic scale $p^{(0)}=1 / 2 \kappa^{(0)}=\hbar^{2} n_{2}^{2} / m$ of the zero point pressure at density $n_{2}$. This leads to $X_{-1}=\tilde{p}$ and $X_{1}=\tilde{\kappa} / 2$ in the notation of Ref. [240].

[^28]:    ${ }^{28}$ In one dimension, only two representations with $k_{0}=1 / 4$ and $k=3 / 4$ are necessary, which account for the even and the odd-parity levels, see Ref.

[^29]:    ${ }^{29}$ The definition of the scattering amplitude $f(k, \theta)$ follows the convention in Refs. [49 265]. It differs by a factor $-1 / \sqrt{8 \pi}$ from the one used by Adhikari [266]. Note that $f(k)$ vanishes logarithmically as $k \rightarrow 0$ but the total scattering cross section $\sigma_{\text {tot }}=-\operatorname{Im} f(k, \theta=0) / k \rightarrow\left|f_{0}(k)\right|^{2} / 4 k$ diverges in the low energy limit.

[^30]:    ${ }^{30}$ The case of two-component Fermi gases differs by a factor two, giving rise to a prefactor $\hbar^{2} /(4 \pi m)$ in Eq. 153], see [269]. Note that in contrast to the situation in 1D, the local energy density operator $\hat{\varepsilon}(\boldsymbol{R})$ cannot be split into a kinetic and an interaction part since $\bar{g}_{d}(\Lambda)$ depends on the cutoff $\Lambda$ for dimensions $d \geq 2$.

[^31]:    ${ }^{31}$ In a superfluid, there are actually three different bulk viscosities $\zeta_{1,2,3}$ [279] 280]. The combination of scale and conformal invariance requires two of them to vanish. In particular, $\zeta_{2} \equiv 0$, which is the one which takes the role of the standard bulk viscosity in the normal fluid.

[^32]:    ${ }^{32}$ Note that our notation for the Mori-product $(\hat{A}(t) \mid \hat{A})=\int_{0}^{\beta} d \alpha\langle\delta \hat{A}(t) \delta \hat{A}(t=i \hbar \alpha)\rangle_{\text {eq }}$ with $\delta \hat{A}=\hat{A}-\langle\delta \hat{A}\rangle$ differs from the one in Refs. [280 287] by a factor $\beta$.

[^33]:    ${ }^{33}$ For a discussion of the subtle connection between the rigidity parameter $\gamma$ which defines the superfluid density $n_{s}$ and the condensate density $n_{0}$, which appears in the definition of ODLRO in Eq. (1), see the Appendix in the review by Bloch et al. [49].

[^34]:    ${ }^{34}$ In a paper in Progress of Theoretical Physics Supplement No. 86 p. 43 (1986), S. Weinberg shows that the existence of a finite stiffness with respect to gradients of the two-particle phase arises as a consequence of the fact that in superconductors the global $U(1)$ gauge symmetry associated with number conservation is broken down to a remaining discrete $Z_{2}$ group, i.e. the multiplication of field operators with a global factor $\pm 1$.

[^35]:    ${ }^{35}$ Both the left and the right hand side of this equation are invariant under time dependent gauge transformations $\boldsymbol{A}^{\prime}=\boldsymbol{A}+\nabla_{\chi}, \Phi^{\prime}=\Phi-\partial_{t} \chi / c$. Indeed, $\varphi_{2}^{\prime}=\varphi_{2}-2 e \chi /(\hbar c)$ implies that $\hbar \partial_{t} \varphi_{2}-2 e \Phi=\hbar \partial_{t} \varphi_{2}^{\prime}-2 e \Phi^{\prime}$.

