Covariant quantization of a relativistic string

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Plan of the talk

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2. The bosonic string in the conformal gauge
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We have seen that a free string is described by the Nambu-Goto action:

\[ S_{NG}(x^\mu) = -T \int d\tau \int d\sigma \sqrt{-\det(\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu})} \]

where \( \xi^\alpha \equiv (\tau, \sigma) \) and \( \partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha} \).

This Lagrangian is very non-linear and not easy to treat if we want to quantize the string using the path integral formalism.

On the other hand, there exists an alternative to the Nambu-Goto action that was constructed by [Brink, Deser, DV, Howe and Zumino] in 1976.

It was then used by Polyakov in 1982 for quantizing the string with the path integral formalism.

For this reason it is called Polyakov action.
It is given by:

\[ S(x^\mu, g_{\alpha\beta}) = -\frac{T}{2} \int d\tau \int_0^\pi d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \]

- \( x^\mu(\sigma, \tau) \) is the coordinate of the string (\([\mu, \nu = 0, 2, \ldots d - 1]\)).
- \( T = \frac{1}{2\pi\alpha'} \) is a dimensional (Energy per unit length) parameter called the string tension.
- \( g^{\alpha\beta}(\sigma, \tau) \) is the two-dimensional world-sheet metric tensor with \( g = \det(g_{\alpha\beta}) \).
- \( \eta^{\mu\nu} = (-1, 1\ldots1, 1) \) is the d-dimensional target space metric.
- Viewed as a two dimensional field theory, it describes the interaction of a set of \( d \) massless fields with an external gravitational field \( g_{\alpha\beta} \).
- From this point of view the d-dimensional Lorentz index plays the role of a flavour index.
It can be easily shown that the two actions are equivalent.

We can immediately write the algebraic equation of motion for the world-sheet metric:

$$\theta_{\alpha\beta} \equiv \partial_{\alpha} x \cdot \partial_{\beta} x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_{\gamma} x \cdot \partial_{\delta} x = 0$$

where we have used

$$\frac{\delta \sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} g_{\alpha\beta} \sqrt{-g}$$

From it we get:

$$\det (\partial_{\alpha} x \cdot \partial_{\beta} x) = \frac{g}{4} \left( g^{\alpha\beta} \partial_{\alpha} x \cdot \partial_{\beta} x \right)^2$$

Inserting it in the Polyakov action one gets the Nambu-Goto action

$$\Rightarrow \text{the two actions are equivalent} !!$$
The bosonic string in the conformal gauge

- Let us start from the Polyakov action:

$$S(x^\mu, g_{\alpha\beta}) = -\frac{T}{2} \int d\tau \int_0^\pi d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

- It is invariant under an arbitrary reparametrization ($\xi \rightarrow \xi'(\xi)$) of the world-sheet coordinates $\xi^\alpha \equiv (\tau, \sigma)$:

$$x^\mu(\xi) = x'^\mu(\xi'), \quad g_{\alpha\beta}(\xi) = \frac{\partial \xi'^\gamma}{\partial \xi^\alpha} \frac{\partial \xi'^\delta}{\partial \xi^\beta} g'_{\gamma\delta}(\xi')$$

- The second equation implies:

$$d^2 \xi \sqrt{-g} = d^2 \xi' \sqrt{-g'}$$

- For infinitesimal transformations $\xi' = \xi - \epsilon$ we get:

$$\delta x^\mu = \epsilon^\alpha \partial_\alpha x^\mu; \quad \delta g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma}$$
It is also invariant under Weyl rescaling of the metric:
\[
g_{\alpha\beta}(\xi) \rightarrow \Lambda^2(\xi) g_{\alpha\beta}(\xi) \ ; \ x^\mu(\xi) \rightarrow x^\mu(\xi)
\]

From the string action we can derive the Euler-Lagrange equations of motion:
\[
-\frac{2}{T\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} \equiv \theta_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0
\]
for the two-dimensional world-sheet metric.

This equation implies that the two-dimensional world-sheet energy-momentum tensor is identically vanishing.

The eq. of motion for the string coordinate is instead
\[
\partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta x^\mu \right) = 0
\]
It is still non-linear.
In order to solve the previous equations and find the most general classical motion of a string it is convenient to choose a gauge where the previous equation of motion linearizes.

A convenient Lorentz covariant gauge is the conformal gauge where the world-sheet metric tensor is taken to be of the form:

$$g_{\alpha\beta} = \rho(\xi)\eta_{\alpha\beta} \quad \eta_{11} = -\eta_{00} = 1$$

This gauge choice does not fix completely the gauge.

We can still perform conformal transformations that leave the metric in the same form, but with a rotated $\rho$.

They are characterized by the following equation:

$$\partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha - \eta^{\alpha\beta} \partial^\gamma \epsilon_\gamma = 0$$

Under the previous infinitesimal transformation we get:

$$g_{\alpha\beta} + \delta g_{\alpha\beta} = (\rho + \partial_\gamma (\epsilon^\gamma \rho)) \eta_{\alpha\beta}$$
The conditions of the conformal gauge are more transparent if we introduce light-cone coordinates:

\[ \xi^\pm = \xi^0 \pm \xi^1, \quad \epsilon^\pm = \epsilon^0 \pm \epsilon^1, \quad \frac{\partial}{\partial \xi^\pm} = \frac{1}{2} \left( \frac{\partial}{\partial \xi^0} \pm \frac{\partial}{\partial \xi^1} \right) \]

In terms of those variables, they reduce to

\[ \frac{\partial}{\partial \xi^-} \epsilon^+ = \frac{\partial}{\partial \xi^+} \epsilon^- = 0 \implies \epsilon^+(\xi^+) ; \quad \epsilon^-(\xi^-) \]

In the conformal gauge the equation of motion becomes:

\[ \left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) x^\mu(\sigma, \tau) = 0 \]

Boundary conditions for open string

\[ \frac{\partial}{\partial \sigma} x^\mu(\tau, \sigma) \big|_{\sigma=0,\pi} = 0 \]

Boundary condition for closed string:

\[ x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + \pi) \]
The most general solution for open string:

\[ x^\mu(\tau, \sigma) = q^\mu + 2\alpha' p^\mu \tau + i \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma \]

and for closed string

\[ x^\mu(\tau, \sigma) = q^\mu + 2\alpha' p^\mu \tau + i \frac{\sqrt{2\alpha'}}{2} \sum_{n \neq 0} \left[ \frac{\alpha_n^\mu}{n} e^{-2in(\tau+\sigma)} + \tilde{\alpha}_n^\mu n e^{-2in(\tau-\sigma)} \right] \]

Here \( \alpha_n \) and \( \tilde{\alpha}_n \) are just constant parameters.

We must impose the vanishing of the two independent components of the world-sheet energy-momentum tensor:

\[ \theta^{00} \pm \theta^{01} \sim \frac{1}{2} (\dot{x} \pm x')^2 = 0 \]

\[ \dot{x} \equiv \partial_\tau x \quad ; \quad x' \equiv \partial_\sigma x \]
The operators $\alpha_n$ and $\tilde{\alpha}_n$ are related to the harmonic oscillators and the center of mass variables by:

$$\alpha_n^\mu = \begin{cases} 
\sqrt{n}a_n^\mu & \text{if } n > 0 \\
2\alpha'\hat{p}^\mu & \text{if } n = 0 \\
\sqrt{|n|}a_n^+\mu & \text{if } n < 0 
\end{cases}$$

for the open string, and by

$$\alpha_n^\mu = \begin{cases} 
\sqrt{n}a_n^\mu & \text{if } n > 0 \\
\frac{\sqrt{2\alpha'}}{2}\hat{p}^\mu & \text{if } n = 0 \\
\sqrt{|n|}a_n^+\mu & \text{if } n < 0 
\end{cases}$$

; \quad \tilde{\alpha}_n^\mu = \begin{cases} 
\sqrt{n}\tilde{a}_n^\mu & \text{if } n > 0 \\
\frac{\sqrt{2\alpha'}}{2}\hat{p}^\mu & \text{if } n = 0 \\
\sqrt{|n|}\tilde{a}_n^+\mu & \text{if } n < 0 
\end{cases}$$

for the closed string.
In the case of the open string they give the same condition, namely:

\[ L_n = \frac{1}{4\pi \alpha'} \int_0^{\pi} d\sigma \, e^{in(\tau \pm \sigma)} (\dot{x} \pm x')^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m = 0 \]

where \( \alpha_0 \equiv \sqrt{2\alpha'} p \)

In the case of a closed string we get instead:

\[ L_n = \frac{1}{4\pi \alpha'} \int_0^{\pi} d\sigma \, e^{in(\tau \pm \sigma)} (\frac{\dot{x} + x'}{2})^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m \cdot \alpha_{n-m} = 0 \]

\[ \tilde{L}_n = \frac{1}{4\pi \alpha'} \int_0^{\pi} d\sigma \, e^{in(\tau - \sigma)} (\frac{\dot{x} - x'}{2})^2 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0 \]

\[ \alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{2\alpha'} \frac{p^\mu}{2} \]
The theory is quantized by imposing the following commutation relations:

\[ [\alpha_n^\mu, \alpha_m^\nu] = n \eta^{\mu\nu} \delta_{n+m,0} \quad ; \quad [\hat{q}^\mu, \hat{p}^\nu] = i \eta^{\mu\nu} \]

for an open string.

In the case of a closed string, one must also impose the commutation relations for the other infinite set of oscillators:

\[ [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{n+m,0} \quad \text{that commute with the oscillators of the previous set.} \]

In the quantum theory, the operators \( L_n \) are defined with the normal ordering:

\[ L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \cdot \alpha_m : \]

that, however, regards only \( L_0 = \alpha' \hat{p}^2 + \sum_{n=1}^{\infty} na_n^\dagger \cdot a_n \).
In the quantum theory, the vanishing of $L_n$ for all $n$ is too restrictive.

One can only impose their vanishing between physical states.

In other words one can define a physical subspace where:

$$\langle \text{Phys.}, P | (L_n - \alpha_0 \delta_{n0}) | \text{Phys.'}, P \rangle = 0 \ ; \ -\infty < n < +\infty$$

$\alpha_0$ is a constant to be determined.

They are satisfied if

$$(L_0 - \alpha_0) | \text{Phys.}, P \rangle = L_n | \text{Phys.}, P \rangle = 0 \ ; \ n = 1, 2, \ldots$$

Those conditions are exactly those obtained from the analysis of the residues of the poles in the $N$-point dual amplitude.

except that there and in the light-cone gauge $\alpha_0 = 1$, while here there is no obvious way to compute it.
In the present covariant way of quantizing the string, we cannot reproduce two properties of the string that we have obtained in the light-cone gauge, namely

- the fact that the intercept of the Regge trajectory $\alpha_0 = 1$
- and the critical dimension $d = 26$ that in the light-cone was essential to have a Lorentz invariant theory.

On the other hand, one expects that, quantizing the theory in two different gauges, one would get the same result.

Here conformal invariance is a gauge symmetry because it comes from the invariance under reparametrizations.

Therefore, we expect the energy momentum tensor to transform as a two-index tensor without an anomaly term:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{d}{12}n(n^2 - 1)\delta_{n+m;0}$$

corresponding to the $c$-number of the Virasoro algebra.

What is wrong in our present treatment of the conformal gauge?
Before this, let us consider shortly the case of the closed string.
In this case we have two sets of Virasoro operators $L_n$ and $\tilde{L}_n$.
The equations that characterize the on-shell physical states are:

$$(L_0 - 1)|\text{Phys.}\rangle = (\tilde{L}_0 - 1)|\text{Phys.}\rangle = 0$$

$L_n|\text{Phys.}\rangle = \tilde{L}_n|\text{Phys.}\rangle = 0$ ; $n = 1, 2 \ldots$

with

$$L_0 = \alpha' \left( \frac{\hat{p}}{2} \right)^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n ; \quad \tilde{L}_0 = \alpha' \left( \frac{\hat{p}}{2} \right)^2 + \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n$$

The mass spectrum is given by $(\hat{p}^2 = -m^2)$:

$$\frac{\alpha'}{2} m^2 = \sum_{n=1}^{\infty} n \left( a_n^\dagger \cdot a_n + \tilde{a}_n^\dagger \cdot \tilde{a}_n \right) - 2 ; \quad \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n = \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n$$
The lowest state is the ground state $|0, P\rangle$ with mass $-P^2 = m^2 = -\frac{4}{\alpha'} \implies$ it is a tachyon.

The state contributing to the next massless level is the following:

$$a_{1\mu}^\dagger \tilde{a}_{1\nu}^\dagger |0, P\rangle$$

The symmetric and traceless part corresponds to a massless spin 2 $\implies$ graviton $G_{\mu\nu}$.

The trace part corresponds to a scalar particle called dilaton $\phi$.

The antisymmetric part corresponds to a 2-index antisymmetric tensor $B_{\mu\nu}$.

In the open string we have a massless gauge boson, while in the closed string we have a massless graviton together with a massless dilaton and a massless $B_{\mu\nu}$. 
The physical states are a subset of the previous states that satisfy the conditions:

\[ L_n |\text{Phys.}\rangle = \tilde{L}_n |\text{Phys.}\rangle = 0 \]

The analysis at this level proceeds as at the massless level of the open string.

In the reference frame where the momentum of the state is \( P_\mu = (P, \ldots, P) \), after the elimination of the zero norm states, the only physical states are:

\[ a_{1,i}^\dagger \tilde{a}_{1,j}^\dagger |0, P\rangle \quad ; \quad i, j = 1 \ldots (d - 2) \]

In conclusion, one gets \( \frac{(d-2)(d-1)}{2} - 1 \) physical states for the graviton, \( \frac{(d-2)(d-3)}{2} \) physical states for the two-index antisymmetric tensor and one state associated to the dilaton.

The total number of physical states at this level is therefore \( (d - 2)^2 \).
The Polyakov path integral

The most convenient way to find what is lacking in the old covariant quantization is to compute the string partition function using the string path integral formalism:

\[
\int Dx^\mu Dg_{\alpha\beta} e^{-S(x^\mu, g_{\alpha\beta})}
\]

The string action in Euclidean space is equal to

\[
S(x^\mu, g_{\alpha\beta}) \equiv \frac{T}{2} \int d^2 \xi \sqrt{g} \; g^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x
\]

It is invariant under world-sheet reparametrizations that act on the world-sheet metric and on the string coordinates as follows:

\[
x^\mu(\xi) = (x')^\mu(\xi') \quad ; \quad g_{\alpha\beta}(\xi) = \frac{\partial \xi'^\gamma}{\partial \xi^\alpha} \frac{\partial \xi'^\delta}{\partial \xi^\beta} g'_{\gamma\delta}(\xi')
\]

For infinitesimal transformations \(((\xi'^\alpha = \xi^\alpha - \epsilon^\alpha(\xi))\) they become

\[
\delta x^\mu = \epsilon^\alpha \partial_\alpha x^\mu \quad ; \quad \delta g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma} \partial_\beta \epsilon^\gamma + g_{\beta\gamma} \partial_\alpha \epsilon^\gamma = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha
\]
It is also invariant under Weyl transformations (rescaling of the metric):

\[ x^\mu(\xi) \rightarrow x^\mu(\xi) ; \quad g_{\alpha\beta}(\xi) \rightarrow \Lambda^2(\xi) g_{\alpha\beta}(\xi) \]

These two invariances involve three arbitrary functions \( \epsilon^\alpha(\xi) \) with \( \alpha = 1, 2 \) and \( \Lambda(\xi) \).

The metric tensor has also three independent components.

Locally, one can always choose a suitable reparametrization and a Weyl transformation that lead to a flat metric or to the one in the conformal gauge where

\[ \hat{g}_{\alpha\beta} = \delta_{\alpha\beta} ; \quad \hat{g}_{\alpha\beta} = \rho(\xi) \delta_{\alpha\beta} \]

if reparametrization and Weyl invariances are maintained at the quantum level.

Because of these two local invariances, the path integral is ill defined being the volume of the reparametrizations and Weyl transformations infinite.
We can define the functional integral by dividing by the volume of the reparametrizations and Weyl rescalings:

\[
\int \frac{Dg_{\alpha\beta} \, D\!x^\mu}{V_{\text{rep.}} \times V_{\text{Weyl}}} e^{-S(x,g)}
\]

In order to extract from \( Dg \) the two volumes, we perform the Faddeev and Popov procedure that can be applied to any theory with local gauge invariance.

Starting from a fixed fiducial metric \( \hat{g}_{\alpha\beta}(\xi) \) we can obtain the most general metric by transforming it by a reparametrization and a Weyl transformation:

\[
\hat{g}^\zeta_{\alpha\beta}(\xi') = e^{2\omega(\xi)} \frac{\partial \xi^\gamma}{\partial \xi'_{\alpha}} \frac{\partial \xi^\delta}{\partial \xi'_{\beta}} \hat{g}^{\gamma\delta}(\xi) \quad ; \quad \zeta \equiv (\xi'(\xi), \omega(\xi))
\]
In order to extract the volume of the reparametrization and Weyl transformations, we change integration variables from the original $g_{\alpha\beta}$ to the parameters of those transformations $\omega(\xi)$ and $\xi'^\alpha(\xi)$.

The integral over the parameters of the reparametrization and Weyl transformations gives the volume $V_{\text{rep.}} \times V_{\text{Weyl}}$ that cancels the volume in the denominator.

One is left with the jacobian of the transformation from $g_{\alpha\beta}$ to the parameters of the invariance group, called the determinant of Faddeev-Popov.

This procedure is explained in detail in a section at the end of this lecture.

Here we only give the final result:

$$\int D\!x^\mu \, \Delta_{FP}(\hat{g}) \, e^{-S(x,\hat{g})}$$
The determinant of the Faddev-Popov can be expressed in terms of a functional integral over the ghost fields $b^{\alpha\beta}$ (traceless) and $c^\alpha$ obtaining:

$$Z(\hat{g}) = \int Dx \, Db \, Dc \, e^{-S_x - S_{gh}}$$

where

$$S_{gh} = \frac{1}{2\pi} \int d^2 \xi \, \sqrt{\hat{g}} \, b_{\alpha\beta} \, \hat{\nabla}^\alpha \, c^\beta ; \quad S_x = \frac{1}{2\pi} \int d^2 \xi \, \sqrt{\hat{g}} \, \hat{g}^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x$$

We call them ghosts because they anti-commute (they are Grassmann variables), but not Dirac fermions.

We have made $x$ dimensionless by dividing it by $\sqrt{2\alpha'}$. 

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In the conformal gauge and world-sheet light-cone coordinates \( z = \xi^1 + i\xi^2 \) and \( \bar{z} = \xi^1 - i\xi^2 \) where

\[
g_{\alpha\beta} = \rho(\xi) \delta_{\alpha\beta} \implies g_{z\bar{z}} = g_{\bar{z}z} = \frac{\rho}{2} ; \quad g_{zz} = g_{\bar{z}\bar{z}} = 0
\]

the ghost action becomes:

\[
S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}} b_{\alpha\beta} \hat{g}^{\alpha\gamma} \hat{\nabla}_\gamma c^\beta = \frac{1}{2\pi} \int d^2\xi \left[ b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}} \right]
\]

In the present derivation we have ignored the possibility of anomalies.

It can be shown that, in general, we can have a Weyl anomaly that disappears, however, if the space-time dimension \( d = 26 \).
One can quantize the theory preserving reparametrization invariance.

But then, in general, one cannot preserve Weyl invariance.

How does a quantum violation of Weyl invariance manifest itself?

On the fact that the functional integral over $x^\mu, b, c$ will depend on $\rho \to e^\varphi$ one does not get anymore the volume of the Weyl group.

It turns out that the contribution of the functional integral over $x^\mu, b, c$ gives:

$$e^{\frac{1}{12\pi}(\frac{d}{2}-13)} \int d^2 \xi \left[ \frac{1}{2} \partial \varphi \partial \varphi + \mu^2 e^\varphi \right] ; \quad \rho \equiv e^\varphi$$

The dependence on $\varphi$ disappears only if $d = 26$.

Only for $d = 26$ one has a Weyl invariant quantum theory.
Conformal invariance

- Introducing the simpler notation:
  \[ b \equiv b_{zz} ; \quad \bar{b} \equiv b_{\bar{z}\bar{z}} ; \quad \bar{c} \equiv c_{\bar{z}} ; \quad c \equiv c^{z} ; \quad \partial \equiv \partial_{z} ; \quad \bar{\partial} \equiv \partial_{\bar{z}} \]

- The action becomes:
  \[ S = \frac{1}{\pi} \int d^{2}\xi \left[ \frac{1}{2} \partial x \cdot \bar{\partial} x + b \bar{\partial} c + \bar{b} \partial \bar{c} \right] \]

- This action is conformal invariant if we assume that \( x, b, c \) transform as conformal fields with dimension respectively equal to 0, 2, \(-1\), namely:
  \[
  \begin{align*}
  \delta x &= \epsilon \partial x + \bar{\epsilon} \bar{\partial} x \\
  \delta b &= \epsilon \partial b + 2 \partial \epsilon \ b \quad ; \quad \delta c = \epsilon \partial c - \partial \epsilon \ c \\
  \delta \bar{b} &= \bar{\epsilon} \bar{\partial} \bar{b} + 2 \bar{\partial} \bar{\epsilon} \ \bar{b} \quad ; \quad \delta \bar{c} = \bar{\epsilon} \bar{\partial} \bar{c} - \bar{\partial} \bar{\epsilon} \ \bar{c}
  \end{align*}
  \]
Each of the three pieces of the previous Lagrangian transforms as a total derivative (it is a conformal tensor with dimension $\Delta = 1$) under the conformal transformations with parameters $\epsilon$ and $\bar{\epsilon}$:

$$\delta \left( \frac{1}{2} \partial x \cdot \bar{\partial} x \right) = \partial \left( \epsilon \frac{1}{2} \partial x \cdot \bar{\partial} x \right) + \bar{\partial} \left( \bar{\epsilon} \frac{1}{2} \partial x \cdot \bar{\partial} x \right)$$

$$\delta \left( b \bar{\partial} c \right) = \partial \left( \epsilon b \bar{\partial} c \right)$$

$$\delta \left( \bar{b} \partial \bar{c} \right) = \bar{\partial} \left( \bar{\epsilon} \bar{b} \partial \bar{c} \right)$$

But now the energy-momentum tensor and the corresponding operators $L_n$ get also a contribution from the ghosts!!

In particular, one get:

$$L_n = \oint_0 dz \, z^{n+1} \, T(z) = \oint_0 dz \, z^{n+1} \left( T^x(z) + T^{gh}(z) \right)$$

where

$$T^x(z) = -\frac{1}{2} \left( \frac{\partial x}{\partial z} \right)^2 ; \quad T^{gh}(z) =: cb' + 2c'b :$$
It can be shown that the new operators $L_n$ satisfy the following algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{d - 26}{12}\delta_{n+m;0}n(n^2 - 1)$$

The c-number of the Virasoro algebra is vanishing at the critical dimension $D=26$.

as it must happen in any theory where the conformal symmetry is a gauge symmetry obtained after a partial fixing of the reparametrization invariance.

This is the first sign that also in the covariant quantization we need to have $d = 26$ as in the light-cone gauge.
Some details of the previous calculation

- Using the following contraction rules:

\[ < x^\mu(z)x^\nu(\zeta) > = -\eta^{\mu\nu} \log(z - \zeta) \quad ; \quad < b(z)c(\zeta) > = \frac{1}{z - \zeta} \]

- It can be shown that the transformation properties of a conformal tensor with dimension \( \Delta \) are completely equivalent to the following singular terms in the OPE of the energy-momentum tensor with the conformal field:

\[ T(z)\phi(w) \sim \frac{\partial \phi}{\partial w} \frac{z}{z - w} + \Delta \frac{\phi(w)}{(z - w)^2} + \ldots \]

- In fact, from it we get:

\[ \delta \phi \sim [L_n, \phi(w)] = \oint_w dz z^{n+1} T(z)\phi(w) = w^{n+1} \frac{\partial \phi(w)}{\partial w} + \Delta(n + 1)w^n \phi(w) \]
In particular, we can compute the OPE between two energy-momentum tensors (conformal fields with $\Delta = 2$):

$$T(z) T(\zeta) = \frac{\partial}{\partial \zeta} \frac{T(\zeta)}{z - \zeta} + 2 \frac{T(\zeta)}{(z - \zeta)^2} + \frac{D-26}{2} \frac{1}{(z - \zeta)^4} + \ldots$$

and from it we get:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{d - 26}{12} \delta_{n+m;0} n(n^2 - 1)$$

Remember:

$$L_n = \int_0^1 dz \, z^{n+1} T(z)$$
**BRST invariance**

- By fixing the gauge, we have lost the invariance under reparametrizations and Weyl transformations.
- But we are left with BRST invariance.
- It is straightforward to show that, under the following transformations:

\[
\begin{align*}
\delta x &= \lambda c \partial x \\
\delta c &= \lambda c \partial c \\
\delta b &= -\frac{1}{2} \lambda (\partial x)^2 + \lambda [c \partial b + 2 \partial cb]
\end{align*}
\]

- the gauge fixed Lagrangian transforms as a total derivative:

\[
\delta L = \partial [\lambda c L]
\]

- \(\lambda\) is a constant Grassmann parameter.
- It is generated by the following operator:

\[
Q = \int_0 dz : c(z) [T^x(z) + \frac{1}{2} T^{gh}(z)] :
\]
Because of its Grassmann character, in the classical theory the product of two BRST transformations is identically vanishing.

In the quantum theory the square of the BRST charge is given by:

\[\{Q, Q\} = \frac{1}{12} (d - 26) \int_0^1 d\zeta c'''(\zeta)c(\zeta)\]

The square of the BRST charge is vanishing only if \(d=26\).

This is another sign that our covariant quantization is consistent only for the critical dimension \(d=26\).
Physical states

In terms of the oscillators the BRST charge is given by:

\[ Q = \sum_{n=1}^{\infty} [c_n L_n^x + c_n^\dagger L_n^x] + c_0 [L_0^x + L_0^g] + \tilde{Q} \]

where

\[ \tilde{Q} = \sum_{n,m=1}^{\infty} m [c_n^\dagger c_m^\dagger b_{n+m} - c_n c_m b_{n+m}] - 2b_0 \sum_{n=1}^{\infty} n c_n^\dagger c_n + \]

\[ + \sum_{n,m=1}^{\infty} (n + 2m) [c_m^\dagger c_{n+m} b_n^\dagger + c_{n+m}^\dagger c_m b_n] \]

The ghost fields have the following expansion in terms of the harmonic oscillators:

\[ b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-2} \quad c(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n+1} \]
The oscillators satisfy the algebra:

\[ \{ c_n, b_m \} = \delta_{n+m,0} \quad ; \quad \{ c_n, c_m \} = \{ b_n, b_m \} = 0 \]

In the BRST quantization the physical states are defined as those annihilated by the BRST charge:

\[ Q|\text{Phys.}\rangle = 0 \]

This is the residual invariance left from having fixed the gauge.

The generators of this invariance must annihilate the physical states.

What are the states that satisfy this equation?

In order to answer this question we have to introduce and discuss the ghost number current.
The ghost Lagrangian is invariant under a $U(1)$ current that acts on the ghost fields as follows:

$$\delta b = i \alpha b \quad \delta c = -i \alpha c$$

The generator corresponding to this invariance can be constructed in terms of the ghost number density:

$$j(z) =: c(z)b(z) :$$

The ghost number is given by

$$q = \int_0^\infty dz \, j(z) = \sum_{n=-\infty}^{\infty} : c_n b_{-n} :$$

$$= c_0 b_0 + c_1 b_{-1} + c_{-1} b_1 + \sum_{n=2}^{\infty} (c_{-n} b_n - b_{-n} c_n)$$
It turns out that the ghost number current is anomalous and requires the following unconventional normal ordering for the ghost oscillators:

$$: c_n b_{-n} := \begin{cases} 
  c_n b_{-n} & \text{if } n \leq 1 \\
  -b_{-n} c_n & \text{if } n \geq 2 
\end{cases}$$

or equivalently

$b_{-1}, b_0, b_1, b_2 \ldots c_2, c_3 \ldots$ are "annihilation operators"

$b_{-2}, b_{-3}, b_{-4} \ldots c_1, c_0, c_{-1} \ldots$ are "creation operators".

In particular, a state that satisfies the following equations:

$$\langle \ldots b_2, b_1, b_0, b_{-1} \rangle |q = 0 \rangle = (\ldots c_3, c_2) |q = 0 \rangle = 0$$

has ghost number zero.

It plays the role of the vacuum because it is annihilated by all "annihilation operators".

The state with $q = 1$ is what "one would normally call a vacuum":

$$|q = 1 \rangle \equiv c_1 |q = 0 \rangle \implies (\ldots b_2, b_1, b_0) |q = 1 \rangle = (\ldots c_3, c_2, c_1) |q = 1 \rangle = 0$$
A detailed analysis shows that the on-shell physical states must have the following form [Freeman and Olive, 1986]:

\[ |\text{Phys.}\rangle = |q = 1; \psi_a\rangle \]

where the state \( |\psi_a\rangle \) is constructed only in terms of the oscillators of the string coordinate \( x \).

Remembering the form of \( Q \) in terms of the oscillators we see that

\[ \tilde{Q}|q = 1\rangle = 0 \]

and the action of \( Q \) on the physical state is then given by:

\[ Q|q = 1; \psi_a\rangle = \left[ \sum_{n=1}^{\infty} [c_n L_{-n}^x + c_n^\dagger L_n^x] + c_0 [L_0^x + L_0^g] \right] |q = 1; \psi_a\rangle \]

\[ = \left[ \sum_{n=1}^{\infty} c_n^\dagger L_n^x + c_0 (L_0^x - 1) \right] |q = 1; \psi_a\rangle = 0 \]
We have used the two identities:

\[ c_n |q = 1\rangle = 0 \quad ; \quad n = 1, 2 \ldots \quad ; \quad L^g_0 |q = 1\rangle = -|q = 1\rangle \]

The second equation follows from the following expression for \( L^g_0 \):

\[
L^g_0 = \sum_{n=-\infty}^{\infty} n : b_{-n} c_n := \sum_{n=2}^{\infty} n(b_{-n} c_n + c_{-n} b_n) + c_{-1} b_1 - c_1 b_{-1}
\]

In conclusion, we correctly reproduce the conditions for on physical states:

\[
L^x_n |\psi_a\rangle = (L^x_0 - 1)|\psi_a\rangle = 0
\]

The most general physical state has therefore the following form:

\[
|\text{Phys.}\rangle = |q = 1, \psi_a\rangle + Q|\lambda\rangle
\]

where \(|\lambda\rangle\) is an arbitrary state.
Conclusions

- Quantizing correctly the bosonic string in a covariant gauge we have obtained the same results as in the light-cone gauge!
- namely the correct values for the Regge intercept and the critical dimensions:
  \[ \alpha_0 = 1 \quad d = 26 \]

- It turns out the equations characterizing the on-shell physical states are precisely those obtained in 1970 from factorizing the \( N \)-point amplitude without knowing that there was an underlying string theory!!

- The new feature is the presence in the covariant gauge of the reparametrization ghosts \( b \) and \( c \).
- They are, however, in practice not relevant if we limit ourselves to the computation of the spectrum and of tree diagrams.
They are, instead, essential for computing one-loop and especially multiloop diagrams.

If one computes loop diagrams in the light-cone gauge one has only the physical transverse states circulating in the loop.

In a covariant formulation one must keep all string oscillators and not just the physical transverse ones.

One has then too many states circulating in the loops.

The ghost degrees of freedom that are fermions, are there to cancel the contribution of the non-physical states kept in order to have a manifest Lorentz invariant formulation of the string theory.
The material that follows is for helping those interested in understanding some of the more technical details.
Faddeev-Popov procedure

▶ We define the functional integral by dividing by the volume of the reparametrizations and Weyl rescalings:

\[
\int \frac{Dg\ Dx}{V_{\text{rep.}} \times V_{\text{Weyl}}} e^{-S(x,g)}
\]

▶ In order to extract from \(Dg\) the two volumes, we perform the Faddeev and Popov procedure that can be applied to any theory with local gauge invariance.

▶ Starting from a fiducial metric \(\hat{g}_{\alpha\beta}(\xi)\) we can transform it by a reparametrization and a Weyl transformation:

\[
\hat{g}_{\xi\rho}(\xi') = e^{2\omega(\xi)} \frac{\partial \xi_\gamma}{\partial \xi'_{\alpha}} \frac{\partial \xi_\delta}{\partial \xi'_{\beta}} \hat{g}_{\gamma\delta}(\xi) \quad ; \quad \zeta \equiv (\epsilon, \omega)
\]

▶ We define the Faddeev-Popov measure by

\[
1 = \Delta_{FP}(g) \int D\zeta \delta(g - \hat{g}^\zeta)
\]
\( D\zeta \) is the invariant measure of the reparametrizations plus Weyl transformations.

- We can insert 1 in the functional integral, integrate over \( h \) and rename the dummy variable \( x \rightarrow x\zeta \):
  \[
  \int \frac{D\zeta \ Dx^\zeta}{V_{rep.} \times V_{Weyl}} \Delta_{FP}(\hat{g}^\zeta) \ e^{-S(x^\zeta, \hat{g}^\zeta)}
  \]

- Using the gauge invariance of the action, of the measure and of \( \Delta_{FP} \) one gets:
  \[
  \int \frac{D\zeta \ Dx}{V_{rep.} \times V_{Weyl}} \Delta_{FP}(\hat{g}) \ e^{-S(x, \hat{g})}
  \]

- Nothing depends on \( \zeta \) and therefore we can integrate on it producing the volume of the invariance groups that cancels the volume in the denominator:
  \[
  \int Dx \ \Delta_{FP}(\hat{g}) \ e^{-S(x, \hat{g})}
  \]
\[ \Delta_{FP} \] can be computed for \( \zeta \) near the identity where:

\[
\hat{g}_{\alpha\beta} - \hat{g}^\zeta_{\alpha\beta} \sim 2\delta\omega g_{\alpha\beta} - \nabla_{\alpha}\epsilon_{\beta} - \nabla_{\beta}\epsilon_{\alpha}
\]

\[
= (2\delta\omega - \nabla_\gamma\epsilon_\gamma)g_{\alpha\beta} - 2(P_1\epsilon)_{\alpha\beta}
\]

and

\[
(P_1\epsilon)_{\alpha\beta} = \frac{1}{2} \left( \nabla_{\alpha}\epsilon_{\beta} + \nabla_{\beta}\epsilon_{\alpha} - g_{\alpha\beta} \nabla_\gamma\epsilon_\gamma \right)
\]

Near the identity we can compute the Faddeev-Popov determinant:

\[
\Delta_{FP}^{-1}(\hat{g}) = \int D\epsilon D\delta\omega \delta \left( -2(\delta\omega - \hat{\nabla}\cdot\epsilon)\hat{g} + 2\hat{P}_1\epsilon \right)
\]

\[
= \int D\epsilon D\delta\omega D\beta \ e^{2\pi i \int d^2\xi \sqrt{\hat{g}}\epsilon_{\alpha\beta} \left( -2(\delta\omega - \hat{\nabla}\cdot\epsilon)\hat{g} + 2\hat{P}_1\epsilon \right)_{\alpha\beta}}
\]

The integration over \( \delta\omega \) forces \( \beta_{\alpha\beta} \) to be traceless and one gets:

\[
\Delta_{FP}^{-1}(\hat{g}) = \int D\epsilon D\beta \ e^{4\pi i \int d^2\xi \sqrt{\hat{g}}\epsilon_{\alpha\beta} \left( \hat{P}_1\epsilon \right)_{\alpha\beta}}
\]

In this way we have computed the inverse determinant.
In order to obtain directly the Faddeev-Popov determinant we have to replace any bosonic with a fermionic field:

\[ \beta^{\alpha\beta} \rightarrow b^{\alpha\beta} ; \quad \epsilon^{\alpha} \rightarrow c^{\alpha} \]

obtaining

\[ \Delta_{FP}(\hat{g}) = \int Dc Db \ e^{4\pi i \int d^2\xi \sqrt{\hat{g}} b^{\alpha\beta} (\hat{P}_1 c)_{\alpha\beta}} \]

where \( b \) is traceless.

We call them ghosts because they are Grassmann, but not Dirac fermions.

In conclusion, with a convenient normalization of the two ghost fields we obtain the following gauge fixed partition function:

\[ Z(\hat{g}) = \int Dx \ Db \ Dc \ e^{-S_x - S_{gh}} \]

where

\[ S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}} b_{\alpha\beta} \hat{\nabla}^{\alpha} c^{\beta} ; \quad S_x = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}} \hat{g}^{\alpha\beta} \partial_{\alpha} x \cdot \partial_{\beta} x \]
Determinants in the numerator or in denominator

- If we have a gaussian integral with bosonic complex variables we get:

\[
\int \prod_i d^2z_i e^{-\sum_{i,j} \bar{z}_i M_{ij} z_j} = \frac{1}{\det M}
\]

- Instead, if we have a gaussian integral involving fermionic (Grassmann) complex variables we get:

\[
\int \prod_i d^2\psi_i e^{-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j} = \det M
\]

- Remember that Grassmann variables anticommute:

\[
\psi_i \psi_j = -\psi_j \psi_i ; \quad \psi_i \bar{\psi}_j = -\bar{\psi}_j \psi_i \quad \Rightarrow \quad \psi_i^2 = 0
\]

- The determinant is computed using the following integration rules:

\[
\int d\psi = 0 ; \quad \int d\psi \psi = 1
\]
We have made $x$ dimensionless by dividing it by $\sqrt{2\alpha'}$.

In the conformal gauge and world-sheet light-cone coordinates $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$ where

$$g_{\alpha\beta} = \rho(\xi)\delta_{\alpha\beta} \implies g_{z\bar{z}} = g_{\bar{z}z} = \frac{\rho}{2}; \quad g_{zz} = g_{\bar{z}\bar{z}} = 0$$

The ghost action becomes:

$$S_{gh} = \frac{1}{2\pi} \int d^2\xi \sqrt{\hat{g}} \, b_{\alpha\beta} \hat{g}^{\alpha\gamma} \hat{\nabla}_\gamma c^\beta = \frac{1}{2\pi} \int d^2\xi \left[ b_{zz} \partial_z c^z + b_{z\bar{z}} \partial_{\bar{z}} c^{\bar{z}} \right]$$

In the present derivation we have ignored the possibility of anomalies.

It can be shown that, in general, we can have a Weyl anomaly that disappears, however, if the space-time dimension $d = 26$. 
Conformal tensors

- Consider string theory in the conformal gauge, characterized by the following choice of the Euclidean world-sheet metric tensor:

\[ g_{\alpha\beta} = \rho(\xi) \delta_{\alpha\beta} ; \quad \rho = e^{\varphi(\xi)} \]

- We have seen that the conformal transformations leave in the conformal gauge.

- It is convenient to work with light-cone coordinates:

\[ z = \xi^1 + i\xi^2 ; \quad \bar{z} = \xi^1 - i\xi^2 \]

- In these coordinates the invariant length is defined by:

\[ (ds)^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = \frac{\rho}{2} [dzd\bar{z} + d\bar{z}dz] \]

- implying the following light-cone coordinates for the metric tensor:

\[ g^{zz} = g^{\bar{z}\bar{z}} = g_{zz} = g_{\bar{z}\bar{z}} = 0 \]
\[ g^{z\bar{z}} = g^{\bar{z}z} = 2/\rho \quad g_{z\bar{z}} = g_{\bar{z}z} = \rho/2 \]
In terms of the light-cone components of a vector:

\[
\begin{align*}
\epsilon^z &= \epsilon^1 + i\epsilon^2 \\
\bar{\epsilon}^z &= \epsilon^1 - i\epsilon^2 \\
\epsilon_z &= \frac{1}{2}(\epsilon^1 - i\epsilon^2) \\
\bar{\epsilon}_z &= \frac{1}{2}(\epsilon^1 + i\epsilon^2)
\end{align*}
\]

one can define the scalar product between two vectors:

\[
A^\alpha B_\alpha = [A^z B_z + A^{\bar{z}} B^{\bar{z}}] = [A^z B_z + A_z B^{\bar{z}}] = A_z B^{\bar{z}} + A_{\bar{z}} B^z
\]

where the indices are lowered and raised by means of the metric tensor as follows:

\[
\begin{align*}
A^z &= g^{z\bar{z}} A_{\bar{z}} \\
A_z &= g_{z\bar{z}} A^{\bar{z}} \\
A^{\bar{z}} &= g^{\bar{z}z} A_z \\
A_{\bar{z}} &= g_{\bar{z}z} A^z
\end{align*}
\]

The covariant derivatives are given by:

\[
\nabla_\alpha \epsilon^\beta = \partial_\alpha \epsilon^\beta + \Gamma^\beta_{\alpha\gamma} \epsilon^\gamma, \quad \nabla_\alpha \epsilon = \partial_\alpha \epsilon - \Gamma^\gamma_{\alpha\beta} \epsilon_{\gamma}
\]

where the Christoffel symbols are given in the conformal gauge by:

\[
\Gamma^\gamma_{\alpha\beta} = \frac{g^{\gamma\delta}}{2} \left[ \partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta} \right] = \left[ \partial_\alpha \delta^\gamma_{\beta} + \partial_\beta \delta^\gamma_{\alpha} - \partial^\gamma \delta_{\alpha\beta} \right] \frac{\log \rho}{2}
\]
Only two non-vanishing components:

\[ \Gamma_{zz}^z = \rho^{-1} \partial_z \rho \quad , \quad \Gamma_{z\bar{z}}^{\bar{z}} = \rho^{-1} \partial_{\bar{z}} \rho \]

One gets:

\[ \nabla_{\bar{z}} \epsilon^z = \partial_{\bar{z}} \epsilon^z \quad ; \quad \nabla_{\bar{z}} \epsilon^{\bar{z}} = \rho^{-1} \partial_{\bar{z}} \rho \epsilon^{\bar{z}} \]
\[ \nabla_{z} \epsilon^{\bar{z}} = \partial_{z} \epsilon^{\bar{z}} \quad ; \quad \nabla_{z} \epsilon^z = \rho^{-1} \partial_{z} \rho \epsilon^z \]

Raising the index of the covariant derivative with the metric tensor one gets:

\[ \nabla^z \epsilon^z = \frac{2}{\rho} \partial_z \epsilon^z \quad ; \quad \nabla^{\bar{z}} \epsilon^{\bar{z}} = \frac{2}{\rho^2} \partial_{\bar{z}} \rho \epsilon^{\bar{z}} \]
\[ \nabla^{\bar{z}} \epsilon^z = \frac{2}{\rho} \partial_{\bar{z}} \epsilon^z \quad ; \quad \nabla^z \epsilon^{\bar{z}} = \frac{2}{\rho^2} \partial_z \rho \epsilon^{\bar{z}} \]
The action of the covariant derivative on a conformal tensor $T^{z\cdots z}$ with rank $n$ is given by:

\[
\nabla^{n}_{\bar{z}} T^{z\cdots z} = \partial_{\bar{z}} T^{z\cdots z} \\
\nabla_{z}^{n} T^{z\cdots z} = \rho^{-n} \partial_{\bar{z}} \rho^{n} T^{z\cdots z} \\
\n\nabla^{z} T^{z\cdots z} = \frac{2}{\rho} \partial_{\bar{z}} T^{z\cdots z} \\
\n\nabla_{\bar{z}}^{z} T^{z\cdots z} = 2\rho^{-1-n} \partial_{\bar{z}} \rho^{n} T^{z\cdots z}
\]

Under a general relativity transformation a vector transforms as follows:

\[
e^{\mu}(\xi) \rightarrow \frac{\partial \xi^{\mu}}{\partial \xi'^{\nu}} e^{\nu}(\xi')
\]

In terms of light-cone coordinates one gets:

\[
\epsilon^{z}(z, \bar{z}) \rightarrow \frac{\partial z}{\partial w} \epsilon^{w} = \frac{1}{w'(z)} \epsilon^{w} \\
\epsilon^{\bar{z}}(z, \bar{z}) \rightarrow \frac{\partial \bar{z}}{\partial \bar{w}} \epsilon^{\bar{w}} = \frac{1}{\bar{w}'(\bar{z})} \epsilon^{\bar{w}}
\]

We have restricted us to conformal transformations for which:

\[
\frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0
\]
A conformal tensor of rank $n$ transforms as follows under a conformal transformation:

\[
T^{z\cdots z}(z) \to \frac{1}{[w'(z)]^n} T^{w\cdots w}(w) ;
T^{\bar{z}\cdots \bar{z}}(\bar{z}) \to \frac{1}{[\bar{w}'(\bar{z})]^n} T^{\bar{w}\cdots \bar{w}}(\bar{w})
\]

\[
T_{z\cdots z}(z) \to [w'(z)]^n T_{w\cdots w}(z) ;
T_{\bar{z}\cdots \bar{z}}(\bar{z}) \to [\bar{w}'(\bar{z})]^n T_{\bar{w}\cdots \bar{w}}(\bar{w})
\]

We have lowered the indices with the metric tensor and we have used the transformation of $\rho$ under a conformal transformation:

\[
\rho(z, \bar{z}) \to w'(z) \bar{w}'(\bar{z}) \rho(w, \bar{w})
\]

The covariant derivative $\nabla^z_n$ applied to a conformal tensor of rank $n$ gives a conformal tensor of rank $n + 1$:

\[
\nabla^z_n T^{z\cdots z}_{(n)}(z) \equiv \frac{2}{\rho} \partial \bar{z} T^{z\cdots z}_{(n)}(z) \to [w'(z)]^{-n-1} \nabla^w T^{w\cdots w}_{(n)}(w)
\]
The covariant derivative $\nabla^n_z$ applied to a conformal tensor of rank $n$ gives a conformal tensor with rank $n-1$:

$$\nabla^n_z T_{(n)}^{z_1 \ldots z_n}(z) \equiv \rho^{-n}(z) \partial_z \rho^n(z) T_{(n)}^{z_1 \ldots z_n}(z) \to [w'(z)]^{1-n} \nabla_w T_{(n)}^{w_1 \ldots w_n}(w)$$

In conclusion, the action of the covariant derivative on a conformal tensor of rank $n$ gives the following tensors:

\[
\begin{align*}
T_{(n)} & \xrightarrow{\nabla^n_z} T_{(n+1)} \xrightarrow{\nabla_{z+1}} T_{(n)} \\
T_{(n)} & \xrightarrow{\nabla^n_z} T_{(n-1)} \xrightarrow{\nabla_{n-1}} T_{(n)}
\end{align*}
\]

In terms of the covariant derivatives we can define the following Laplacians:

$$\Delta^{(+)}_n = -\nabla_{z+1} \nabla^n_z \quad \Delta^{(-)}_n = -\nabla_{n-1} \nabla^n_z$$
They satisfy the relation:

\[ \Delta_n^{(+)} - \Delta_n^{(-)} = \frac{n}{2} R \]

where \( R \) is the scalar curvature:

\[ R = \frac{4}{\rho} \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \]
The ghost number current

- The ghost Lagrangian is invariant under a $U(1)$ current that acts on the ghost fields as follows:

\[ \delta b = i \alpha b \quad \delta c = -i \alpha c \]

- The generator corresponding to this invariance can be constructed in terms of the the ghost number density:

\[ j(z) =: c(z)b(z) : \]

- Using the $b - c$ contraction one can to compute the following OPE’s:

\[ j(z)j(\zeta) = \frac{1}{(z - \zeta)^2} \]

\[ T^g(z)j(\zeta) = \frac{\partial j(\zeta)}{\partial \zeta} + \frac{j(z)}{(z - \zeta)^2} - \frac{3}{(z - \zeta)^3} \]

- $j(z)$ is a conformal field with dimension $\Delta = 1$, but there is an extra term that makes the analysis more complicated.
The ghost coordinates $b(z)$ and $c(z)$ are conformal fields with conformal dimension $\Delta = 2$ and $-1$ respectively.

Their expansion in term of the harmonic oscillators is given by:

$$b(z) = \sum_{-\infty}^{\infty} b_n z^{-n-2} \quad c(z) = \sum_{-\infty}^{\infty} c_n z^{-n+1}$$

The oscillators satisfy the algebra:

$$\{ c_n, b_m \} = \delta_{n+m,0} \quad ; \quad \{ c_n, c_m \} = \{ b_n, b_m \} = 0$$

Introduce the Fourier components of $j(z)$ and $T^g(z)$

$$j_n = \oint dzz^n j(z) = \sum_m : c_{n-m} b_m :$$

$$L^g_n = \oint dzz^{n+1} T^g(z) = \sum_m (m + n) : b_{n-m} c_m :$$
They satisfy the algebra:

\[
[j_n, j_m] = n \delta_{n+m,0} \ ; \quad [L^g_n, j_m] = -mj_{n+m} - \frac{3}{2} n(n + 1) \delta_{n+m,0}
\]

\[
[L^g_n, L^g_m] = (n - m)L^g_{n+m} - \frac{26}{12} n(n^2 - 1) \delta_{n+M,0}
\]

It can be reproduced in terms of the oscillators only if the normal ordering is defined in the following non-conventional way:

\[
: c_n b_{-n} := \begin{cases} 
c_n b_{-n} & \text{if } n \leq 1 \\
-b_{-n} c_n & \text{if } n \geq 2
\end{cases}
\]

From the algebra it turns out that \( j_0 \) is not anti-hermitian as \( j_n \) for \( n \neq 0 \), but it satisfies the more complicated relation:

\[
j_0 + j_0^\dagger - 3 = 0
\]
Therefore if $|q\rangle$ is an eigenstate of the ghost number operator

$$j_0|q\rangle = q|q\rangle$$

the previous equation implies that

$$\langle q'|q\rangle \sim \delta_{q,3-q'}$$

It can be checked that the state defined by

$$b_n|q\rangle = 0 \quad \text{if} \quad n > q - 2$$

$$c_n|q\rangle = 0 \quad \text{if} \quad n \geq -q + 2$$

is an eigenstate of the ghost number operator with ghost number equal to $q$.

It satisfies also the equation:

$$L_0|q\rangle = \frac{1}{2}q(q-3)|q\rangle$$
Among those eigenstates of $j_0$ the only one, that is annihilated by the three generators of the projective group is $|q = 0 >$:

$$L_0|q = 0 > = L_1|q = 0 > = L_{-1}|q = 0 > = 0$$

$|q = 0 >$ is therefore projective invariant.

The non-anti-hermicity of $j_0$ implies that, if we compute any matrix element containing objects with a definite ghost number, we will get zero unless the total ghost charge is equal to 3.

In particular, in order to compute $b - c$ the contraction, we must compute the following matrix element:

$$< q = 3 | b(z)c(\zeta) | q = 0 > = \frac{1}{z - \zeta}$$