Particules Élémentaires, Gravitation et Cosmologie

Année 2005-2006

Interactions fortes et chromodynamique quantique

II: aspects non-perturbatifs

Cours IV: 28 fevrier 2006

Jouant avec N_c, N_f

- Reminder from previous lecture
- Motivations for large-N
- Counting powers of N:
  1. N-vector-valued fields
  2. N x N matrix-valued fields, QCD
- Masses, couplings, anomaly in large-N QCD

Reminder from last week

- We considered first QCD with N_f quark flavours in an ideal world where its full global classical symmetry

$$G_F = SU(N_f)_L \otimes U(1)_L \otimes SU(N_f)_R \otimes U(1)_R$$

is not explicitly broken (no masses, no anomaly)

- We assumed that G_F is spontaneously broken down to its diagonal subgroup $U(N_f)_V$ by a BEC:

$$\langle 0|\bar{\psi}_i \psi_j|0\rangle = c\, \delta_{ij}\, \Lambda^3 \equiv \delta_{ij} \langle \bar{\psi}\psi \rangle$$

- Then, according to Goldstone's theorem, there must be N_f^2 NG bosons, $\pi_{ij} \sim q^*_i q_j$ that couple to as many spontaneausly broken symmetry currents (no sum over i,j)

$$\langle 0|J^{ij}_{\mu 5}|\pi_{ij}\rangle = -iF_\pi q_\mu$$

- We then considered the effect of (small) quark masses m_i while still neglecting the anomaly, and found that our NGB become PNGB with masses given to first order in m_i by:

$$\mu_{ij}^2 = -F_\pi^{-2}(m_i + m_j)\langle\bar{\psi}\psi\rangle \; ; \; i,j = 1,2,\ldots N_f$$

In this (no anomaly) limit the mass (energy) eigenstates are «pure» (unmixed) in the quark basis.

This works perfectly well for the 6 flavourful PNGB:

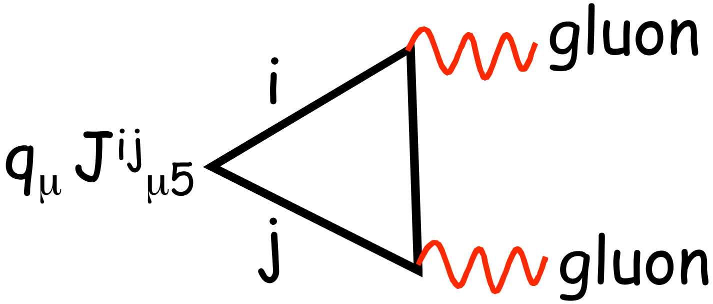
$\pi^{\pm}$ ~ ud*, du* ; K^+,K^- ~ us*, su* ; K^0,K^{0*} ~ ds*, sd*

(their measured masses already tell us much about the relative values of the corresponding quark masses)

This does not work at all for the 3 flavour-less PNGB

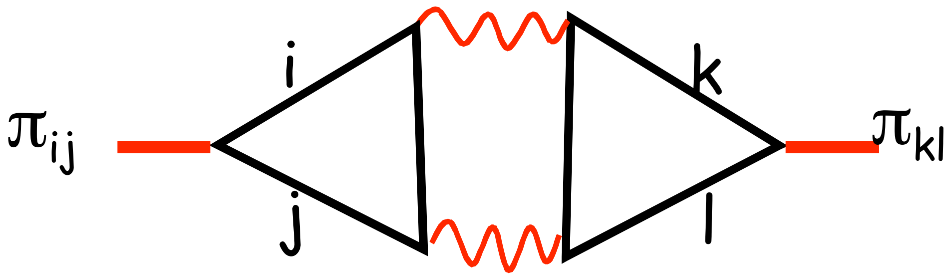
uu* = ?? , dd* = ?? , ss* = ??

- We suspected this problem to have to do with the ABJ anomaly coming from the diagram (GV2, KK1)



$\sim \delta_{ij}$

Indeed, this diagram only contributes to the flavourless channels (recall: gluons do not carry flavour!). In fact the contribution of the anomaly to the flavourless 3x3 mass matrix is "completely democratic" in flavour space:



$= a\, \delta_{ij}\, \delta_{kl}$

NB: $a \neq 0$ requires $\nu \neq 0$, instantons (KK2)

Can the anomaly save us?

- Let us combine the two contributions to the PNGB masses: a) from quark masses, and b) from the anomaly. We get:

M^2 =

	π_{uu}	π_{dd}	π_{ss}
π_{uu}	μ_{uu}^2 $+a$	a	a
π_{dd}	a	μ_{dd}^2 $+a$	a
π_{ss}	a	a	μ_{ss}^2 $+a$

Two limiting cases:

1. If $a=0$ we have "pure quark" eigenstates (X)
2. If $m_i = 0$ we have two massless eigenstates and one, the flavour singlet of mass^2 = 3a

Realistic case

If $m_u = m_d = 0$ and $\mu_{ss}^2 \ll a$, we have approximate eigenvalues 0, $2/3\mu_{ss}^2$, $3a + 1/3\mu_{ss}^2$ and the right flavour-mixing pattern..

$$\pi^0 \sim \frac{1}{\sqrt{2}}[uu^* - dd^*]$$

$$\eta \sim \frac{1}{\sqrt{6}}[uu^* + dd^* - 2ss^*] \,, \eta' \sim \frac{1}{\sqrt{3}}[uu^* + dd^* + ss^*]$$

The problem in dealing quantitatively with the anomaly and instantons is that, unlike the explicit breaking due to quark masses, there is apparently no sense in which the anomaly can be treated as a perturbation.
This is true, a priori, of all non-perturbative phenomena in QCD, almost by definition.
Fortunately, we can find appropriate limits in which the anomaly can be regarded as being small..

The large-N idea

- When the coupling constant α_s is large, as it is our case, there is no reason to stop perturbation theory (i.e. the loop expansion) at any finite order
- Terms of order $\exp(-1/\alpha_s)$ are invisible in PT
- Can we find, in this regime, another small expansion parameter?
- At first sight the answer looks negative: QCD depends just on one dimensionful constant, Λ_{QCD}
- It turns out, however, that a trick to generate another small parameter does exist
- It is the so-called large-N expansion of QCD

- Consider a whole infinite family of QCD-like theories that differ from QCD by having SU(N) (rather than SU(3)) as gauge symmetry.
- Unlike powers of α_s, powers of 1/N do not correspond to the number of loops, but to some more global, topological, property of the Feynman diagrams
- Actually, 1/N methods are of much more general applicability (also in statistical mechanics where they were first discovered).
- There are essentially two kinds of 1/N expansions:

1. When fields are N-dim vectors of some O(N) symmetry
2. When fields are NxN matrices of some SU(N) symmetry

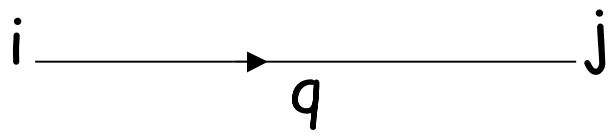
The former case is much easier; unfortunately, QCD belongs to the second..

Vector fields in O(N)

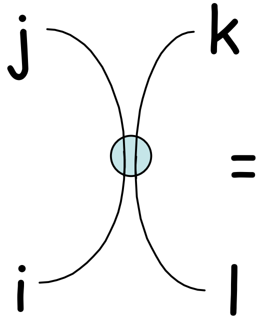
- A prototype model is as follows (sum over i understood):

$$S = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi^i \partial_\mu \phi^i - m^2 \phi^i \phi^i) - g(\phi^i \phi^i)^2 \right]$$

Its Feynman rules are simple:

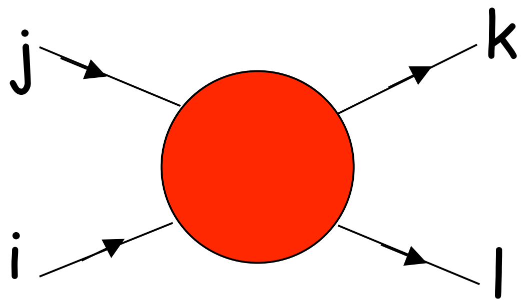


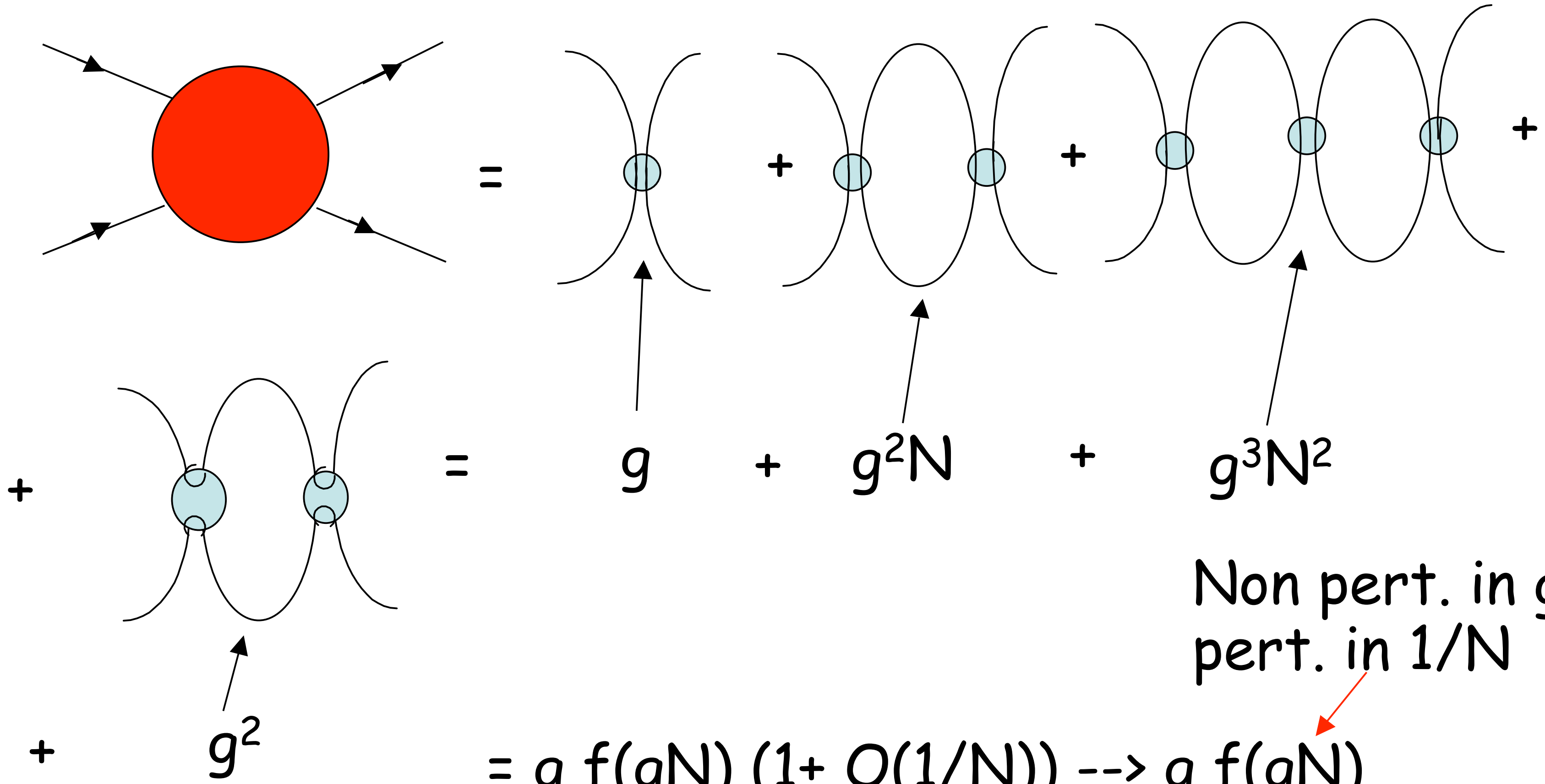
$= \delta_{ij}/(q^2-m^2)$



$= g\, \delta_{ij}\, \delta_{kl}$

Consider now, for instance, a 2->2 scattering process:





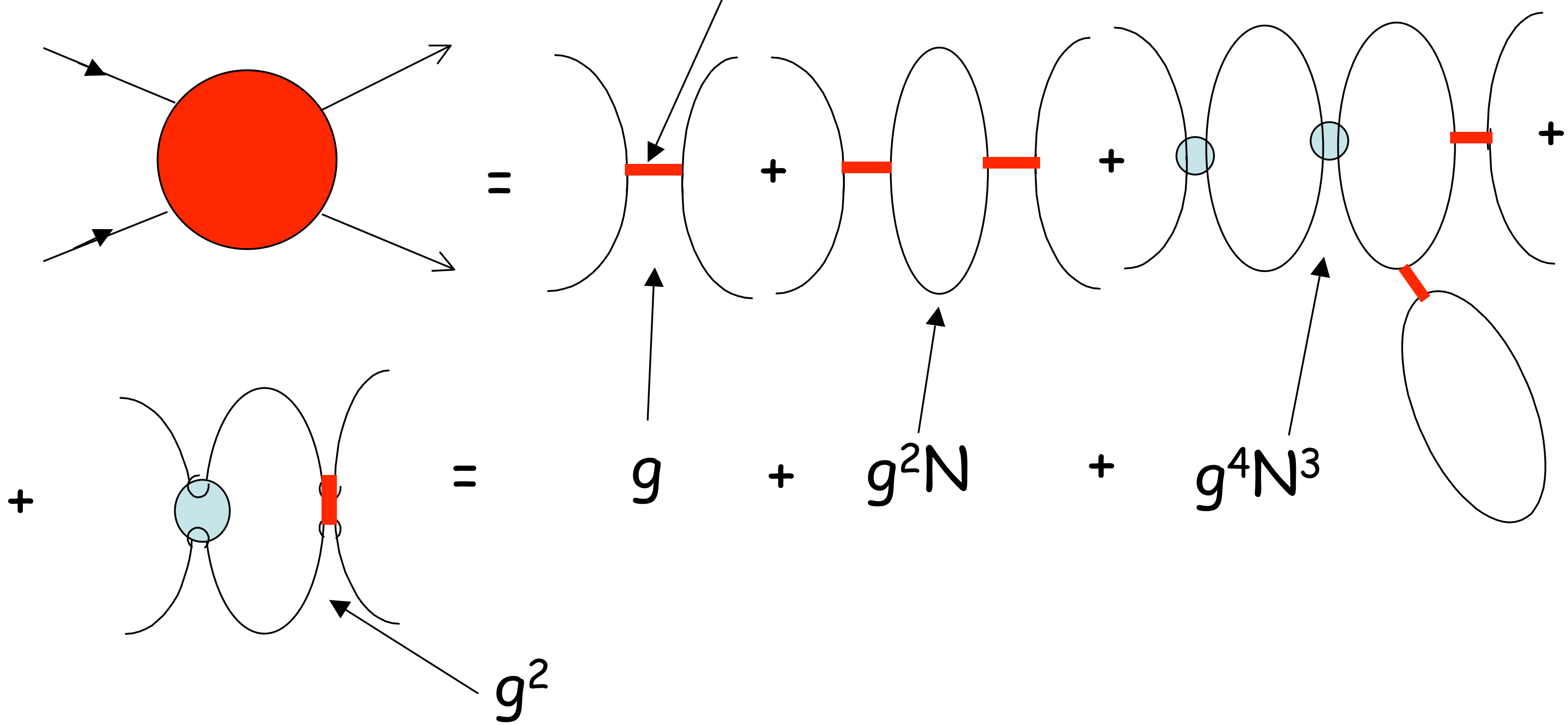
Non pert. in g,
pert. in $1/N$

$= g\, f(gN)\, (1+ O(1/N)) \dashrightarrow g\, f(gN)$

Q: What characterizes the leading diagrams?

A: Certainly not the number of loops, but some global topological property. How can we characterize them?

Let us introduce a “fake” heavy singlet and redraw diagrams

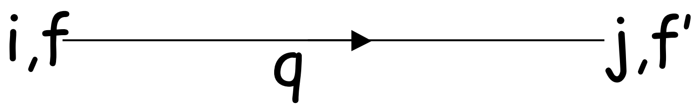


A: At large N the leading diagrams are those that become disconnected as soon as one heavy-singlet line is cut. They are “trees” from the point of view of the heavy singlet

This is what makes life “easier” (see seminar by P. Di Vecchia)

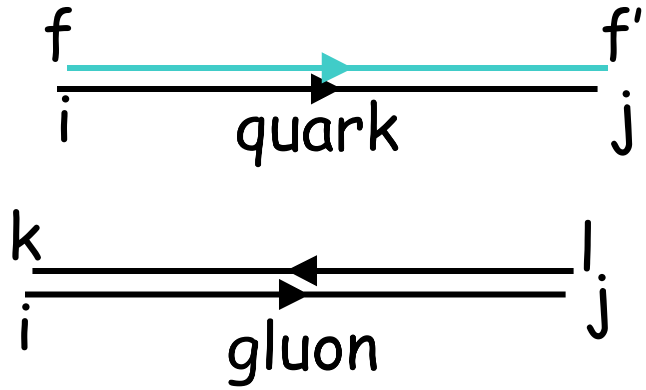
Matrix-valued fields, QCD

- The gauge fields of QCD, A_μ^a can be rewritten as NxN matrices through: $A_{\mu}{}^{i}{}_{j} = \Sigma_a (T^a)^i{}_j A_\mu{}^a$
- A gauge transformation acts as $A_\mu \to U A_\mu U^+$ + derivatives
- Quark fields are instead N-dim vectors of SU(N) and transform as $\psi \dashrightarrow U\psi$, $\psi^+ \dashrightarrow \psi^+ U^+$
- However they also carry a flavour label ($f = 1,2,..N_f$)
- The essential points of the 1/N expansion can be easily understood, once more, if we «decorate» the FD by adding lines that keep track of these colour/flavour indices:



$= \delta_{ij}\delta_{ff'} (q^2 - m^2)^{-1}$

Drawing Feynman diagrams à la 't Hooft



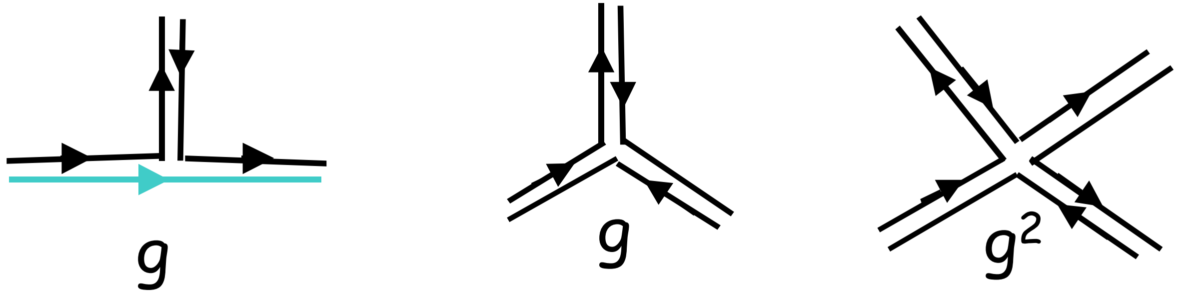
$= \delta_{ij}\delta_{ff'} \, (q^2-m^2)^{-1}$

$= \delta_{ij}\delta_{kl} \, q^{-2}$

propagators

quark-gluon

gluon-gluon

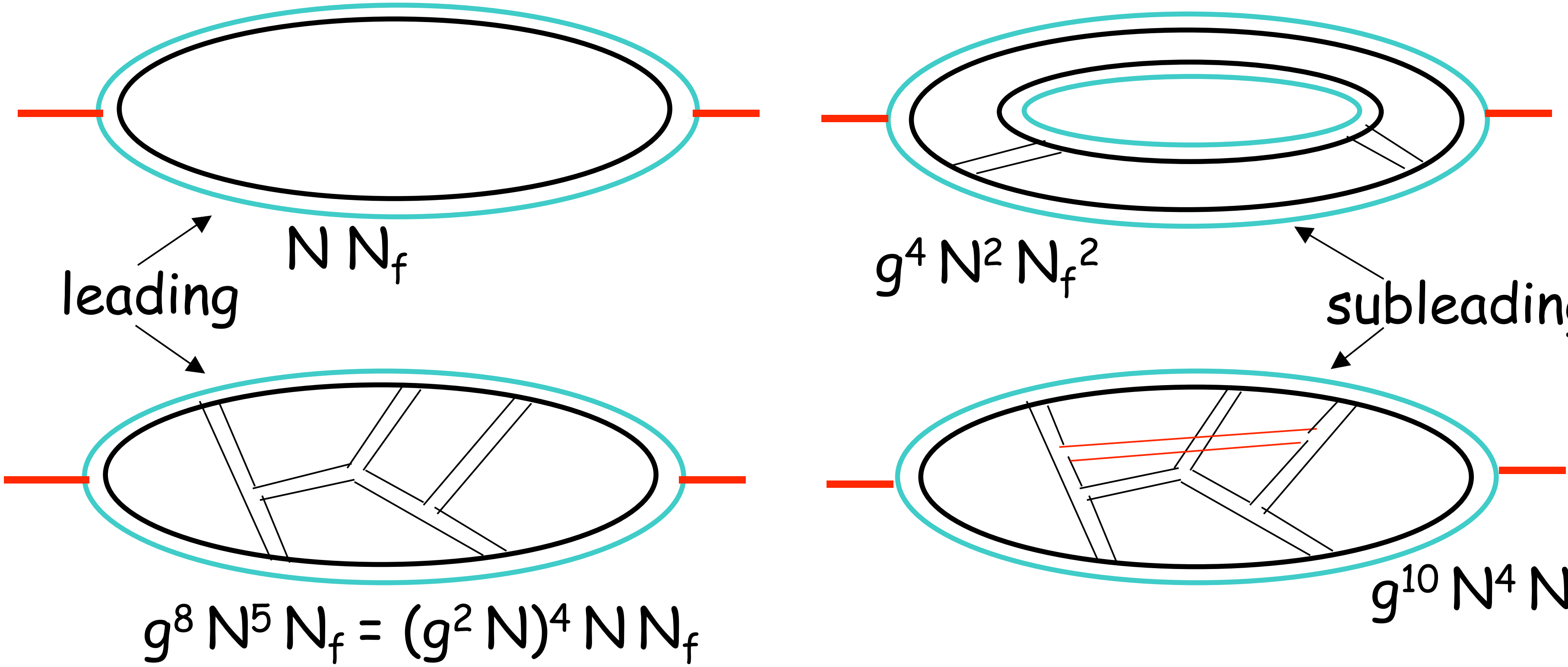


vertices

Counting powers of g, N and N_f is now straightforward

Counting powers of g, N, N_f, graph topology

Easiest to give examples:



Apart for powers of g^2N, FDs carry an extra power of N_f/N per quark loop and an extra power of $1/N^2$ per handle (degree of non-planarity)

Different large-N limits

This means that we can consider the following large N limits:

1. Take N -> infinity while keeping g^2N and N_f fixed: both quark loops and non planar diagrams are suppressed (this is the original 't-Hooft large-N expansion)

2. Take N -> infinity while keeping g^2N and N_f /N fixed: quark loops are included, but non planar diagrams are suppressed (so-called topological expansion)

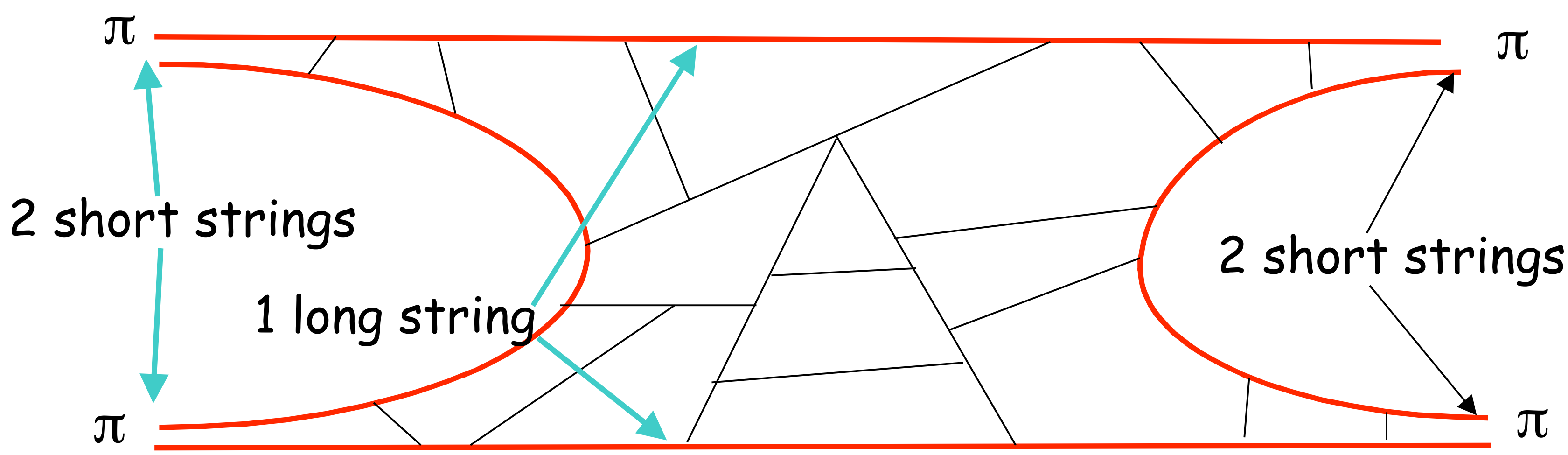
But why should one be interested in these limits?

Masses, widths, scattering at large N

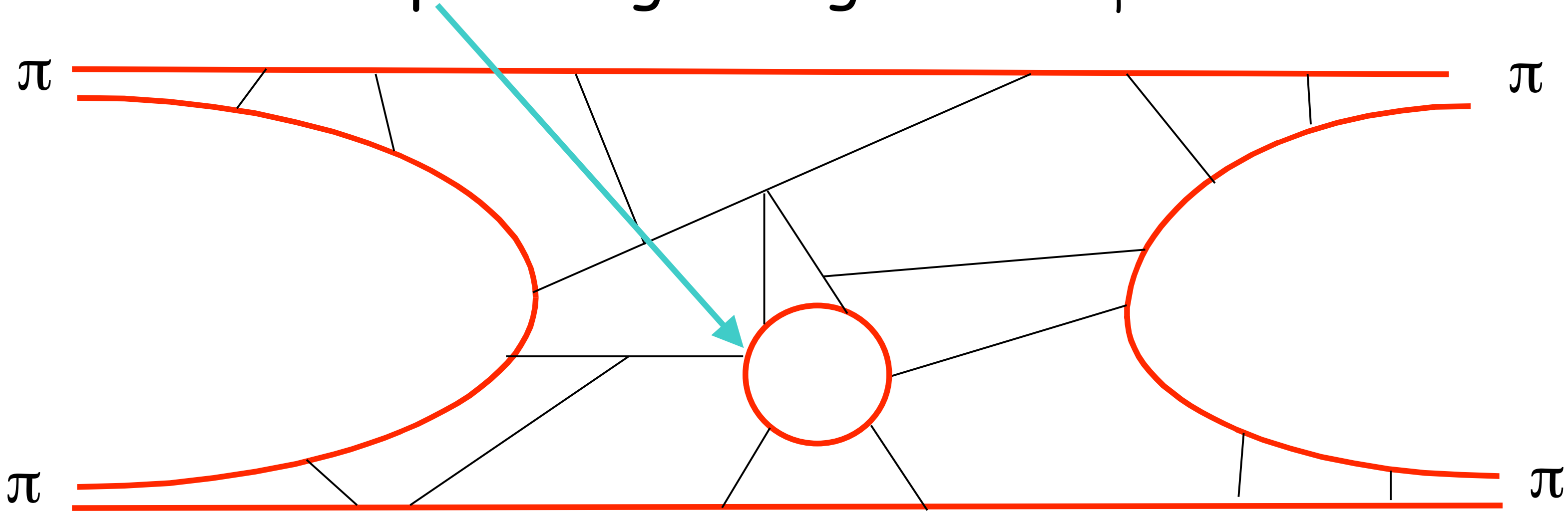
Let us keep the Λ_{QCD} parameter fixed as we send N to infinity. One can figure out how different quantities scale at large N

1. Masses of hadrons become N-independent in Λ_{QCD} units
2. Hadronic couplings, widths, scattering amplitudes vanish in Λ_{QCD} units
3. Couplings to singlet currents, leptonic, photon widths, blow up in Λ_{QCD} units (e.g. $\pi^0 \to \gamma\gamma$, $F_\pi \sim N$)
4. Topology of the leading Feynman diagrams at large N is the same as that describing the scattering of strings (reason for discovery of string theory in the sixties?)

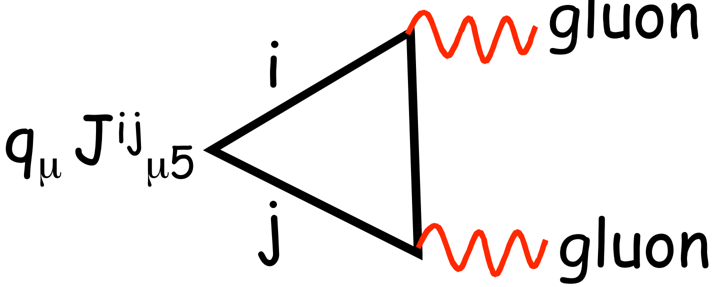
— =quark **Omitting double lines** — =gluon



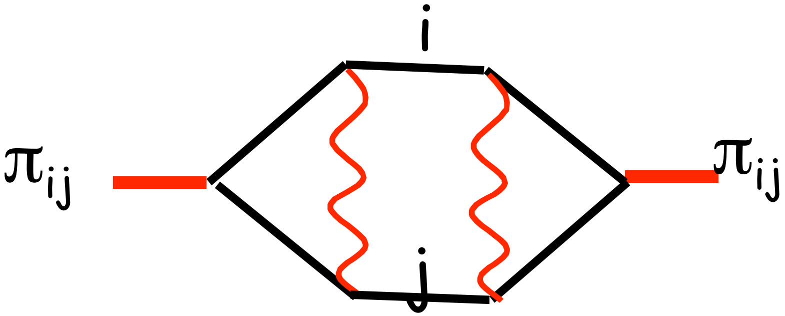
Breakup of long string costs N_f/N



The axial anomaly at large N

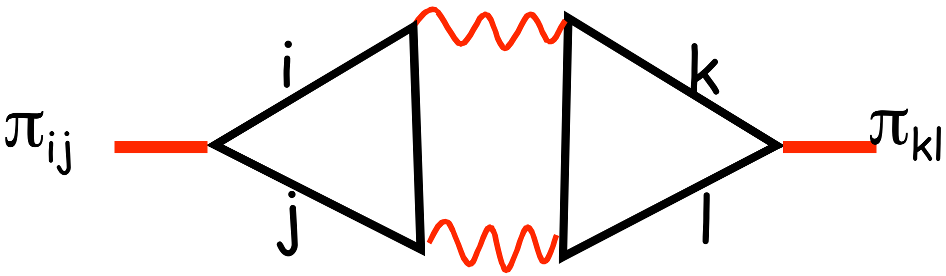


$\sim \delta_{ij}\, g^2\, Tr[F_{\mu\nu}\, {}^*F_{\mu\nu}] = O(1/N)$



$\sim g^4 N^3 = O(N)$

Anomaly contribution down by N_f/N



$\sim\ \delta_{ij}\, \delta_{kl}\, g^4\, N^2 = O(N_f)$

In conclusion

Using the large-N idea (at fixed N_f) we managed to «buy» a new small parameter controlling the anomaly-induced breaking of $U(1)_A$. The parameter a turns out to be $O(N_f/N)$

Recall that the PNGB mass matrix contains quantities like μ_{ij}^2, which are proportional to the quark masses, and a.
=> We are able to use perturbation theory in both if we consider both the small-mass and the large-N limits

This is what we shall do next time in order to achieve a quantitative solution of the U(1) problem