AN APPLICATION OF THE SIMPLICITY OF $Diff^k_+(T^1)$ [1]

MICHAEL ROBERT HERMAN

We suppose that k is an integer, $k \neq 0, 2$; but k can be equal to $+\infty$ or ω. In these cases it is known that $Diff^k_+(T^1)$ is simple[2].

Proposition. *There exists no non-trivial algebraic group homomorphism*

$$\rho : Diff^k_+(T^1) \to Diff^1_+(D^2).$$

Observe that the standard cone construction embeds $Diff^k_+(T^1)$ into $Homeo_+(D^2)$.

Proof. Assume the existence of ρ. By simplicity of $Diff^k_+(T^1)$, ρ has to be injective. Let R be the rotation of T^1 defined by $R(x) = x + \frac{1}{2}\ \ (mod.1)$. Then $r = \rho(R)$ has to be an involution of D^2 orientation preserving. Recall that any involution of a manifold is an isometry for some Riemannian metric. Moreover, any isometry of a connected Riemannian manifold is the identity as soon as it has a fixed point where the derivative is the identity. Therefore, $r|T^1$ is not the identity. Then the rotation number of $r|T^1$ is $\frac{1}{2}$ and r has no fixed points on T^1.

Lemma 1. *The involution r has a unique fixed point x_0 in the interior of D^2.*

Proof. At each fixed point y of r, the derivative is not the identity, hence is equal to $-Id$. It follows that r is C^1-conjugate to the rotation $z \mapsto -z$ in the neighborhood of y. This implies that the fixed points of r are isolated, hence, since there are no fixed points on the boundary, in finite number. Moreover, each fixed point has index 1 . Take $S^2 = D^2 \cup D^2$ and double r. One gets a homeomorphism $\bar{r}$ of S^2 whose index is twice the number of fixed points of r. The lemma follows from $Index(\bar{r}) = 2$. □

Let us consider $Z = \{G \in Diff^k_+(T^1) \mid R \circ G = G \circ R\}$. Observe that the center of Z is the cyclic group Z_0 of order 2 generated by R. Each $g = \rho(G)$, $G \in Z$, commutes with r. Then $g(x_0) = x_0 = r(x_0)$. Consider the group homomorphism

$$\begin{aligned} D_{x_0} :\ \ & \rho(Z) \to GL_+(2, \mathbb{R})\,. \\ & g \mapsto Dg(x_0) \end{aligned}$$

It is a non-trivial homomorphism since $D_{x_0}(r) = -Id$. Notice that one has the following exact sequences (given by double covering).

[1]Ce document extrait des archives de Michel Herman a été préparé par F. Laudenbach et J.C. Yoccoz.
[2]See [1] and [2] when $k = \infty$ or ω; [3] and [4] in the other cases.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Z_0 & \xrightarrow{\pi} & Z & \longrightarrow & Diff_+^k(T^1) & \longrightarrow & 1 \\
 & & \uparrow \cong & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & SL(2,\mathbb{R}) & \longrightarrow & PSL(2,\mathbb{R}) & \longrightarrow & 1
\end{array}
$$

Lemma 2. *The subgroup Z_0 is the only non-trivial normal subgroup of Z.*

Proof. Let N be a normal subgroup of Z. Then $\pi(Z)$ is a normal subgroup of $Diff_+^k(T^1)$. As the latter is simple, $\pi(N)$ is a trivial subgroup of $Diff_+^k(T^1)$.

If $\pi(N) = \{Id\}$, N is equal to $\{Id\}$ or Z_0.

Assume $\pi(N) = Diff_+^k(T^1)$. We claim that N contains Z_0, which implies $N = Z$ and proves the lemma.

To prove the claim notice that $N_0 = N \cap SL(2,\mathbb{R})$ is a normal subgroup of $SL(2,\mathbb{R})$ which maps onto $PSL(2,\mathbb{R})$. If N does not contain Z_0, then N_0 does not contain $-Id$ and $SL(2,\mathbb{R})$ is the product $\{\pm Id\} \times PSL(2,\mathbb{R})$, which is not true (indeed, the image of a rotation of order 4 in $SL(2,\mathbb{R})$ has order 2 in $PSL(2,\mathbb{R})$). $\square$

Proof of the proposition. The kernel $K = \ker D_{x_0} \circ \rho$ is a normal subgroup of Z. By lemma 2, it is equal to $\{Id\}$, Z_0 or Z. As $R \notin K$, the two last cases are excluded. We are going to prove that the first case is also impossible. Let $R_4 \in Z$ be the rotation $x \mapsto x + \frac{1}{4}$. Then $r_4 = D_{x_0} \circ \rho(R_4)$ is of order 4 in $GL_+(2,\mathbb{R})$. Therefore it is a rotation of angle $\pm\frac{\pi}{2}$ w.r.t. a unique (up to scaling) scalar product on $\mathbb{R}^2$. The subset $A = \{G \in Z \mid G^2 = R_4\}$ is uncountable. For $G \in A$, the image $g = D_{x_0} \circ \rho(G)$ satisfies $g^2 = r_4$. There are only two elements in $GL_+(2,\mathbb{R})$ with this property. Then $D_{x_0} \circ \rho$ is not injective. $\square$

References

[1] M.R. Herman, *Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n*, C. R. Acad. Sci. Paris, Sér. A-B, 273 (1971), A232-A234.

[2] M.R. Herman, *Sur le groupe des difféomorphismes du tore*, Ann. Inst. Fourier (Grenoble) 3 (1973), 75-86.

[3] J. N. Mather, *Simplicity of certain groups of diffeomorphisms*, Bull. Amer. Math. Soc. 80 (1974), 271-273.

[4] J. N. Mather, *Commutators of Diffeomorphisms*, Comment. Math. Helv. 49 (1974), 512-528.