Some questions and remarks about $SL(2, \mathbf{R})$ cocycles

to Anatole Katok for his 60^{th} birthday

0. There have been many deep results about cocycle maps in recent years, especially in the quasiperiodic case with the achievements of Eliasson ([E]), Bourgain ([B], [B-G]) and Krikorian ([K1], [K2]) amongst others. The questions that we address here are much more elementary : most of the time, we will be interested in locally constant cocycles with values in $SL(2, \mathbf{R})$ over a transitive subshift of finite type ; we want to determine whether this cocycle map is uniformly hyperbolic and how it can bifurcate from uniform hyperbolicity. In our setting, parameter space is finite-dimensional, and we would like to describe it in the same way that one does for polynomials or rational maps, where hyperbolic components play a leading role in the picture. It appears that even in this very much simplified situation, several interesting questions appear.

I am very much indebted to Artur Avila, who explained to me the statement and the proof of proposition 6 below. I thank Raphael Krikorian for many helpful conversations, and also several participants from the July 2003 Oberwolfach meeting in Dynamical Systems, Artur Avila, Bassam Fayad, Federico Hertz, for fruitful discussions. This is in a sense very appropriate, since Anatole Katok has been for the last twenty years a most illuminating presence in this series of meetings which happen every odd year.

1. Recall that a matrix $A_0 \in SL(2, \mathbf{R})$ is said to be positive hyperbolic (resp. negative hyperbolic, resp. hyperbolic, resp. positive parabolic, resp. negative parabolic, resp. parabolic, resp. elliptic) if $trA_0 > 2$ (resp. $trA_0 < -2$, resp. $|trA_0| > 2$, resp. $trA_0 = 2$, resp. $trA_0 = -2$, resp. $|trA_0| = 2$, resp. $|trA_0| < 2$).

If $f : X \to X$ is a continuous map and $A : X \to SL(2, \mathbf{R})$ is another continuous map, the cocycle (f, A) defined by these data is the map :

$$\begin{aligned}(f, A) : \quad & X \times \mathbf{R}^2 \to X \times \mathbf{R}^2 \\ & (x, v) \mapsto (f(x), A(x)v) \ .\end{aligned}$$

We have $(f, A)^n = (f^n, A_n)$ for $n \geq 0$, where

$$A_n(x) = A(f^{n-1}(x)) \dots A(x) .$$

We will use the same symbol (f, A) to denote the quotient map at the projective level from $X \times \mathbf{P}^1(\mathbf{R})$ to itself.

Assume that X is compact.

A cocycle map (f, A) is said to be uniformly hyperbolic if there exists a (necessarily unique) continuous section

$$\begin{aligned} e_s : \quad & X \to \mathbf{P}^1(\mathbf{R}) \\ & x \mapsto e_s(x) \end{aligned}$$

which is invariant and repelling (in the $\mathbf{P}^1(\mathbf{R})$ direction). This is equivalent to satisfy the usual cone condition. When f is an homeomorphism, one also obtains another continuous section e_u which is invariant and contracting, with $e_u(x) \neq e_s(x)$ for all $x \in X$.

We will say that two cocycle maps associated to $A, \hat{A}$ are conjugated if there exists a continuous map $B : X \to SL(2, \mathbf{R})$ such that

$$(1_X, B) \circ (f, A) \circ (1_X, B)^{-1} = (f, \hat{A}) ,$$

i.e.

$$\hat{A}(x) = B(f(x))A(x)(B(x))^{-1}, \forall x \in X .$$

2. The following result is reminiscent of a classical theorem of Gottschalk and Hedlund ([G-H], [H]). Assume that $f : X \to X$ is a minimal homeomorphism of the compact metric space X. Let $A : X \to SL(2, \mathbf{R})$ a continuous map.

PROPOSITION 1 — *Assume that, for some $x_0 \in X$, the sequence of matrices $(A_n(x_0))_{n \geq 0}$ is bounded. Then (f, A) is conjugated to a cocycle map $(f, \hat{A})$ with $\hat{A}$ taking values in $SO(2, \mathbf{R})$.*

Proof — Consider the upper half-plane $\mathbf{H}$ with the usual $SL(2, \mathbf{R})$ action and define the fibered map

$$F(x, z) := (f(x), A(x).z)$$

from $X \times \mathbf{H}$ to itself. We equip $\mathbf{H}$ with its hyperbolic metric. The map A takes its values in $SO(2, \mathbf{R})$ iff the constant section $z \equiv i$ is invariant. We have to show that F leaves invariant some continuous section $\sigma : X \to \mathbf{H}$.

The hypothesis in the proposition means that the positive orbit $(F^n(x_0, i))_{n\geq 0}$ is relatively compact in $X \times \mathbf{H}$. Therefore we may choose a minimal compact non empty invariant subset L contained in the closure of this orbit.

For $x \in X$, denote by $L_x \subset \mathbf{H}$ the fiber of L over x. This is a compact subset of $\mathbf{H}$. As f is minimal, L_x is non empty for all $x \in X$. We have, for all $x \in X$:

$$A(x)L_x = L_{f(x)} \ .$$

Consider the function

$$x \mapsto \ \text{diam} \ \ L_x \ .$$

As f is minimal and $SL(2, \mathbf{R})$ acts on $\mathbf{H}$ by isometries, this function is contant. If the constant value is 0, L is the graph of a continuous section σ and we obtain the required result. We nous assume that we have

$$\text{diam} \ \ L_x \equiv D > 0 \ .$$

Set

$$\begin{aligned} L_x^{(2)} = & \ \ \{(z, z') \in L_x \times L_x \ , d_{\mathbf{H}}(z, z') = D\} \ , \\ L^{(2)} = & \ \ \{(x, z, z') \in X \times \mathbf{H}^2 \ , (z, z') \in L_x^{(2)}\} \ ; \end{aligned}$$

then $L^{(2)}$ is compact, each $L_x^{(2)}$ is compact non empty and we have

$$(A(x) \times A(x)) \ (L_x^{(2)}) = L_{f(x)}^{(2)} \ , \ \forall x \in X \ .$$

Denote by $m(z, z')$ the midpoint of the geodesic segment $[z, z']$. Define

$$\begin{aligned} L'_x = & \ \ m(L_x^{(2)}) \ , \\ L' = & \ \ \{(x, w) \in X \times \mathbf{H}, w \in L'_x\} \ . \end{aligned}$$

Then L' is compact, each L'_x is compact, non empty, contained in the convex closure $\hat{L}_x$ of L_x, and we have (as $SL(2, \mathbf{R})$ acts by isometries)

$$A(x) \ (L'_x) = L'_{f(x)} \ , \ \ \forall x \in X \ .$$

Claim — One has diam $L'_x \leq D' < D$ with

$$chD' = \frac{chD}{chD/2} \ .$$

Proof of claim — In a triangle in hyperbolic space with a right angle, adjacent sides d, d' and opposite side h, we have "Pythagoras theorem" :

$$ch(h) = ch(d)ch(d')$$

Let $(z_\varepsilon, z'_\varepsilon) \in L^{(2)}_x$, $\varepsilon = 0, 1$ and $w_\varepsilon = m(z_\varepsilon, z'_\varepsilon)$. From $d_{\mathbf{H}}(z_0, z_1) \leq D$, $d_{\mathbf{H}}(z'_0, z_1) \leq D$, we have $d_{\mathbf{H}}(w_0, z_1) \leq D$. Similarly we have $d_{\mathbf{H}}(w_0, z'_1) \leq D$. This gives

$$\begin{aligned} chD \ &\geq ch(\max(d_{\mathbf{H}}(w_0, Z_1) \ , \ d_{\mathbf{H}}(w_0, Z'_1))) \\ &\geq chD/2 \ chd(w_0, w_1) \ . \end{aligned}$$

□

Let $L^{(1)}$ be a minimal non empty compact invariant subset of L' ; the fibers $L^{(1)}_x$ are contained in $\hat{L}_x$ and their constant diameter satisfy $D^{(1)} < D$. Among all minimal non empty compact invariant subsets $\tilde{L}$, whose fibers $\tilde{L}_x$ are contained in $\hat{L}_x$, choose one for which the (constant) diameter of the fibers is minimal. By the argument above, this diameter has to be 0 and we find the required invariant section. □

3. The following elementary result is probably well-known, but I could not find a reference.

Let $f : X \to X$ be a continuous self-map of the compact metric space X and let $A : X \to SL(2, \mathbf{R})$ be continuous.

PROPOSITION 2 — *The cocycle map (f, A) is uniformly hyperbolic iff the matrices $A_n(x)$ are uniformly exponentially increasing : there exists $c > 0, \lambda > 1$ such that*

$$\|A_n(x)\| \geq c\lambda^n$$

for all $n \geq 0, x \in X$.

Proof — If (f, A) is uniformly hyperbolic, A_n is uniformly exponentially increasing. Let us show the converse.

The norm of matrices is the operator norm for matrices acting on the Euclidean plane. For a matrix $M \in SL(2, \mathbf{R})$ with $\|M\| > 1$ (i.e. $M \notin SO(2, \mathbf{R})$), we denote by $e(M)$ the (unique) maximally dilated direction. We have the

LEMMA 1 — *Let $M, M' \in SL(2, \mathbf{R}),\ C_1 > 0$ such that*

$$C_1^{-1} \quad \|M\| \le \|M'\| \le C_1\|M\| \ ,$$
$$\|M\| > 1 \ \ , \ \ \|M'M\| > 1 \ .$$

The angle between $e(M), e(M'M)$ *is* $\le C_0\|M'M\|^{-1}$, *with* C_0 *depending only on* C_1.

Proof — It is sufficient to consider the case where

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad M' = \begin{pmatrix} \lambda' & 0 \\ 0 & \lambda'^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ,$$

where $\lambda = \|M\|$, $\lambda' = \|M'\|$. Then we have

$$M'M = \begin{pmatrix} \lambda'\lambda\cos\theta & -\lambda'\lambda^{-1}\sin\theta \\ \lambda\lambda'^{-1}\sin\theta & \lambda'^{-1}\lambda^{-1}\cos\theta \end{pmatrix}$$

If $\|M'M\|$ is not large ($\le C_0\pi^{-1}$), there is nothing to prove. Otherwise we have

$$\|M'M\| = \lambda\lambda'|\cos\theta| + O(1) \ .$$

Maximizing the square norm of $M'M\begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix}$ over ω is now an easy computation leading to the required result. □

Let $n > 0, x \in X, C_1 = \max_x \|A(x)\|$; choose $m > 0$ such that

$$C_1^{-1}\|A_n(x)\| \le \|A_m(f^n(x))\| \le C_1\|A_n(x)\| \ .$$

We have then

$$C_1^{-3}\|A_{n+1}(x)\| \le \|A_{m-1}(f^{n+1}(x))\| \le C_1^3\|A_{n+1}(x)\| \ .$$

From the lemma, there exists $C_0 > 0$ such that

$$\mathrm{angle}(e(A_n(x)),\ e(A_{m+n}(x))) \le C_0\|A_{m+n}(x)\|^{-1} \ ,$$

$$\mathrm{angle}(e(A_{n+1}(x)),\ e(A_{m+n}(x))) \le C_0\|A_{m+n}(x)\|^{-1}$$

if n is large enough (to have $c\lambda^n > 1$). We conclude that

$$\text{angle}(e(A_n(x)),\ e(A_{n+1}(x))) \leq 2C_0(C\lambda^n)^{-1}$$

The sequence of continuous sections

$$x \mapsto e(A_n(x))\ ,\ n \geq n_0, x \in X$$

converges uniformly exponentially fast to a continuous section $x \mapsto e_\infty(x)$. The direction $e_s(x)$ orthogonal to $e_\infty(x)$ is easily checked to be the required stable direction. $\square$

4. In the rest of the paper, we will concentrate on more specific cocycles.

Let $\mathcal{A}$ be a finite alphabet and let $X \subset \mathcal{A}^{\mathbf{Z}}$ be a transitive subshift of finite type (Most of the time, we will be satisfied with the case of the full shift on two letters 0, 1). We will restrict our discussion to maps $A : X \to SL(2, \mathbf{R})$ which **only depend** on the letter in position 0. Such a map is thus determined by a family $(A^\alpha)_{\alpha\in\mathcal{A}} \in (SL(2, \mathbf{R}))^{\mathcal{A}}$. This last space is the **parameter space** for the cocycle maps we are interested in.

We denote by $\mathcal{H} \subset (SL(2, \mathbf{R}))^{\mathcal{A}}$ the open set of parameters for which the associated cocycle map is uniformly hyperbolic.

The finite group $\{+1, -1\}^{\mathcal{A}}$ acts on the parameter space through

$$(\varepsilon_\alpha) \cdot (A^\alpha) = (\varepsilon_\alpha A^\alpha)$$

and $\mathcal{H}$ is invariant under this action.

5. We denote by P_X the set of periodic orbits of X. Let $(A^\alpha) \in \mathcal{H}, \pi \in P_X, x \in \pi, n$ the minimal period of π. The matrix $A_n(x)$ is hyperbolic and its trace depends only on π, not on the choice of $x \in \pi$. We thus define a map

$$\begin{aligned} \tau :\ & \mathcal{H} \to \{+1, -1\}^{P_X} \\ & (A^\alpha) \mapsto (sgn(tr(A_n(x))))_{\pi\in P_X} \end{aligned}$$

which is constant on each connected component of $\mathcal{H}$. This map is covariant with the actions of $\{+1, -1\}^a$ if one defines

$$(\varepsilon_\alpha)_{\alpha\in a} \cdot (\tau_\pi)_{\pi\in P_X} = (\prod_\alpha \varepsilon_\alpha^{n_\alpha(\pi)} \cdot \tau_\pi)_{\pi\in P_X}\ ,$$

where $n_\alpha(\pi)$ is the number of times the letter α appears in one period of π.

Question 1 : Is the quotient map $\bar{\tau}$ from $\pi_0(\mathcal{H})$ into $\{+1,-1\}^{P_X}$ essentially injective ?

We will see in the sequel that $\bar{\tau}$ is not always injective : there may exist a component $\mathcal{U}_0$ and a matrix $A^* \in GL(2,\mathbf{R})$, $\det A^* < 0$, such that the component $A^*\mathcal{U}_0(A^*)^{-1} = \mathcal{U}_1$ is distinct from $\mathcal{U}_0$. Then $\mathcal{U}_1$ and $\mathcal{U}_0$ have the same image under $\bar{\tau}$. Is this the only reason for the lack of injectivity of $\bar{\tau}$?

Problem 1 : Describe the image of τ.

6. Let A^* be a positive hyperbolic matrix. The parameter such that $A^\alpha = A^*$ for all $\alpha \in \mathcal{A}$ belongs to $\mathcal{H}$. The open subset of $SL(2,\mathbf{R})$ formed by the positive hyperbolic matrices is connected. We call **principal component** the connected component of $\mathcal{H}$ which contains these uniformly hyperbolic parameters. The image of this component under τ satisfies $\tau_\pi = +1$ for all $\pi \in P_X$.

Question 1' : Is there another component with the same image under τ ?

When X is the full shift $\mathcal{A}^{\mathbf{Z}}$, it is possible to give a nice description of the principal component and its boundary.

We assume till further notice, that $X = \mathcal{A}^{\mathbf{Z}}$. Then, a necessary condition for the parameter $(A^\alpha)_{\alpha\in\mathcal{A}}$ to be uniformly hyperbolic is that each matrix A^α should be hyperbolic. If $(A^\alpha)_{\alpha\in\mathcal{A}}$ belongs to the principal component, we conclude that each A^α should be positive hyperbolic. Denote by $e_s(A^*)$ (resp. $e_u(A^*)$) the stable (resp. unstable) direction of an hyperbolic matrix A^*.

PROPOSITION 3 — *A parameter $(A^\alpha)_{\alpha\in\mathcal{A}}$ belongs to the principal component iff the following conditions are satisfied*
(i) each A^α is positive hyperbolic ;
(ii) there exist two disjoint intervals I_s, I_u in $\mathbf{P}^1(\mathbf{R})$ such that $e_s(A^\alpha) \in I_s, e_u(A^\alpha) \in I_u$ for every $\alpha \in \mathcal{A}$.

Remark : When X is not a full shift, it may happen that $(A^\alpha)_{\alpha\in\mathcal{A}}$ is uniformly hyperbolic but not all A^α are hyperbolic. Consider for instance $\mathcal{A} = \{0,1,2\}, A^0 = 1, A^1 = A^2 = A^*$ hyperbolic, and X the subshift of $\mathcal{A}^{\mathbf{Z}}$ which allows all transitions except consecutive 0's.

Proof of proposition — We first observe that conditions (i) and (ii) imply uniform hyperbolicity : indeed, condition (ii) allows to define a constant cone field which is sent strictly inside itself by each $A^\alpha, \alpha \in \mathcal{A}$.

It is also very easy to check that the open set of parameters U defined by conditions (i), (ii) is connected.

To end the proof, we will show that any point in the boundary ∂U cannot belong to $\mathcal{H}$. let $(A^\alpha)_{\alpha\in\mathcal{A}} \in \partial U$. If A^α is not positive hyperbolic for some $\alpha \in \mathcal{A}$, it is positive parabolic. The fixed point $\ldots\alpha\alpha\alpha\ldots$ of $X = \mathcal{A}^{\mathbf{Z}}$ then prevents uniform hyperbolicity. Assume now that A^α is positive hyperbolic for each $\alpha \in \mathcal{A}$. Then condition (ii) is not satisfied by $(A^\alpha)_{\alpha\in\mathcal{A}}$ but is satisfied by some arbitrarily small perturbations ; this means that there exist distinct $\alpha, \beta \in \mathcal{A}$ with $e_s(A^\alpha) = e_u(A^\beta)$. Looking at the heteroclinic orbit $\ldots\beta\beta\beta\alpha\alpha\alpha\ldots$, it is clear that the parameter is not uniformly hyperbolic. □

The proof has also shown the

PROPOSITION 4 — *Let $(A^\alpha)_{\alpha\in\mathcal{A}}$ be a parameter on the boundary of the principal component of $\mathcal{H}$. Either some A^α is positive parabolic or there exist distinct $\alpha, \beta \in \mathcal{A}$ with $e_u(A^\beta) = e_s(A^\alpha)$ (or both).*

Problem 2 : Describe the principal component and its boundary when X is a general subshift of finite type.

7. In this section and the next one, we assume that $X = \{0,1\}^{\mathbf{Z}}$ is the full shift on two letters. We will describe components of $\mathcal{H} \subset (SL(2,\mathbf{R}))^2$ which are not principal (nor deduced from the principal one by the action of $\{+1,-1\}^2$!).

Let $(A^0, A^1) \in (SL(2,\mathbf{R}))^2$ such that

$(*)$ A^0, A^1 are positively hyperbolic and A^0A^1 (and thus also A^1A^0) is negatively hyperbolic.

PROPOSITION 5 — *Condition (*) defines the union of two connected components of $\mathcal{H}$.*

Remark : As A^0A^1 is negatively hyperbolic while A^0, A^1 are positively hyperbolic, we see that these components are not principal nor deduced from the principal one by the action of an element of $\{-1,+1\}^2$.

Proof — Let $\mathcal{U}$ be the open subset of $(SL(2,\mathbf{R}))^2$ defined by (*). We first observe that for $(A^0, A^1) \in \mathcal{U}$, the directions $e_u(A^0), e_u(A^1), e_s(A^0), e_s(A^1)$ are always distinct : indeed otherwise A^0, A^1 would have a common eigenvector ; as A^0, A^1 are positive hyperbolic, the corresponding eigenvalues would be positive ; but then A^0A^1 would also be positive parabolic or hyperbolic.

For the cyclic order of $\mathbf{P}^1(\mathbf{R})$, we may only have

$$e_u(A^0) < e_s(A^1) < e_u(A^1) < e_s(A^0) < e_u(A^0) \tag{1}$$

or

$$e_u(A^0) < e_s(A^0) < e_u(A^1) < e_s(A^1) < e_u(A^0)\ , \tag{2}$$

the other possibilities being forbidden by proposition 3. By symmetry, it is sufficient to consider the first of the two cases above.

Choose vectors e_0, e_1 in the directions of $e_u(A^0), e_u(A^1)$ such that (e_0, e_1) is an oriented basis of $\mathbf{R}^2$. In the basis (e_0, e_1), the matrices A^0, A^1 are rewritten as

$$\tilde{A}^0 = \begin{pmatrix} \lambda_0 & u_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad , \quad \lambda_0 > 1, \;\; u_0 > 0 \tag{3}$$

$$\tilde{A}^1 = \begin{pmatrix} \lambda_1^{-1} & 0 \\ u_1 & \lambda_1 \end{pmatrix} \quad , \quad \lambda_1 > 1, \;\; u_1 < 0 \tag{4}$$

We have then

$$Tr(\tilde{A}^0 \tilde{A}^1) = \lambda_0 \lambda_1^{-1} + \lambda_0^{-1} \lambda_1 + u_0 u_1$$

and the condition that $A^0 A^1$ is negatively hyperbolic is equivalent to

$$|u_0 u_1| > \lambda_0 \lambda_1^{-1} + \lambda_0^{-1} \lambda_1 + 2 \; . \tag{5}$$

It is now clear that conditions (1), (3), (4), (5) define a connected subset of $(SL(2,\mathbf{R}))^2$, and therefore that $\mathcal{U}$ has two connected components associated to the cases (1), (2) above.

Let

$$A^0(\lambda) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda^{-1} \end{pmatrix} \quad , \quad A^1(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ -\lambda & \lambda \end{pmatrix} \; .$$

For $\lambda > 2$, $(A^0(\lambda), A^1(\lambda))$ belongs to the component of $\mathcal{U}$ associated to (1).

We have

$$A^0(\lambda) A^1(\lambda) = \begin{pmatrix} 1-\lambda^2 & \lambda^2 \\ -1 & 1 \end{pmatrix}$$

$$A^1(\lambda) A^0(\lambda) = \begin{pmatrix} 1 & 1 \\ -\lambda^2 & 1-\lambda^2 \end{pmatrix}$$

Computing the stable and unstable directions, we obtain, for the cyclic order of $\mathbf{P}^1(\mathbf{R})$:

$$\begin{aligned} &e_u(A^0(\lambda)) < e_u(A^0(\lambda)A^1(\lambda)) < e_s(A^0(\lambda)A^1(\lambda)) < e_s(A^1(\lambda)) < e_u(A^1(\lambda)) < \\ &< e_u(A^1(\lambda)A^0(\lambda)) < e_s(A^1(\lambda)A^0(\lambda)) < e_s(A^0(\lambda)) < e_u(A^0(\lambda)) \; . \end{aligned} \tag{6}$$

This cyclic ordering of directions must be valid in all the component of $\mathcal{U}$ associated to (1) : indeed, we have seen that in $\mathcal{U}$, A^0 and A^1 cannot have a common eigenvector and therefore $A^0 A^1$ or $A^1 A^0$ cannot have an eigenvector which is also an eigenvector of A^0 or A^1. Therefore all inequalities in (6) must remain strict through the component of $\mathcal{U}$.

From (6), it is easy to see that parameters in the component of $\mathcal{U}$ associated to (1) are uniformly hyperbolic : choose directions e_0, e_1, e_{01}, e_{10} such that

$$\begin{aligned} &e_s(A^0) < e_0 < A^0(e_0) < e_u(A^0) \ , \\ &e_s(A^1) < e_1 < A^1(e_1) < e_u(A^1) \ , \\ &e_u(A^0A^1) < A^0(e_{10}) < e_{01} < e_s(A^0A^1) \ , \\ &e_u(A^1A^0) < A^1(e_{01}) < e_{10} < e_s(A^1A^0) \ , \end{aligned}$$

(observe that $A^1(e_u(A^0A^1)) = e_u(A^1A^0)$, ...).

Set

$$\begin{aligned} \mathcal{C}_0 &= \{e_0 < e < e_{10}\} \ , \\ \mathcal{C}_1 &= \{e_1 < e < e_{01}\} \ , \end{aligned}$$

we have

$$\begin{aligned} A^0\mathcal{C}_0 &\subset\subset \mathcal{C}_0 \cap \mathcal{C}_1 \ , \\ A^1\mathcal{C}_1 &\subset\subset \mathcal{C}_0 \cap \mathcal{C}_1 \ , \end{aligned}$$

and thus we can construct conefields which satisfy the conditions of uniform hyperbolicity.

On the other hand, it is obvious that a parameter on the boundary of $\mathcal{U}$ cannot be uniformly hyperbolic. The proof of the proposition is complete. □

On the boundary of any of the two components of $\mathcal{H}$ defined by (*), one of the three matrices A^0, A^1, A^0A^1 must be parabolic. This is in sharp contrast with the boundary of the principal component : let $A^0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}$, $A^1 = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_1 \end{pmatrix}$, with $\lambda_0, \lambda_1 > 1$ and $\frac{\log \lambda_0}{\log \lambda_1}$ irrational. Then (A^0, A^1) is on the boundary of the principal component of $\mathcal{H}$, but there is no periodic orbit of $\{0,1\}^{\mathbf{Z}}$ for which the corresponding product is parabolic.

Let A^* be any matrix in $GL(2, \mathbf{R})$ with negative determinant. Then A^* conjugates each of the two components of $\mathcal{H}$ contained in $\mathcal{U}$ to the other (see the comment after question 1). The two components have the same image under $\bar{\tau}$, that we will now determine.

Let π be a periodic orbit of the shift, which is not a fixed point ; let n be its period. We can choose $x \in \pi$ such that the period of π starting at x is

$$0^{m_1} \ 1^{n_1} \ 0^{m_2} \ 1^{n_2} \ \dots \ 0^{m_l} \ 1^{n_l}$$

(with $n = m_1 + n_1 + \ldots + m_l + n_l$). An easy calculation shows that, for $A^\varepsilon = A^\varepsilon(\lambda)$ as in the proof of proposition 5, we have

$$Tr \ A_n(x, \lambda) = (-1)^l \lambda^n + 0(\lambda^{n-2}) \ .$$

We conclude that the image of $\mathcal{U}$ under $\bar{\tau}$ is given by

$$\tau_\pi = (-1)^l \quad .$$

8. We describe in a shorter way another pair of components of $\mathcal{H}$ when X is the full shift on two letters.

For $(A^0, A^1) \in (SL(2,\mathbf{R}))^2$, we now consider the condition

$(**)$ A^0, A^1, A^0A^1 are positive hyperbolic, $A^0(A^1)^2$ is negative hyperbolic.

This open set is made of two symmetric parts corresponding to cyclic orderings (1) or (2). We consider the case where (1) holds.

After conjugacy in $SL(2,\mathbf{R})$, we can replace A^0, A^1 by

$$\tilde{A}^0 = \begin{pmatrix} \lambda_0 & u_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad , \quad \tilde{A}^1 = \begin{pmatrix} \lambda_1^{-1} & 0 \\ -u_1 & \lambda_1 \end{pmatrix}$$

with $\lambda_0, \lambda_1 > 1, u_0, u_1 > 0$. Condition $(**)$ is then equivalent to

$$(**') \;\; \lambda_0\lambda_1^{-1} + \lambda_0^{-1}\lambda_1 - 2 > u_0u_1 > \frac{\lambda_0\lambda_1^{-2} + \lambda_0^{-1}\lambda_1^2 + 2}{\lambda_1 + \lambda_1^{-1}} \quad .$$

One checks that this last relation defines a connected set in $\{\lambda_0 > 1, \lambda_1 > 1, u_0 > 0, u_1 > 0\}$. Set

$$A^0(\lambda) = \begin{pmatrix} \lambda^2 & 1 \\ 0 & \lambda^{-2} \end{pmatrix} \quad , \quad A^1(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ -1 & \lambda \end{pmatrix} .$$

For $\lambda > 1, \lambda + \lambda^{-1} > 4$, this value of parameter satisfies $(**')$ (or $(**)$). One then computes, for large λ, the cyclic ordering of the stable and unstable directions for $A^0, A^1, A^0A^1, A^1A^0, A^0(A^1)^2, A^1A^0A^1, (A^1)^2A^0$ (we did not indicate the dependance on λ). One obtains

$$\begin{gathered} e_u(A^0) < e_u(A^0(A^1)^2) < e_s(A^0(A^1)^2) < e_s(A^1) < e_u(A^1) < \\ < e_u((A^1)^2A^0) < e_s((A^1)^2A^0) < e_s(A^0) < e_s(A^1A^0) < e_u(A^1A^0) < \\ < e_u(A^1A^0A^1) < e_s(A^1A^0A^1) < e_s(A^0A^1) < e_u(A^0A^1) < e_u(A^0) \quad . \end{gathered}$$

Condition $(**)$ implies that A^0, A^1 cannot have a common eigenvector. This implies that none of the following pairs can have a common eigenvector : $(A^0, A^0(A^1)^2), (A^0(A^1)^2, A^1)$, $(A^1, (A^1)^2A^0), ((A^1)^2A^0, A^0), (A^0, A^1A^0), (A^1A^0, A^1A^0A^1), (A^1A^0A^1, A^0A^1), (A^0A^1, A^0)$.

This implies in turn that the cyclic ordering above is valid through the connected set defined by $(**)$ and (1).

The cyclic ordering as above allows to construct cone fields with the properties required for uniform hyperbolicity.

One chooses $e_1, e_{01}, e_{10}, e_{110}, e_{101}, e_{011}$ such that

$$\begin{aligned}
&e_s(A^1) < e_1 < A^1(e_1) < e_u(A^1) \ , \\
&e_s(A^0A^1) < e_{01} < A^0(e_{10}) < e_u(A^0A^1) \ , \\
&e_s(A^1A^0) < e_{10} < A^1(e_{01}) < e_u(A^1A^0) \ , \\
&e_s((A^1)^2A^0) < e_{110} < A^1(e_{101}) < e_u((A^1)^2A^0) \ , \\
&e_s(A^1A^0A^1) < e_{101} < A^1(e_{011}) < e_u(A^1A^0A^1) \ , \\
&e_s(A^0(A^1)^2) < e_{011} < A^0(e_{110}) < e_u(A^0(A^1)^2) \ .
\end{aligned}$$

Next we set

$$\begin{aligned}
\mathcal{C}_0 &= \{e_{10} < e < e_{110}\} \ , \\
\mathcal{C}_{01} &= \{e_{01} < e < e_{011}\} \ , \\
\mathcal{C}_{011} &= \{e_{10} < e < e_{101}\} \ , \\
\mathcal{C}_{111} &= \{e_1 < e < e_{110}\} \ ,
\end{aligned}$$

with the following properties

$$\begin{aligned}
&A^0\mathcal{C}_0 \subset\subset \mathcal{C}_{01} && , && \\
&A^1\mathcal{C}_{01} \subset\subset \mathcal{C}_{011} && , && \mathcal{C}_{01} \subset \mathcal{C}_0 \ , \\
&A^1\mathcal{C}_{011} \subset\subset \mathcal{C}_{111} && , && \mathcal{C}_{011} \subset \mathcal{C}_0 \ , \\
&A^1\mathcal{C}_{111} \subset\subset \mathcal{C}_{111} && , && \mathcal{C}_{111} \subset \mathcal{C}_0 \ .
\end{aligned}$$

For $\underline{x} = (x_n)_{n\in\mathbf{Z}} \in \{0,1\}^{\mathbf{Z}}$, we set

$$\mathcal{C}(\underline{x}) = \begin{cases} \mathcal{C}_0 & \text{if} \quad x_0 = 0 \ , \\ \mathcal{C}_{01} & \text{if} \quad x_0 = 1 \ , \ x_{-1} = 0 \ , \\ \mathcal{C}_{011} & \text{if} \quad x_0 = x_{-1} = 1 \ , \ x_{-2} = 0 \ , \\ \mathcal{C}_{111} & \text{if} \quad x_0 = x_{-1} = x_{-2} = 1 \ . \end{cases}$$

The required properties are satisfied. The open connected set defined by $(**)$ and (1) is a component of $\mathcal{H}$. On the boundary of this component, one of the matrices $A^0, A^1, A^0A^1, A^0(A^1)^2$ has to be parabolic.

The image under $\bar{\tau}$ of this component is as follows : let π be a periodic orbit ; write its minimal period, after cyclic permutation if necessary, as

$$0^{m_0} \ 1^{n_0} \ 0^{m_1} \ 1^{n_1} \ \ldots \ 0^{m_l} \ 1^{n_l}$$

with $m_i, n_i > 0$ (we assume that π is not a fixed point) ; let l^* be the number of indices i such that $n_i > 1$. We have

$$\tau_\pi = (-1)^{l^*}$$

The verification is left to the reader.

9. Consider again general transitive subshifts of finite type. For the components of $\mathcal{H}$ that we have considered, the boundary points are of one (at least) of the two following types

- ("parabolic periodic") there exists a periodic orbit π in X for which the associated product matrix is parabolic ;
- ("heteroclinic connexion") there exist periodic points x^+, x^- (not necessarily distinct), a point $\underline{x} \in W^s(x^+) \cap W^u_{\mathrm{loc}}(x^-)$ and a large integer M such that

$$A_M(\underline{x}) e_u(x^-) = e_s(x^+) \ .$$

Each of the two phenomena obviously prevents uniform hyperbolicity.

Question 2 : Are there boundary points of $\mathcal{H}$ which are not of one of these two types ?

There are many other questions relative to the boundary points of $\mathcal{H}$.

Question 3 : Let $(A^\alpha)_{\alpha \in A}$ be a boundary point of $\mathcal{H}$. Does there exist a component of $\mathcal{H}$, and a neighbourhood of this boundary point, such that the intersection of $\mathcal{H}$ with this neighbourhood is contained in the component ?

A weaker version of this question is :

Question 3' : Is any boundary point of $\mathcal{H}$ a boundary point of a connected component of $\mathcal{H}$?

Let $\mathcal{E}$ be the set of parameters such that there exists a periodic point x in X for which the matrix $A_n(x)$ is elliptic (n being the period of x). This is an open subset disjoint of $\mathcal{H}$.

Question 4 : Does one have

$$\partial \mathcal{H} = \partial \mathcal{E} = (\mathcal{H} \cup \mathcal{E})^c \ ?$$

A point in $\partial \mathcal{H}$ of the "parabolic periodic" type belongs to $\partial \mathcal{E}$; we will also see in the last section some evidence in the "heteroclinic connexion" case. The following result was explained to me by Artur Avila [A].

PROPOSITION 6 — *$\mathcal{H} \cup \mathcal{E}$ is dense in parameter space. More precisely, one has $\mathcal{H}^c = \bar{\mathcal{E}}$.*

Proof — Let A^* be a matrix in $SL(2, \mathbf{C})$ with $tr A^* \in \mathbf{C} - [-2, 2]$. Then A^* has two eigenvalues λ_u and $\lambda_s = \lambda_u^{-1}$ satisfying $|\lambda_u| > 1 > |\lambda_s|$.

As a Möbius transformation of the Riemann sphere, A^* has two fixed points with multipliers λ_u^2, λ_s^2. The eigenvalues λ_u, λ_s depend holomorphically on A^* ; therefore, if $A^* = A^*(\theta)$ depend holomorphically on a complex parameter θ, the function $\theta \mapsto \log |\lambda_u(A^*(\theta))|$ is positive harmonic. These considerations apply to prove the following :

LEMMA 2 — *Let $A^1, \ldots, A^n \in SL(2, \mathbf{R})$; for $\theta \in \mathbf{R}$, define $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $A^*(\theta) = A^1 R_\theta A^2 R_\theta \ldots A^n R_\theta$. Assume that, for some $\theta_0 > 0$, $A^*(\theta)$ is hyperbolic for all $|\theta| < \theta_0$. Then we have*

$$\|A^*(0)\| \geq \ \exp\,(\frac{1}{2\sqrt{3}}\,\theta_0 n)\ .$$

Proof of lemma 2 — Consider the action of $SL(2, \mathbf{R})$ on the Poincaré disk ; then R_θ acts by $w \mapsto e^{2i\theta} w$. For $Im\theta > 0, R_\theta \in SL(2, \mathbf{C})$ still acts by $w \mapsto e^{2i\theta} w$ and thus contracts the Poincaré metrics by a factor equal or smaller than $\exp(-2\ Im\ \theta)$. On the other hand, $A^1, \ldots, A^n$ act by isometries ; therefore, the Poincaré metrics is contracted by a factor equal or smaller than $\exp(-2n\ Im\ \theta)$ by $A^*(\theta)$; the multiplier λ_s^2 then satisfies $|\lambda_s^2| \leq \exp(-2n\ Im\ \theta)$, and we obtain

$$\log|\lambda_u(A^*(\theta))| \geq n\ Im\ \theta\ .$$

Similarly, one obtains

$$\log|\lambda_u(A^*(\theta))| \geq n\ |Im\ \theta|$$

for $Im\ \theta < 0$. The matrix $A^*(\theta)$ satisfies

$$Tr(A^*(\theta)) \notin [-2, +2]$$

in the disk $\{|\theta| < \theta_0\} \subset \mathbf{C}$. Harmonicity of $\log|\lambda_u|$ now gives

$$\begin{aligned} \log\|A^*(0)\| &\geq \log|\lambda_u(A^*(0))| \\ &= \tfrac{1}{2\pi} \int_0^{2\pi} \log|\lambda_u(A^*(\theta_0 e^{it}))|\ dt \\ &\geq \tfrac{1}{2\sqrt{3}}\ \theta_0 n\ . \end{aligned}$$

□

Let now $(A^\alpha)_{\alpha\in\mathcal{A}}$ be a parameter which is not in the closure of $\mathcal{E}$. We show, using proposition 2, that it belongs to $\mathcal{H}$. Set $A^\alpha(\theta) = A^\alpha R_\theta$; by hypothesis, there exists θ_0 such that $(A^\alpha(\theta))_{\alpha\in\mathcal{A}}$ does not belong to the closure of $\mathcal{E}$ for $|\theta| < \theta_0$. Let y be a periodic point of X, of period n. A rotation number argument shows that $A_n(y,\theta)$ must be hyperbolic for $|\theta| < \theta_0$; then by the lemma, we get

$$\|A_n(y)\| \geq\ \exp\left(\frac{1}{2\sqrt{3}}\ \theta_0 n\right)$$

From this inequality, we immediately deduce that there exists $c > 0$ such that, for all $x \in X, n \geq 0$:

$$\|A_n(x)\| \geq c\ \exp\left(\frac{1}{2\sqrt{3}}\ \theta_0 n\right)$$

□

10. We now give some further evidence for a positive answer to questions 3, 4. We assume again that X is the full shift on two letters 0, 1.

The principal component of the locus of uniform hyperbolicity $\mathcal{H}$ was described in proposition 3, and its boundary in proposition 4. Here we want to consider a parameter (A_0^0, A_0^1) on the boundary such that A_0^0, A_0^1 are positive hyperbolic and $e_u(A_0^1) = e_s(A_0^0)$. After conjugating in $GL(2, \mathbf{R})$, we can assume that

$$A_0^0 = \begin{pmatrix} \bar\lambda_0 & 0 \\ 0 & \bar\lambda_0^{-1} \end{pmatrix} \quad , \quad A_0^1 = \begin{pmatrix} \bar\lambda_1^{-1} & 0 \\ \bar u_1 & \bar\lambda_1 \end{pmatrix},$$

with $\bar\lambda_0, \bar\lambda_1 > 1$ and $\bar u_1 \geq 0$. We assume that A_0^0, A_0^1 do not commute, i.e. $\bar u_1 > 0$.

PROPOSITION 7 — *The boundary of the principal component of $\mathcal{H}$ is locally near (A_0^0, A_0^1) an hypersurface which is the complement of $\mathcal{H} \cup \mathcal{E}$ in a neighbourhood of (A_0^0, A_0^1). In other terms, a "half-neighbourhood" of (A_0^0, A_0^1) is contained in $\mathcal{E}$.*

Proof — Let (A^0, A^1) be close to (A_0^0, A_0^1). In the basis $(e_u(A^0), e_u(A^1))$, we have

$$A^0 = \begin{pmatrix} \lambda_0 & u_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad , \quad A^1 = \begin{pmatrix} \lambda_1^{-1} & 0 \\ u_1 & \lambda_1 \end{pmatrix}$$

with λ_0 close to $\bar\lambda_0$, λ_1 close to $\bar\lambda_1$, u_1 close to $\bar u_1$ and u_0 close to 0.

If $u_0 > 0$, the parameter (A^0, A^1) is uniformly hyperbolic and belongs to the principal component of $\mathcal{H}$. If $u_0 = 0$, it belongs to the boundary of this component. We will show that for $u_0 < 0$, $|u_0|$ small enough, the parameter (A^0, A^1) belongs to $\mathcal{E}$.

LEMMA 3 — *Let $m_0, n_0, m_1, n_1 \geq 0$. The trace of $(A^0)^{m_0}(A^1)^{n_0}(A^0)^{m_1}(A^1)^{n_1}$ is equal to $P + u_0 u_1 Q + u_0^2 u_1^2 R$, with*

$$\begin{aligned}
P &= \lambda_0^{m_0+m_1}\lambda_1^{-(n_0+n_1)} + \lambda_0^{-(m_0+m_1)}\lambda_1^{n_0+n_1}, \\
R &= \frac{\lambda_0^{m_0} - \lambda_0^{-m_0}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_0^{m_1} - \lambda_0^{-m_1}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_1^{n_0} - \lambda_1^{-n_0}}{\lambda_1 - \lambda_1^{-1}} \; \frac{\lambda_1^{n_1} - \lambda_1^{-n_1}}{\lambda_1 - \lambda_1^{-1}}, \\
Q &= \lambda_0^{m_0}\lambda_1^{-n_0} \; \frac{\lambda_0^{m_1} - \lambda_0^{-m_1}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_1^{n_1} - \lambda_1^{-n_1}}{\lambda_1 - \lambda_1^{-1}} \\
&+ \lambda_0^{m_1}\lambda_1^{-n_1} \; \frac{\lambda_0^{m_0} - \lambda_0^{-m_0}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_1^{n_0} - \lambda_1^{-n_0}}{\lambda_1 - \lambda_1^{-1}} \\
&+ \lambda_0^{-m_0}\lambda_1^{n_1} \; \frac{\lambda_0^{m_1} - \lambda_0^{-m_1}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_1^{n_0} - \lambda_1^{-n_0}}{\lambda_1 - \lambda_1^{-1}} \\
&+ \lambda_0^{-m_1}\lambda_1^{n_0} \; \frac{\lambda_0^{m_0} - \lambda_0^{-m_0}}{\lambda_0 - \lambda_0^{-1}} \; \frac{\lambda_1^{n_1} - \lambda_1^{-n_1}}{\lambda_1 - \lambda_1^{-1}}
\end{aligned}$$

Proof — Compute ! □

Set $u = u_0 u_1 (\lambda_0 - \lambda_0^{-1})^{-1}(\lambda_1 - \lambda_1^{-1})^{-1}$. We rewrite $P + u_0 u_1 Q + u_0^2 u_1^2 R$ as $P + \bar Q u + \bar R u^2$ with

$$\bar{Q} = Q(\lambda_0 - \lambda_0^{-1})(\lambda_1 - \lambda_1^{-1})$$
$$\bar{R} = (\lambda_0^{m_0} - \lambda_0^{-m_0})(\lambda_0^{m_1} - \lambda_0^{-m_1})(\lambda_1^{n_0} - \lambda_1^{-n_0})(\lambda_1^{n_1} - \lambda_1^{-n_1}) \; .$$

Set

$$Q^* = \lambda_0^{m_0+m_1}\lambda_1^{-n_0+n_1} + \lambda_0^{m_0+m_1}\lambda_1^{n_0-n_1} + \lambda_0^{-m_0+m_1}\lambda_1^{n_0+n_1} + \lambda_0^{m_0-m_1}\lambda_1^{n_0+n_1} \; ,$$
$$R^* = \lambda_0^{m_0+m_1}\lambda_1^{n_0+n_1} \; .$$

For $\lambda_0^{m_0}, \lambda_0^{m_1}, \lambda_1^{n_0}, \lambda_1^{n_1} > 2$, one obtains

$$(\tfrac{3}{4})^2 Q^* < \bar{Q} < Q^* \; ,$$
$$(\tfrac{3}{4})^4 R^* < \bar{R} < R^* \; .$$

We assume now that

(7)
$$\lambda_0^{m_0} \leq \frac{1}{100} \min(\lambda_0^{m_1}, \lambda_1^{n_0}, \lambda_1^{n_1}) \; .$$

Then, we have, with $\hat{Q} = \lambda_0^{m_1-m_0}\lambda_1^{n_0+n_1}$

$$\hat{Q} \leq Q^* \leq (1 + 3.10^{-4})\hat{Q} \; .$$

We will choose m_0, m_1, n_0, n_1 in order to have (7) and also

(8)
$$1/2 < \lambda_0^{m_0+m_1}\lambda_1^{-(n_0+n_1)} < 2 \; ,$$

which is equivalent to

(9)
$$2 \leq P < 5/2 \; .$$

Then we obtain :

$$\bar{R}u^2 \leq R^* u^2 \leq 2.10^{-4}\lambda_0^{4m_1} u^2 \; , \text{ by } (7), (8) \; ;$$
$$\bar{Q}|u| \leq (1 + 3.10^{-4})\hat{Q}|u| \leq 2(1 + 3.10^{-4}) \; \lambda_0^{2m_1}|u| \; , \text{ by } (8) \; ;$$
$$\bar{Q}|u| \geq (\tfrac{3}{4})^2 \tfrac{1}{2} \; \lambda_0^{2m_1}|u| \; , \text{ by } (8) \; .$$

We conclude that, for

(10)
$$4(P-2)\lambda_0^{-2m_1} \leq -u \leq \lambda_0^{-2m_1} \; ,$$

the matrix $(A^0)^{m_0}(A^1)^{n_0}(A^0)^{m_1}(A^1)^{n_1}$ is elliptic.

The last step in the proof of the proposition is the :

Claim — For every $\varepsilon > 0$, there exists $m_1(\varepsilon)$ such that, for $m_1 \geq m_1(\varepsilon)$, one can find m_0, n_0, n_1 such that $\lambda_0^{m_0} > 2, P < 2 + \varepsilon$ and (7) is satisfied.

Choose $m(\varepsilon)$ such that, for any $m^* \in \mathbf{Z}$, there exists $n \in \mathbf{Z}, m \in \mathbf{Z}$ with $m^* \leq m < m^* + m(\varepsilon)$ and $1 < \lambda_0^m \lambda_1^{-n} < 1 + \varepsilon$. Let m_0^* such that $\lambda_0^{m_0^*} > 2$ and $m_1(\varepsilon) >> m_0^* + m(\varepsilon)$. For $m_1 \geq m_1(\varepsilon)$, one can find $n > 0$ and $m_0 \in [m_0^*, m_0^* + m(\varepsilon))$ such that $1 < \lambda_0^{m_0+m_1}\lambda_1^{-n} < 1 + \varepsilon$. It is now sufficient to write n as $n_0 + n_1$ with $|n_0 - n_1| \leq 1$.

Take $\varepsilon \leq \lambda_0^{-2}$. For any m_1 large enough, one can find m_0, n_0, n_1 such that the matrix $(A^0)^{m_0}(A^1)^{n_0}(A^0)^{m_1}(A^1)^{n_1}$ is elliptic for $\lambda_0^{-(2m_1+2)} \leq -u \leq \lambda_0^{-2m_1}$. This ends the proof of the proposition. □

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